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Kh.H.Karayan[1], L.G.Mardoyan[1], V.M.Ter-Antonyan[2]

THE EULERIAN BOUND STATES: 5D COULOMB PROBLEM

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[1]Yerevan State University, Armenia
[2]E-mail: terant@thsun1.jinr.dubna.su

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1. Introduction

The Hurvitz transformation approach to the bound states physics distinguished 5D Hydrogen atom problem as the exact solvable quantum model with the non–abelian constraints [1]

The eigenvalues of the energy ($n = 0, 1, 2, \cdots$)

$$E_n = -\frac{\mu_0 \alpha^2}{2\hbar^2 (n+2)^2}$$

are degenerated with multiplicity

$$g_n = \frac{(n+1)(n+3)(n+2)^2}{12}.$$

These properties have the most simple expression in terms of the Eulerian coordinates [2]. We undertake, in present paper, the Eulerian analyses of the 5D Hydrogen bound states problem. In this way, we achieved the 5D generalization of some results of the usual Hydrogen atom theory [3], [4].

The material of the present paper is arranged as follows. The subject matter of the sections 2 and 3 is the calculation of the spherical bound states of the 5D Hydrogen atom. The two subsequent sections deal with the same topics for the parabolic states. The expressions for the generalized Park–Tarter's amplitudes are derived though sections 6 and 7. In next two sections we investigate the spheroidal states as the spherical and parabolic superpositions. The trinominal recurrent relations have been established for the generalized Coulson–Joseph's amplitudes. Finally, we conclude that the 5D extension for the well-known results of the ordinary theory has been searched for thanks to employment of the Eulerian coordinates.

2 Spherical 5D coordinates

Eulerian spherical coordinates for the 5D space are defined by

$$\begin{aligned}
x_0 &= r\cos\varphi \\
x_1 &= r\sin\varphi\cos\beta/2\cos(\alpha+\gamma)/2 \\
x_2 &= r\sin\varphi\cos\beta/2\sin(\alpha+\gamma)/2 \\
x_3 &= r\sin\varphi\sin\beta/2\cos(\alpha-\gamma)/2 \\
x_4 &= r\sin\varphi\sin\beta/2\sin(\alpha-\gamma)/2
\end{aligned} \tag{1}$$

This coordinates may vary in the ranges

$$0 \le r < \infty, \qquad 0 \le \varphi \le 2\pi, \qquad 0 \le \beta \le \pi, \qquad 0 \le \alpha < 4\pi, \qquad 0 \le \gamma < 2\pi.$$

Coulomb potential, differential elements of the length and volume and the Laplace operator have the next forms

$$V = -\frac{\alpha}{r}$$

$$dl_5^2 = dr^2 + r^2 d\varphi^2 + \frac{r^2}{4}\sin^2\varphi dl^2$$

$$dV_5 = r^4 dr \sin^3\varphi d\varphi d\Omega$$

$$\Delta_5 = \frac{1}{r^4}\frac{\partial}{\partial r}\left(r^4\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin^3\varphi}\frac{\partial}{\partial\varphi}\left(\sin^3\varphi\frac{\partial}{\partial\varphi}\right) - \frac{4}{r^2\sin^2\varphi}\hat{J}^2 ,$$

where

$$dl^2 = d\alpha^2 + d\beta^2 + d\gamma^2 + 2\cos\beta d\alpha d\gamma$$

$$d\Omega = \frac{1}{8}\sin\beta d\beta d\alpha d\gamma$$

$$\hat{J}^2 = -\left[\frac{\partial^2}{\partial\beta^2} + \cot\beta\frac{\partial}{\partial\beta} + \frac{1}{\sin^2\beta}\left(\frac{\partial^2}{\partial\alpha^2} - 2\cos\beta\frac{\partial^2}{\partial\alpha\partial\beta} + \frac{\partial^2}{\partial\gamma^2}\right)\right].$$

3 Spherical Bound States

In the coordinates (1) the scheme of separation of the variables corresponds to the factorization

$$\Psi^{sph} = R(r)\psi(\varphi)\mathcal{D}^j_{mm'}(\alpha,\beta,\gamma), \tag{2}$$

where $\mathcal{D}^j_{mm'}$ is Wigner's function [5]. Taking into account that

$$\hat{J}^2\mathcal{D}^j_{mm'}(\alpha,\beta,\gamma) = j(j+1)\mathcal{D}^j_{mm'}(\alpha,\beta,\gamma)$$

we arrive to the two coupled differential equations

$$\frac{1}{\sin^3\varphi}\frac{d}{d\varphi}\left(\sin^3\varphi\frac{d\psi}{d\varphi}\right) - \frac{1}{\sin^2\varphi}4j(j+1)\psi + \tau(\tau+3)\psi = 0, \tag{3}$$

$$\frac{1}{r^4}\frac{d}{dr}\left(r^4\frac{dR}{dr}\right) - \frac{\tau(\tau+3)}{r^2}R + \frac{1}{a}\left(\frac{2}{r} - \frac{1}{a(n+2)}\right)R = 0. \tag{4}$$

Here $\tau(\tau+3)$ is a separation constant and

$$n+2 = \frac{h}{\sqrt{-2\mu_0 E}}, \qquad a = \frac{h^2}{\mu_0\alpha}.$$

(μ_0 is the mass of the electron). Equation (3) determines the Gegenbauer polynoimials [6]

$$C_n^\nu = \frac{\Gamma(n+2\nu)}{\Gamma(2\nu)\Gamma(n+1)} {}_2F_1\left(-n, n+2\nu; \nu+\frac{1}{2}; \frac{1-x}{2}\right)$$

only to the factor $(\sin\varphi)^{2j}$. We choose the following normalization condition

$$\int_0^\pi |\psi_{\tau j}(\varphi)|^2 \sin^3\varphi d\varphi = 1$$

and find

$$\psi_{\tau j}(\varphi) = 2^{2j+1}\Gamma\left(2j+\frac{3}{2}\right)\sqrt{\frac{(2\tau+3)(\tau-2j)!}{2\pi(\tau+2j+2)!}}(\sin\varphi)^{2j} C_{\tau-2j}^{2j+3/2}(\cos\varphi) \cdot \tag{5}$$

The radial function R, under the normalization condition

$$\int_0^\infty R_{n\tau}^2(r) r^4 dr = 1$$

can be expressed in terms of the confluent hypergeometric function

$$R_{n\tau}(r) = C_{n\tau}(\varepsilon r)^\tau e^{-\frac{\varepsilon r}{2}} F(-n+\tau; 2\tau+4; \varepsilon r) \tag{6}$$

$$C_{n\tau} = \frac{4}{a^{5/2}(n+2)^3} \frac{1}{(2\tau+3)!}\sqrt{\frac{(n+\tau+3)!}{(n-\tau)!}}. \tag{7}$$

Here $n = 0, 1, \cdots$ and the parameter ε is defined by

$$\varepsilon = \frac{2}{a(n+2)} \cdot \tag{8}$$

Thus, the normalized spherical wavefunction for the 5D Hydrogen atom has the next form

$$\Psi^{sph} = \sqrt{\frac{2j+1}{2\pi^2}} R_{n\tau}(r)\psi_{\tau j}(\varphi)\mathcal{D}^j_{mm'}(\alpha,\beta,\gamma). \tag{9}$$

The spherical basis (9) is the common eigenfunction of the operators $\{\hat{H}, \hat{J}^2, \hat{J}_z, \hat{J}'_z\}$ and the operator $\hat{T}^2$ of the global angular momentum

$$\hat{T}^2 = -\frac{1}{\sin^3\varphi}\frac{\partial}{\partial\varphi}\left(\sin^3\varphi\frac{\partial}{\partial\varphi}\right) + \frac{4}{\sin^2\varphi}\hat{J}^2 \tag{10}$$

with

$$\hat{T}^2\Psi^{sph} = \tau(\tau+3)\Psi^{sph}.$$

In Cartesian coordinates:

$$\hat{T}^2 = -r^2\Delta_5 + x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + 4x_i\frac{\partial}{\partial x_i}, \tag{11}$$

where $i, j = 0, 1, 2, 3, 4$.

4 Parabolic 5D Coordinates

Denote the Eulerian 5D parabolic coordinates by $\mu, \nu, \alpha, \beta, \gamma$ and assume that $\mu, \nu \in [0, \infty)$ and α, β, γ have the same meaning as for Eulerian spherical coordinates. By definition

$$\begin{aligned} x_0 &= (\mu - \nu)/2 \\ x_1 &= \sqrt{\mu\nu} \cos \beta/2 \cos(\alpha + \gamma)/2 \\ x_2 &= \sqrt{\mu\nu} \cos \beta/2 \sin(\alpha + \gamma)/2 \\ x_3 &= \sqrt{\mu\nu} \sin \beta/2 \cos(\alpha - \gamma)/2 \\ x_4 &= \sqrt{\mu\nu} \sin \beta/2 \sin(\alpha - \gamma)/2 . \end{aligned} \tag{12}$$

In the coordinates (12)

$$V = -\frac{2\alpha}{\mu + \nu} .$$

As a consequence of the (12)

$$dl_5^2 = \frac{\mu + \nu}{4\mu} d\mu^2 + \frac{\mu + \nu}{4} d\nu^2 + \frac{\mu\nu}{4} dl^2$$

$$dV_5 = \frac{\mu\nu}{4} (\mu + \nu) d\mu d\nu d\Omega$$

$$\Delta_5 = \frac{4}{\mu + \nu} \left[\frac{1}{\mu} \frac{\partial}{\partial\mu} \left(\mu^2 \frac{\partial}{\partial\mu} \right) + \frac{1}{\nu} \frac{\partial}{\partial\mu} \left(\nu^2 \frac{\partial}{\partial\nu} \right) - \frac{4}{\mu\nu} \hat{J}^2 \right] .$$

The differentials dl^2,$d\Omega$ and the operator $\hat{J}^2$ have been introduced above.

5 Parabolic Bound States

Starting from the representation

$$\Psi^{par} = \sqrt{\frac{2j + 1}{2\pi^2}} C^{par} \Phi_1(\mu) \Phi_2(\nu) \mathcal{D}^j_{mm'}(\alpha, \beta, \gamma) \tag{13}$$

we can simply derive two equations

$$\frac{1}{\mu} \frac{\partial}{\partial\mu} \left(\mu^2 \frac{\partial\Phi_1}{\partial\mu} \right) + \left[\frac{\mu_0 E}{2\hbar^2} \mu - \frac{j(j + 1)}{\mu} + C_1 \right] \Phi_1 = 0, \tag{14}$$

$$\frac{1}{\nu} \frac{\partial}{\partial\nu} \left(\nu^2 \frac{\partial\Phi_2}{\partial\nu} \right) + \left[\frac{\mu_0 E}{2\hbar^2} \nu - \frac{j(j + 1)}{\nu} + C_2 \right] \Phi_2 = 0, \tag{15}$$

where C_1 and C_2 are the separation constants and $C_1 + C_2 = a^{-1}$. Let us introduce the parabolic quantum numbers

$$n_i = -j - 1 + a(n + 2)C_i \tag{16}$$

with $i = 1, 2$ and n as the principal quantum number. Then, it is easy to show that parabolic basis must be given by following expressions

$$\Phi_{n_1 j}(\mu) = \frac{1}{(2j+1)!}\sqrt{\frac{(n_1+2j+1)!}{(n_1)!}}\left(\frac{\varepsilon\mu}{2}\right)^j exp-\frac{\varepsilon\mu}{4}F(-n_1; 2j+1; \frac{\varepsilon\mu}{2}) \tag{17}$$

$$\Phi_{n_2 j}(\nu) = \frac{1}{(2j+1)!}\sqrt{\frac{(n_2+2j+1)!}{(n_2)!}}\left(\frac{\varepsilon\nu}{2}\right)^j exp-\frac{\varepsilon\nu}{4}F(-n_2; 2j+1; \frac{\varepsilon\nu}{2}) \tag{18}$$

$$C^{par} = \frac{\sqrt{2}}{a^{5/2}(n+2)^3} \ . \tag{19}$$

Eliminating the energy E from Eqs. (14) and (15) we get the additional integral of motion

$$\widehat{A}_0 = \frac{2}{\mu+\nu}\left[\frac{\mu}{\nu}\frac{\partial}{\partial\nu}\left(\nu^2\frac{\partial}{\partial\nu}\right) - \frac{\nu}{\mu}\frac{\partial}{\partial\mu}\left(\mu^2\frac{\partial}{\partial\mu}\right)\right] - \frac{2(\mu-\nu)}{\mu\nu}\widehat{J}^2 + \frac{\mu-\nu}{a(\mu+\nu)} \tag{20}$$

with eigenvalues

$$C_1 - C_2 = \frac{n_1 - n_2}{a(n+2)} \ . \tag{21}$$

In Cartesian coordinates, the operator $\widehat{A}_0$ can be rewritten as

$$\widehat{A}_0 = x_0\frac{\partial^2}{\partial x_\sigma \partial x_\sigma} - x_\sigma\frac{\partial^2}{\partial x_0 \partial x_\sigma} - 2\frac{\partial}{\partial x_0} + \frac{x_0}{ar} \tag{22}$$

with $\sigma = 1, 2, 3, 4$. Let us compare (22) with 5D Runge-Lenz vector [7]

$$\widehat{A}_t = \frac{1}{2}(L_{st}P_s + P_s L_{st}) + \frac{x_t}{a(x_s x_s)^{1/2}} \ , \tag{23}$$

where

$$L_{ij} = x_i P_j - x_j P_i, \qquad P_j = -i\hbar\frac{\partial}{\partial x_j} \ .$$

It immediately follows that $\widehat{A}_0$ is the $(t = 0)$ component of $\widehat{A}_t$

6 Tarter's Representation

Now, we can write, for fixed value of the energy, the parabolic bound states (13) as a coherent quantum mixture of the spherical bound states:

$$\Psi^{par} = \sum_{\tau=2j}^{n} W^{\tau}_{n_1 n_2 j}\Psi^{sph}. \tag{24}$$

Expansion (24) is the 5D generalization of the Park's expansion [3] for the Coulomb problem.

The aim of this section will be to obtain the explicit form of the amplitudes $W^{\tau}_{n_1 n_2 j}$. At first, note that

$$\mu = r(1+\cos\varphi), \qquad \nu = r(1-\cos\varphi). \tag{25}$$

Then, substituting $\varphi = 0$, taking into account that

$$C_n^{\lambda}(1) = \frac{(2\lambda)_n}{(n)!} \tag{26}$$

and using the orthogonality relation [8]

$$\int_0^{\infty} r^2 R_{n\tau'}(r) R_{n\tau}(r) dr = \frac{2}{a^2(n+2)^3}\frac{1}{2\tau+3}\delta_{\tau\tau'} \tag{27}$$

we get the following integral representation for the generalized Park-Tarter's amplitudes

$$W^{\tau}_{n_1 n_2 j} = \frac{1}{(2j+1)!(2\tau+3)!} E^{\tau j}_{n_1 n_2} K^{\tau j}_{n n_1}. \tag{28}$$

Here

$$E^{\tau j}_{n_1 n_2} = \left[\frac{(2\tau+3)(\tau-2j)!(n+\tau+3)!(n_1+2j+1)!(n_2+2j+1)!}{(n_1)!(n_2)!(n-\tau)!(\tau+2j+2)!}\right]^{1/2} \tag{29}$$

$$K^{\tau j}_{n n_1} = \int_0^{\infty} x^{\tau+2j+2} e^{-x} {}_1F_1(-n_1, 2j+2; x)\, {}_1F_1(-n+\tau, 2\tau+4; x)\, dx \tag{30}$$

and

$$x = \varepsilon r = \frac{2r}{a(n+2)} \cdot$$

After writing the $F(-n_1; 2j+2; r)$ as a series, integrating according to [9]

$$\int_0^{\infty} e^{-\lambda x} x^{\nu} {}_1F_1(\alpha, \gamma; kx)\, dx = \frac{\Gamma(\nu+1)}{\lambda^{\nu+1}} {}_2F_1\left(\alpha, \nu+1, \gamma; \frac{k}{\lambda}\right). \tag{31}$$

and using the formula

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{32}$$

for the summation of the Gauss hypergeometric function, we obtain

$$K^{\tau j}_{n n_1} = \frac{(n-2j)!(2\tau+3)!(\tau+2j+2)!}{(n+\tau+3)!(\tau-2j)!} \cdot {}_3F_2\left\{\begin{matrix} -n_1, -\tau+2j, \tau+2j+3 \\ 2j+2, -n+2j \end{matrix}\middle| 1\right\}, \tag{33}$$

and thus come to the generalized version of the Tarter's representation [3]:

$$W^{\tau}_{n_1 n_2 j} = \frac{(n-2j)!}{(2j+1)!}\sqrt{\frac{(2\tau+3)(\tau+2j+2)!(n_1+2j+1)!(n_2+2j+1)!}{(n_1)!(n_2)!(\tau-2j)!(n-\tau)!(n+\tau+3)!}}$$

$$\cdot\ {}_3F_2\left\{ \begin{matrix} -n_1, -\tau+2j, \tau+2j+3 \\ 2j+2, -n+2j \end{matrix} \middle| 1 \right\}. \tag{34}$$

7 Park's Representation

The next step is to derive the Park's representation for $W^{\tau}_{n_1 n_2 j}$. It is sufficient, for this purpose, to write the Clebsch-Gordan coefficients for the group $SU(2)$ in term of ${}_3F_2$ - function.

$$C^{c\gamma}_{a\alpha;b\beta} = (-1)^{a-\alpha}\delta_{\gamma,\alpha+\beta}(a+b-\gamma)!(b+c-\alpha)!$$

$$\cdot\left[\frac{(2c+1)(a+\alpha)!(c+\gamma)!}{(a-\alpha)!(b-\beta)!(b+\beta)!(c-\gamma)!(a+b+c+1)!(a+b-c)!(a-b+c)!(b-a+c)!}\right]^{1/2}$$

$$\cdot\ {}_3F_2\left\{ \begin{matrix} -a-b-c-1, -a+\alpha, -c+\gamma \\ -a-b+\gamma, -b-c+\alpha \end{matrix} \middle| 1 \right\}. \tag{35}$$

To use the formula [10]

$${}_3F_2\left\{ \begin{matrix} s, s', -N \\ t', 1-N-t \end{matrix} \middle| 1 \right\} = \frac{(t+s)_N}{(t)_N}\ {}_3F_2\left\{ \begin{matrix} s, t'-s', -N \\ t', t+s \end{matrix} \middle| 1 \right\}, \tag{36}$$

equation (35) can be rewritten in the form

$$C^{c\gamma}_{a\alpha;b\beta} = (-1)^{a-\alpha}\delta_{\gamma,\alpha+\beta}\left[\frac{(2c+1)(b-a+c)!(a+\alpha)!(b+\beta)!(c+\gamma)!}{(a-\alpha)!(b-\beta)!(c-\gamma)!(a+b-c)!(a-b+c)!(a+b+c+1)!}\right]^{1/2}$$

$$\cdot\ \frac{(a+b-\gamma)!}{(b-a+\gamma)!}\ {}_3F_2\left\{ \begin{matrix} -a+\alpha, c+\gamma+1, -c+\gamma \\ \gamma-a-b, b-a+\gamma+1 \end{matrix} \middle| 1 \right\}. \tag{37}$$

By comparing (37) and (34), we finally obtain the desired representation

$$W^{\tau}_{n_1 n_2 j} = (-1)^{n_1} C^{\tau+1,2j+1}_{\frac{n+1}{2},\frac{2j+n_2-n_1+1}{2};\frac{n+1}{2},\frac{2j+n_1-n_2+1}{2}}. \tag{38}$$

The transformation inverse to, (25), namely

$$\Psi^{sph} = \sum_{n_1=0}^{n-2j} \widetilde{W}^{n_1}_{n\tau j}\Psi^{par}, \tag{39}$$

is an immediate consequence of the orthonormality propriety of the $SU(2)$ Clebsch-Gordan coefficients. The expansion coefficients in (39) are thus given by

$$\widetilde{W}^{n_1}_{n\tau j} = (-1)^{n_1} C^{\tau+1,2j+1}_{\frac{n+1}{2},\frac{n+1}{2}-n_1;\frac{n+1}{2},n_1+2j-\frac{n-1}{2}} \tag{40}$$

and may be expressed in terms of the ${}_3F_2$ function through (35) or (37).

8 Spheroidal 5D Coordinates

The spheroidal coordinates, Coulomb potential and some differential constructions we write as:

$$\begin{aligned}
x_0 &= \tfrac{R}{2}(\xi\eta + 1) \\
x_1 &= \tfrac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\cos\beta/2\cos(\alpha+\gamma)/2 \\
x_2 &= \tfrac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\cos\beta/2\sin(\alpha+\gamma)/2 \\
x_3 &= \tfrac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\sin\beta/2\cos(\alpha-\gamma)/2 \\
x_4 &= \tfrac{R}{2}\sqrt{(\xi^2-1)(1-\eta^2)}\sin\beta/2\sin(\alpha-\gamma)/2
\end{aligned} \tag{41}$$

$$1 \le \xi < \infty, -1 \le \eta \le 1$$

$$V = -\frac{2\alpha}{R(\xi+\eta)} \tag{42}$$

$$dl_5^2 = \frac{R^2}{4}(\xi^2-\eta^2)\left(\frac{d\xi^2}{\xi^2-1}+\frac{d\eta^2}{1-\eta^2}\right)+\frac{R^2}{16}(\xi^2-1)(1-\eta^2)dl^2$$

$$dV_5 = \frac{R^5}{32}(\xi^2-\eta^2)(\xi^2-1)(1-\eta^2)d\xi d\eta d\Omega$$

$$\Delta_5 = \frac{4}{R^2(\xi^2-\eta^2)}\left\{\frac{1}{\xi^2-1}\frac{\partial}{\partial\xi}\left[(\xi^2-1)^2\frac{\partial}{\partial\xi}\right]+\frac{1}{1-\eta^2}\frac{\partial}{\partial\eta}\left[(1-\eta^2)^2\frac{\partial}{\partial\eta}\right]\right\}$$
$$-\frac{16\widehat{J}^2}{R^2(\xi^2-1)(1-\eta^2)}\,. \tag{43}$$

Here $R \in [0,\infty)$ is the interfocus distance. In limits $R \to 0$ and $R \to \infty$ we come back to the spherical and parabolic coordinates.

9 Spheroidal Integral of Motion

The separation of variables in spheroidal coordinates

$$\Psi^{spheroidal} = \sqrt{\frac{2j+1}{2\pi^2}}f_1(\xi)f_2(\eta)\mathcal{D}^j_{mm'}(\alpha,\beta,\gamma) \tag{44}$$

produces the separation constant λ (which is a function of the parameter R) and leads to two ordinary differential equations

$$\left[\frac{1}{\xi^2-1}\frac{d}{d\xi}(\xi^2-1)^2\frac{d}{d\xi}-\frac{4j(j+1)}{\xi^2-1}+\frac{R}{a}\xi+\frac{\mu_0R^2E}{2\hbar^2}(\xi^2-1)\right]f_1 = \lambda(R)f_1 \tag{45}$$

$$\left[\frac{1}{1-\eta^2}\frac{d}{d\eta}(1-\eta^2)^2\frac{d}{d\eta}-\frac{4j(j+1)}{1-\eta^2}-\frac{R}{a}\eta+\frac{\mu_0R^2E}{2\hbar^2}(1-\eta^2)\right]f_2 = -\lambda(R)f_2\,. \tag{46}$$

By standard trick we can obtain the Hermitian operator

$$\widehat{\Lambda} = \frac{1}{\xi^2-\eta^2}\left\{\frac{1-\eta^2}{\xi^2-1}\frac{\partial}{\partial\xi}(\xi^2-1)^2\frac{\partial}{\partial\xi} - \frac{\xi^2-1}{1-\eta^2}\frac{\partial}{\partial\eta}(1-\eta^2)^2\frac{\partial}{\partial\eta}\right\} + \frac{4(\xi^2+\eta^2-2)}{(\xi^2-1)(1-\eta^2)}\widehat{J}^2 + \frac{R(\xi\eta+1)}{a(\xi+\eta)} \qquad (47)$$

the eigenvalues of which are $\lambda(R)$ and eigenfunctions of which are $\Psi^{spheroidal}$. Returning to Cartesian coordinates and performing a long calculation we obtain

$$\widehat{\Lambda} = \widehat{T}^2 + R\widehat{A}_0 \, . \qquad (48)$$

Thus, $\widehat{\Lambda}$ is a simple linear combination of spherical and parabolic integrals of motion.

10 Generalized Coulson-Joseph's Amplitudes

In this section we briefly discuss the eigenvalue problem

$$\widehat{\Lambda}\Psi^{spheroidal} = \lambda_q(R)\Psi^{spheroidal} \, . \qquad (49)$$

The integer-valued index q labels the eigenvalues of the operator $\widehat{\Lambda}$ and varies in the range $0 \le q \le n-2j$.

Our strategy will be along the following line:

(a) consider the expansions

$$\Psi^{spheroidal} = \sum_{\tau=2j}^{n} U^{\tau}_{nqj}(R)\Psi^{sph}, \qquad (50)$$

$$\Psi^{spheroidal} = \sum_{n_1=0}^{n-2j} M^{n_1}_{nqj}(R)\Psi^{par}, \qquad (51)$$

(b) act by $\widehat{\Lambda}$ on both sides of (50) and (51)

(c) use the equations

$$\widehat{T}^2\Psi^{sph} = \tau(\tau+3)\Psi^{sph} \qquad (52)$$

$$\widehat{A}_0\Psi^{par} = \frac{n_1-n_2}{a(n+2)}\Psi^{par} \, . \qquad (53)$$

Then, by using (48) we find two sistems of linear homogeneous equations

$$[\lambda_q(R) - \tau(\tau+3)]U^{\tau}_{nqj}(R) = R\sum_{\tau'=2j}^{n} U^{\tau'}_{nqj}(R)(\widehat{A}_0)_{\tau\tau'}, \qquad (54)$$

$$\left[\lambda_q(R) - \frac{R(n_1 - n_2)}{a(n+2)}\right] M^{n_1}_{nqj}(R) = \sum_{n_1'=0}^{n-2j} M^{n_1'}_{nqj}(R)(\widehat{T}^2)_{n_1 n_1'}. \tag{55}$$

Here

$$(\widehat{A}_0)_{\tau\tau'} = \int \Psi^{*sph}_{\tau} \widehat{A}_0 \Psi^{sph}_{\tau'} \, dV. \tag{56}$$

$$(\widehat{T}^2)_{n_1 n_1'} = \int \Psi^{*par}_{n_1} \widehat{T}^2 \Psi^{par}_{n_1'} \, dV. \tag{57}$$

It followes from (25) and (39) that

$$(\widehat{A}_0)_{\tau\tau'} = \frac{1}{a(n+2)} \sum_{n_1=0}^{n-2j} (2n_1 - n + 2j) \widetilde{W}^{n_1}_{n\tau j} \widetilde{W}^{n_1}_{n\tau' j}. \tag{58}$$

If we substitute the expression (40) into (54), take into account the recursion relation [5]

$$C^{c\gamma}_{a\alpha;b\beta} = -\left[\frac{4c^2(2c+1)(2c-1)}{(c+\gamma)(c-\gamma)(-a+b+c)(a-b+c)(a+b-c+1)(a+b+c+1)}\right]^{1/2}$$

$$\cdot\left\{\left[\frac{(c-\gamma-1)(c+\gamma-1)(-a+b+c-1)(a-b+c-1)(a+b-c+2)(a+b+c)}{4(c-1)^2(2c-3)(2c-1)}\right]^{1/2}\right.$$

$$\left.\cdot\, C^{c-2,\gamma}_{a\alpha;b\beta} - \frac{(\alpha-\beta)c(c-1) - \gamma a(a+1) + \gamma b(b+1)}{2c(c-1)} C^{c-1,\gamma}_{a\alpha;b\beta}\right\} \tag{59}$$

and use the orthonormality condition

$$\sum_{\alpha,\beta} C^{c\gamma}_{a\alpha;b\beta} C^{c'\gamma'}_{a\alpha;b\beta} = \delta_{c'c}\delta_{\gamma'\gamma}, \tag{60}$$

we find that

$$(\widehat{A}_0)_{\tau\tau'} = -\frac{1}{a(n+2)}\left(B^{\tau+1}_{nj}\delta_{\tau',\tau+1} + B^{\tau}_{nj}\delta_{\tau',\tau-1}\right), \tag{61}$$

where

$$B^{\tau}_{nj} = \left[\frac{(\tau+2j+2)(\tau-2j)(n-\tau+1)(n+\tau+3)}{(2\tau+1)(2\tau+3)}\right]^{1/2}. \tag{62}$$

Now, combining (61) with (58) we obtain the following trinominal recurrent relation for the coefficients U

$$[\lambda_q(R) - \tau(\tau+3)]\, U^{\tau}_{nqj}(R) + + \frac{R}{a(n+2)}\left[B^{\tau+1}_{nj} U^{\tau+1}_{nqj}(R) + B^{\tau}_{nj} U^{\tau-1}_{nqj}(R)\right] = 0. \tag{63}$$

This system of $n - 2j$ linear homogeneous equations must be solved simultaneously with the normalization condition

$$\sum_{\tau=2j}^{n} |U^{\tau}_{nqj}(R)|^2 = 1. \tag{64}$$

The eigenvalues $\lambda_q(R)$ of the operator $\hat{\Lambda}$ then follow from the vanishing of the determinant for the system (63).

The matrix element $(\hat{T}^2)_{n_1 n_1'}$ can be calculated in the same way as $(\hat{A}_0)_{rr'}$ except that now we must use the relation [5].

$$[(b-a+c)(a-b+c+1)]^{1/2} C^{c\gamma}_{a\alpha;b\beta} = [(a-\alpha+1)(b-\beta)]^{1/2} C^{c\gamma}_{a+1/2,\alpha-1/2;b-1/2,\beta+1/2}$$

$$+ [(a+\alpha+1)(b+\beta)]^{1/2} C^{c\gamma}_{a+1/2,\alpha+1/2;b-1/2,\beta-1/2} \qquad (65)$$

and the orthonormality condition

$$\sum_{c,\gamma} C^{c\gamma}_{a\alpha;b\beta} C^{c\gamma}_{a\alpha';b\beta'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}. \qquad (66)$$

In this way we obtain the expression

$$(\hat{T}^2)_{n_1 n_1'} = [(n_1+1)(n-n_1-2j) + (n-n_1+2)(n_1+2j+1) - 2]\, \delta_{n_1' n_1}$$

$$-[n_1(n-n_1-2j+1)(n-n_1+2)(n_1+2j+1)]^{1/2}\, \delta_{n_1',n_1-1}$$

$$-[(n_1+1)(n-n_1-2j)(n-n_1+1)(n_1+2j+2)]^{1/2}\, \delta_{n_1',n_1+1}. \qquad (67)$$

which leads to the recurrent relation

$$\left[(n_1+1)(n-n_1-2j)+(n-n_1+2)(n_1+2j+1)-2+\frac{R(2n_1-n+2j}{c(n+2)}-\lambda_q(R)\right] M^{n_1}_{nqj}(R) =$$

$$[(n_1+1)(n-n_1-2j)(n-n_1+1)(n_1+2j+2)]^{1/2} M^{n_1+1}_{nqj}(R)$$

$$+ [n_1(n-n_1-2j+1)(n-n_1+2)(n_1+2j+1)]^{1/2} M^{n_1-1}_{nqj}(R). \qquad (68)$$

This equations must be solved together with the normalization conditions

$$\sum_{n_1=0}^{n-2j} |M^{n_1}_{nqj}(R)|^2 = 1 \qquad (69)$$

Mention finally, that the matrices U and W generalized the well-known Coulson-Joseph's amplitudes of the 3D two-center Coulomb problem.

11 Conclusions

In present paper we consider the bound states of the 5D Couloumb problem by separating variables in spherical, parabolic and spheroidal Eulerian coordinates.

The explicit expressions are derived for the normalized wave functions and Park's coefficients, and the trinominal recurrent relations have been established for the 5D Coulson- Joseph's amplitudes.

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