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CAMPBELL-HAUSDORFF FORMULA AND ALGEBRAS WITH OPERATOR

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1. Introduction

It is well known Campbell-Hausdorff formula (*CH* formula) which states that for arbitrary operators A and B $\log e^A e^B$ is expressed through the commutators of the operators A and B:

$$\log e^A e^B = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[B,[B,A]] + \dots \tag{1.1}$$

where $[A,B] = AB - BA$.

Let A be an associative algebra with unity with linear operator Π acting on it such that

$$\Pi 1 = 1, \quad \forall a,b \in A \quad \Pi(a\Pi b) = \Pi(ab), \quad \Pi((\Pi a)b) = \Pi a \Pi b \quad . \tag{1.2}$$

We formulate in this paper a modification of CH-formula for the algebras of this type and discuss one of its applications.

For arbitrary $a_1, \dots, a_n \in A$ we define $\mathcal{G}_\Pi(a_1, \dots, a_n)$ as the Lie algebra generated by the elements $a_1, \dots, a_n$ and by the linear operator Π in the following standard way: Let $[A]$ be commutators Lie algebra— $[A]$ is the algebra A with redefined multiplication

$$[u,v] = uv - vu. \tag{1.3}$$

Then $\mathcal{G}_\Pi(a_1 \dots, a_n)$ is the minimal Lie subalgebra of $[A]$ which obeys to the conditions

$$a_1, \dots a_n \in \mathcal{G}, \quad \text{if} \quad c \in \mathcal{G} \quad \text{then} \quad \Pi c \in \mathcal{G}. \tag{1.4}$$

Theorem 1 For arbitrary $a, b \in A$

$$\log \Pi e^a e^b \in \mathcal{G}_\Pi(a.b) \quad . \tag{1.5}$$

Example 1 In the case if Π is identity operator ($\Pi = \mathbf{id}$) then the conditions (1.2) are fulfilled automatically and we come to CH formula (1.1).

Example 2 Let Γ be a space of the functions and D-the associative algebra of the differentiation operators of all the orders acting on Γ. We consider the projection operator Π acting on D as

$$\Pi a: (\Pi a) f = (a1) f, \tag{1.6}$$

where $a \in D$, a function $f \in \Gamma$, 1 is unity function.—Π is the projection operator which extracts from the differential operator its null degree part. It is easy to see that Π obeys to conditions (1.2).

Before going to detailed proof of this Theorem we briefly recall the algebraic proof of CH-formula (the case where $\Pi = \mathbf{id}$) which is based on the following considerations (See for details for example [1,2]).

Let $\mathcal{G}$ be an arbitrary Lie algebra and $\mathcal{U}(\mathcal{G})$ its universal enveloping algebra.

One can define comultiplication δ on $\mathcal{U}(\mathcal{G})$ - the homomorphism

$$\delta: \mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \tag{1.7}$$

which is correctly and uniquely defined by its values on $\imath\mathcal{G}$:

$$\forall x \in \mathcal{G} \quad \delta \imath x = \imath x \otimes 1 + 1 \otimes \imath x. \tag{1.8}$$

($\imath$ is canonical embedding of $\mathcal{G}$ in $\mathcal{U}$ (monomorphism of $\mathcal{G}$ in $[\mathcal{U}]$)). The elements $a \otimes 1 + 1 \otimes a$ of $U \otimes U$ are called primitive.

The remarkable fact is that comultiplication δ extracts $\mathcal{G}$ from U ([1,2]):

$$\delta a = a \otimes 1 + 1 \otimes a \quad \text{iff} \quad a = \imath x. \tag{1.9}$$

In the case if $A = K(a, b)$ is free associative algebra with unity with two generators a, b then it coincides with the universal enveloping algebra of the Lie algebra $\mathcal{G}(a, b)$ ($\mathcal{G}(a, b) = \mathcal{G}_{\mathbf{id}}(a, b)$ is subalgebra of $[K(a, b)]$ defined by (1.4)). So we can apply (1.9) for proving (1.1) in the case $a, b \in K(a, b)$.

$x \otimes 1$ and $1 \otimes y$ commute in $U \otimes U$ so it is easy to calculate that

$$\delta \log e^a e^b = \log e^{a \otimes 1 + 1 \otimes a} e^{b \otimes 1 + 1 \otimes b} = \log e^a e^b \otimes 1 + 1 \otimes \log e^a e^b \tag{1.10}$$

is primitive in $U \otimes U$. (In (1.10) x and ix are identified). CH-formula is proved for the free algebra $K(a,b)$ hence for arbitrary associative algebra with unity.

Using CH-formula (1.1) we can reformulate the statement of the Theorem 1:

Theorem 1' If A is an associative algebra with unity and with linear operator Π which acts on it obeying to the conditions (1.2) then

$$\forall c \in A \quad \log \Pi e^c \in \mathcal{G}_\Pi(c) \quad . \tag{1.11}$$

$$\left(\log \Pi e^c = \Pi c + \frac{1}{2}\Pi[c,\Pi c] + \frac{1}{6}\Pi[c,\Pi[c,\Pi c]] + \frac{1}{6}\Pi[\Pi c,[\Pi c,c]] + \frac{1}{6}[\Pi c,\Pi[c,\Pi c]] + \ldots\right)$$

($\mathcal{G}_\Pi(c) = \mathcal{G}_\Pi(c,0)$ (1.5), if $c = \log e^a e^b \in \mathcal{G}_id(a,b)$ then $\mathcal{G}_\Pi(c) \subset \mathcal{G}_\Pi(a,b)$.)

To generalize the considerations above for proving the Theorem 1' in the Section 2 we consider the associative algebras provided with additional structure corresponding to the action of operator Π obeying to the conditions (1.2) (CH-algebras) and construct free associative CH-algebra with one generator.

In the Section 3 we introduce Lie CH algebras study their universal enveloping algebras. The main result of this paper is the Theorem 2 which is formulated and proved in this Section and which allows us to prove the Theorem 1' in the same way like the proving of CH-formula ((1.7) - (1.10)).

In the Section 4 considering the algebras like in the Example 2 we use the Theorem 1 for the algebraic reformulation of the conditions of the connectivity of Feynman diagrams corresponding to the logarithm of partition function in quantum field theory. It was the considerations which stimulated us for this algebraic investigation. (The example 2 was the basic example for formulating the conditions (1.2)).

The CH-algebras which are introduced in this paper seems to be interesting in the applications. Professor V.M.Buchstaber offers the general construction for CH-algebras. He noted also that so called Novikov's O-doubles [3] (which are the natural generalization of the algebra constructed in the Example 2) are the interesting examples of CH-algebras. We consider these examples in the Section 5

We have to note that in the formulae (1.1), (1.5), (1.11) the expressions $\log e^a e^b$, $\log \Pi e^a e^b$, $\log \Pi e^c$ are considered as formal power series corresponding to the functions $\log$, e. All the statements have the sense for arbitrary large but finite initial sequences of these formal series.

All the algebras considered here are the linear spaces over the real (or complex numbers). The associative algebras considered here are the associative algebras with unity.

2. CH-algebras

We call the pair (A,Π) associative CH-algebra (with unity) if A is the associative algebra (with unity) and Π is linear operator on it obeying to the conditions (1.2). We say that operator Π provides the algebra A with CH-structure. From (1.2) it is evident that Π is projection operator ($\Pi^2 = \Pi$) and $\mathrm{Im}\Pi$ is sub-algebra in A.

For example the linear operator $\Pi(1.6)$ on the algebra D defined in the example 2 provides this algebra by CH-structure.

Of course every associative algebra can be provided with trivial CH-structure ($\Pi = \mathbf{id}$).

The homomorphism φ of the associative algebra A_1 in the associative algebra A_2 is the morphism of corresponding CH-algebras (CH-morphism) $\varphi : (A_1, \Pi_1) \to (A_2, \Pi_2)$ if

$$\varphi \circ \Pi_1 = \Pi_2 \circ \varphi \quad . \tag{2.1}$$

We need also the construction of tensor product of CH-algebras: $(A_1, \Pi_1) \otimes (A_2, \Pi_2) = (A_1 \otimes A_2, \Pi_1 \otimes \Pi_2)$ where

$$(\Pi_1 \otimes \Pi_2)(a_1 \otimes a_2) = \Pi_1 a_1 \otimes \Pi_2 a_2. \tag{2.2}$$

The CH-algebra $(A_\pi(L), P)$ is free algebra with one generator in the category of CH-associative algebras with unity if for arbitrary CH algebra (B, Π) from this category and for arbitrary $c \in B$ there exists unique CH-morphism φ : $(A_\pi(L), P) \to (B, \Pi)$ such that

$$\varphi(L) = c. \tag{2.3}$$

Proposition 1 There exists unique (up to isomorphism) free CH-associative with unity algebra with one generator.

We give briefly the construction of this algebra.

Let π, L be two formal symbols.

$\mathcal{A}_0$ is the alphabet containing one letter L. Γ_0 is the semigroup of the words on the alphabet $\mathcal{A}_0$. (To the empty word corresponds the unity in Γ_0.) Let $\mathcal{A}_1$ is the alphabet containing the letter L and all the letters πs where $s \in \Gamma_0$ and $s \neq 1$. Γ_1 is the semigroup of the words on the alphabet $\mathcal{A}_1$. By induction the alphabet $\mathcal{A}_{n+1}$ contains all the letters of the alphabet $\mathcal{A}_n$ and the new letters πs where $s \in \Gamma_n \setminus \Gamma_{n-1}$. Γ_{n+1} are the words on the $\mathcal{A}_{n+1}$. The semigroup

$$\Gamma = \Gamma_0 \cup \Gamma_1 \cup \ldots \cup \Gamma_n \cup \ldots \tag{2.4}$$

is the semigroup of the words on the alphabet

$$\mathcal{A} = \mathcal{A}_0 \cup \ldots \cup \mathcal{A}_n \cup \ldots \tag{2.5}$$

The linear combinations of the words of the semigroup Γ with the coefficients from the field of real (or complex) numbers consist the associative with unity algebra $A(\pi, L)$. On this algebra one can consider the linear operator P which is defined on $A(\pi, L)$ by its action on the words from Γ:

$$Pw = \pi w. \tag{2.6}$$

If w is the word from Γ_k then πw is the letter from $\mathcal{A}_{k+1}$(the oneletter word from Γ_{k+1}).

Now we construct the algebra $A_\pi(L)$ as factor algebra of $A(\pi, L)$.

Let us consider the set of ideals in the $A(\pi, L)$:

J_0 is the two–sided ideal generated by all the elements $P(aPb) - P(ab)$, $P(Pab) - PaPb$ ($\forall a, b \in A(\pi, L)$) of the algebra $A(\pi, L)$. J_1 is the two–sided ideal generated by all the

elements Pa_0 ($\forall a_0 \in J_0$). By induction J_{n+1} is the two-sided ideal generated by all the elements Pa_n ($\forall a_n \in J_n$). Then

$$A_\pi(L) = A(\pi, L)/J \tag{2.7}$$

where

$$J = J_0 \dotplus J_1 \dotplus \ldots \dotplus J_n \dotplus \ldots \tag{2.8}$$

It is easy to see that the operator P defined on the algebra $A(\pi, L)$ by (2.6) is correctly defined on the algebra $A_\pi(L)$, provides this algebra with CH-structure and CH-algebra $(A_\pi(L), P)$ is free CH-algebra with one generator.

We can also describe the basis of the algebra $A_\pi(L)$ (considering it as a linear space). We call the letters L, πL and all the letters of the type $\pi(LwL)$ the regular letters (where w is some word in Γ). We call the word in Γ regular if it contains only regular letters. It is easy to see (using the conditions (1.2)) that every element of $A(\pi, L)$ is equivalent in $A_\pi(L)$ to the linear combination of the regular words. Moreover one can show that *the regular words consist the linear basis of the* $A_\pi(L)$. We do not give here the proof of this statement. For our purposes it is enough only that $A_\pi(L)$ is not trivial.

3. Lie CH-algebras

Now we introduce Lie CH-algebras. We say that the pair $(\mathcal{G}, M)$ is Lie CH-algebra if $\mathcal{G}$ is Lie algebra and M is linear operator on it such that

$$M^2 = M, \quad \forall a, b \in \mathcal{G} \quad M[a, b] = M[a, Mb] + M[Ma, b] - [Ma, Mb] \quad . \tag{3.1}$$

From (3.1) it follows that $\mathbf{Im}M$:

$$\tilde{\mathcal{G}} = \{\mathcal{G} \ni a : a = Ma\} \tag{3.2}$$

is subalgebra in $\mathcal{G}$.

In the same way like for associative algebras every Lie algebra can be provided with trivial CH-structure ($M = \mathrm{id}$).

It is easy to see that if (A, Π) is associative CH-algebra then the operator Π on $[A]$ obeys to (3.1) hence $([A], \Pi)$ is Lie CH-algebra ($[A]$ is commutators Lie algebra (1.3)).

In the same way as for associative CH-algebras the homomorphism φ of the Lie algebra $\mathcal{G}_1$ in the Lie algebra $\mathcal{G}_2$ is the morphism of corresponding Lie CH-algebras (CH-morphism) $\varphi : (\mathcal{G}_1, M_1) \to (\mathcal{G}_2, M_2)$ if

$$\varphi \circ M_1 = M_2 \circ \varphi. \tag{3.3}$$

Now we generalize the construction of the universal enveloping algebra for CH-algebras.

Let $(\mathcal{U}, \Pi)$ be associative CH-algebra with unity and $\iota : (\mathcal{G}, M) \to ([\mathcal{U}], \Pi)$ is CH-morphism.

We say that $(\mathcal{U},\Pi)$ with CH-morphism ι is universal enveloping CH-algebra of Lie CH-algebra $(\mathcal{G},M)$ if for arbitrary associative CH-algebra with unity (A,S) and for the arbitrary CH-morphism φ of $(\mathcal{G},M)$ in $([A],S)$ there exists unique (enveloping) CH-morphism ρ of $(\mathcal{U},\Pi)$ in $([A],S)$ such that

$$\rho\circ\iota=\varphi. \tag{3.4}$$

If $(\mathcal{G}_P(L),P)$ is CH-Lie algebra generated in the $([A_\pi(L)],P)$ (See Proposition 1) by the element L and linear operator P (1.4) then it is easy to see that

Proposition 2 The free CH-algebra $(A_\pi(L),P)$ is universal enveloping algebra for Lie CH-algebra $(\mathcal{G}_P(L),P)$, (morphism ι is canonical embedding.)

(The enveloping homomorphism ρ is defined by the condition $\rho(L)=\varphi(L)$.)

Theorem 2 For Lie CH-algebra $(\mathcal{G},M)$ there exists unique (up to isomorphisms) universal enveloping CH-algebra $(\mathcal{U},\Pi)$ with CH morphism $\iota:(\mathcal{G},M)\to([\mathcal{U}],\Pi)$ where $\mathcal{U}$ with ι is universal enveloping algebra for Lie algebra $\mathcal{G}$ in usual sense.

Corollary 1. Let $\mathcal{U}$ be universal enveloping algebra of the Lie algebra $\mathcal{G}$ and ρ be enveloping homomorphism of the homomorphism $\varphi:\mathcal{G}\to[A]$ (for some associative algebra A). If $\mathcal{G}$ and A can be provided with CH-algebras structures in a way that φ becomes CH-morphism, $\varphi:(\mathcal{G},M)\to([A],S)$ then ρ is CH-morphism too: $\rho:(\mathcal{U},\Pi)\to(A,S)$.

Corollary 2 If $(\mathcal{U},\Pi)$ is universal enveloping algebra of CH-Lie algebra $(\mathcal{G},M)$ then comultiplication δ defined on $\mathcal{U}$ by (1.8) commutes with action of operator Π:

$$\delta\Pi=(\Pi\otimes 1+1\otimes\Pi)\delta \quad . \tag{3.5}$$

Indeed comultiplication δ is enveloping homomorphism of the homomorphism φ of $\mathcal{G}$ in $[\mathcal{U}\otimes\mathcal{U}]$: $\varphi(x)=\iota x\otimes 1+1\otimes\iota x$. It is easy to see that φ is morphism of Lie CH-algebra $(\mathcal{G},M)$ in Lie CH-algebra $([\mathcal{U}\otimes\mathcal{U}],\Pi\otimes\Pi)$. It follows from Corollary 1 that δ is CH-morphism of $(\mathcal{U},\Pi)$ in $([\mathcal{U}\otimes\mathcal{U}],\Pi\otimes\Pi)$ hence (3.5) is satisfied.

Now using Theorem and Corollary 2 we can prove Theorem 1′ using (1.9). Indeed from Proposition 2 and Theorem 2 it follows that $A_\pi(L)$ *is universal enveloping algebra for Lie algebra* $\mathcal{G}_P(L)$ *(in usual sense).* Using Corollary 2 we can easy check (like in (1.10) that $\delta\log Pe^L$ is primitive in $\mathcal{U}\otimes\mathcal{U}$. Theorem 1′ is proved for free CH-algebra, so for arbitrary CH-algebra .

Indeed it is easy to see that we proved little more:

$$\log Pe^L\in\hat{\mathcal{G}}_P(L)=\mathrm{Im}P \tag{3.6}$$

Now we prove Theorem 2.

Lemma. If $\mathcal{U}$ is the universal enveloping algebra of the Lie algebra $\mathcal{G}$ and the linear operator M provides Lie algebra $\mathcal{G}$ with CH algebra structure then there exists the linear operator Π providing $\mathcal{U}$ with CH-algebra structure in a way that canonical embedding ι $(\iota:\mathcal{G}\to[\mathcal{U}])$ becomes CH-morphism:

$$\Pi\circ\iota=\iota\circ M. \tag{3.7}$$

Remark If Lie algebra $\mathcal{G}$ is provided with trivial CH–structure M: $\mathbf{Im}M = 0$ then Π on $\mathcal{U}$ is corresponded to augmentation $\varepsilon: \mathcal{U} \to k$ (counity) ($\Pi a = \varepsilon(a) \cdot 1$, $(\Pi \otimes \mathbf{id})\delta a = 1 \otimes a$ where δ is comultiplication (1.7)) on the Hopf algebra $\mathcal{U}$.

We prove Lemma later.

Now we prove that CH-algebra $(\mathcal{U}, \Pi)$ obeying to the conditions of Lemma is the universal enveloping algebra for $(\mathcal{G}, M)$.

Let (A, S) be associative CH-algebra and φ CH-morphism of $(\mathcal{G}, M)$ in $([A], S)$. Because $\mathcal{U}$ is universal enveloping algebra of $\mathcal{G}$ (in usual sense) then there exists unique enveloping homomorphism ρ of $[\mathcal{U}]$ in $[A]$ ($\rho\iota = \varphi$). We have to prove that ρ is CH-morphism of $(\mathcal{U}, \Pi)$ in (A, S):

$$\rho\Pi = S\rho. \tag{3.8}$$

(The uniqueness of the $(\mathcal{U}, \Pi)$ is provided by the fact that if $(\tilde{\mathcal{U}}, \tilde{\Pi})$ is another universal enveloping algebra of the CH-Lie algebra $(\mathcal{G}, M)$ then considering the CH-morphisms $\tilde{\rho}: (\tilde{\mathcal{U}}, \tilde{\Pi}) \to (\mathcal{U}, \Pi)$ and $\rho: \mathcal{U}, \Pi) \to (\tilde{\mathcal{U}}, \tilde{\Pi})$ which envelop correspondingly the embeddings $\iota: (\mathcal{G}, M) \to ([\mathcal{U}], \Pi)$ and $\tilde{\iota}: (\mathcal{G}, M) \to ([\tilde{\mathcal{U}}], \tilde{\Pi})$ we see that $\rho\tilde{\rho} = \mathbf{id}$, $\tilde{\rho}\rho = \mathbf{id}$.)

We prove (3.8) by induction. We consider the linear subspaces $\mathcal{U}_n$ in $\mathcal{U}$: $\mathcal{U}_0 = 1$, $\mathcal{U}_1 = \iota\mathcal{G}$, $\mathcal{U}_n$ ($n \geq 2$) contains all the elements which can be represented as linear combinations of the products of less than $n+1$ elements of $\mathcal{U}_g$.

Inductive suggestion.—For $n \leq k$ ($n \geq 1$)

$$\forall a \in \mathcal{U}_n, \Pi a = \sum \iota M x_i \cdot b_i, \quad \text{where} \quad b_i \in \mathcal{U}_{n-1}, x_i \in \mathcal{G} \tag{3.8a}$$

and

$$\forall a \in \mathcal{U}_n, \quad \rho(\Pi a) = S\rho(a). \tag{3.8b}$$

Here and in the following we use that in a universal enveloping algebra $\iota x \cdot \iota y - \iota y \cdot \iota x = \iota\{x, y\}$. For $k = 1$ (3.8a,3.8b) are evident: $\Pi a = \Pi \iota x = \iota M x$ by (3.7) and $\rho(\Pi a) = \rho\Pi(\iota x) = \rho(\iota M x) = \varphi(M x) = S\varphi(x) = S\rho(\iota x) = S\rho(a)$.

For proving (3.8a) in the case $k \to k+1$ we note using (1.2) that if $\mathcal{U}_{k+1} \ni a = a' \cdot \iota x$ (where $a' \in \mathcal{U}_k$) then $\Pi a = \Pi(a' \cdot \iota x) = \Pi(a' \cdot \Pi \iota x) = \Pi(a' \cdot \iota M x) = \Pi(\iota M x \cdot a' + a'') = \iota M x \cdot \Pi a' + \Pi a''$ (where $a'' \in \mathcal{U}_k$). (3.8a) is proved.

Using (3.8a) we prove (3.8b) for $k \to k+1$. Again using (1.2) we have that if $\mathcal{U}_{k+1} \ni a = \iota x \cdot b$ then $\rho\Pi a = \rho\Pi(\iota x \cdot b) = \rho\Pi(\iota x \cdot \Pi b) =$ (by 3.8a) $= \sum \rho\Pi(\iota x \cdot \iota M y_i \cdot c_i) =$ (where $c_i \in \mathcal{U}_{k-1}$) $= \sum \rho\Pi(\iota[x, M y_i] \cdot c_i) + \sum \rho\Pi(\iota M y_i \cdot \iota x \cdot c_i) = \sum \rho\Pi(\iota[x, M y_i] \cdot c_i) + \sum \rho\iota M y_i \cdot \rho\Pi(\iota x \cdot c_i) =$ (by inductive suggestion (3.8b)) $= \sum S\rho(\iota[x, M y_i] \cdot c_i) + \sum S\rho\iota y_i \cdot S\rho(\iota x \cdot c_i) = \sum S\rho(\iota[x, M y_i] \cdot c_i) + \sum S\rho(\iota M y_i \cdot x \cdot c_i) = S\rho(\iota x \cdot b) = S\rho(a)$.

The Theorem 2 is proved.

Now we prove the Lemma.

Let $\mathcal{G}$ be Lie algebra and the linear operator M provides it by CH–algebra structure (3.1).

Let $\{b_\alpha, e_i\}$ ($\alpha \in I_0, i \in I_1$) be basis in $\mathcal{G}$ such that $\{b_\alpha\}$ is basis in subalgebra $\tilde{\mathcal{G}} = \mathbf{Im}M$ (3.2) ($e_i \notin \tilde{\mathcal{G}}$). We assume that $I = (I_0, I_1)$ is well-ordered and the elements of I_0 precede those of I_1. The monomials $\{b_{\alpha_1 \dots \alpha_n} \cdot e_{i_1 \dots i_m}\}$ where $\alpha_1 \preceq \dots \preceq \alpha_n, i_1 \preceq \dots \preceq i_m$

and $b_{\alpha_1 \dots \alpha_n} = \iota b_{\alpha_1} \cdot \dots \cdot \iota b_{\alpha_n}$, $e_{i_1 \dots i_m} = \iota e_{i_1} \cdot \dots \cdot \iota e_{i_m}$ is the basis (Birchof de Witt basis) of $\mathcal{U}$.

We consider the filtration on $\mathcal{U}$:

$$\mathcal{U}_{(0)} \subset \mathcal{U}_{(1)} \subset \dots \subset \mathcal{U}_{(n)} \subset \dots \tag{3.9}$$

where $\mathcal{U}_{(k)}$ is the linear combination of the basis elements $\{b_{\alpha_1 \dots \alpha_n} \cdot e_{i_1 \dots i_m}\}$ for $m \leq k$.

($\mathcal{U}_{(0)}$ is the universal enveloping algebra of $\hat{\mathcal{G}}$ defined by (3.2).)

We note using (3.1) that Lie algebra $\mathcal{G}$ is the sum (as the linear space) of two subalgebras:

$$\mathcal{G} = \hat{\mathcal{G}} \oplus \dot{\hat{\mathcal{G}}}, \quad (\hat{\mathcal{G}} = \mathbf{Im}M, \dot{\hat{\mathcal{G}}} = \mathbf{ker}M, \hat{\mathcal{G}} \cap \dot{\hat{\mathcal{G}}} = 0). \tag{3.10}$$

If $\{e_i\}$ is a basis in the Lie subalgebra $\dot{\hat{\mathcal{G}}}$:

$$Me_i = 0 \tag{3.11}$$

then it is easy to see that the linear operator Π:

$$\Pi(b_{\alpha_1 \dots \alpha_n} \cdot e_{i_1 \dots i_m}) = 0, \text{ if } m \geq 1, \ \Pi\Big|_{\mathcal{U}_{(0)}} = \mathbf{id}, \ (\Pi : \mathcal{U} \to \mathcal{U}_{(0)}) \tag{3.12}$$

is satisfied to the conditions of Lemma.

(In the case if the condition (3.11) does not hold one can show (by induction) that operator Π can be defined as $\Pi = \Phi^\infty$ where $\Phi(b_{\alpha_1 \dots \alpha_n} \cdot e_{i_1 \dots i_m}) = b_{\alpha_1 \dots \alpha_n} \cdot e_{i_1 \dots i_{m-1}} \cdot \iota M e_{i_m}$, $\Phi\Big|_{\mathcal{U}_{(0)}} = \mathbf{id}$, $(\Phi : \mathcal{U}_{(n+1)} \to \mathcal{U}_{(n)}$ if $n \geq 1$, if $a \in \mathcal{U}_{(k)}$ then $\forall N \geq k \ \Phi^{N+1} a = \Phi^N a \in \mathcal{U}_{(0)}$.)

The proof is finished.

Note Our aime was to give a pure algebraic proof of the formula (3.6). However we want to note that using the above algebraic statements (Proposition 2, Theorem 2) and the formulae (3.10-3.12) one can give another proof of (3.6) which is based on the following fact yielded from CH-formula (1.1): e^x is decomposed into the product $e^{\hat{x}} e^{\dot{\hat{x}}}$ where $\hat{x} \in \mathbf{Im}M$, $\dot{\hat{x}} \in \mathbf{ker}M$.

4.Application

The statement of the Theorem 1 can be used for investigation the problem of the connectivity of the Feynman graphs corresponding to the Green functions in Quantum Fields Theory.

In Quantum Field Theory it is well known the Theorem about the connectivity of the Feynman graphs corresponding to the logarithm of partition function (PFLC Theorem)—generating functional of the Green functions. (See for details for example [4].)

The Green functions of the quantum theory are the vacuum expectation values of the time ordered products of field operators:

$$G(x_1, \dots, x_n) = < T(\hat{\phi}(x_1) \dots \hat{\phi}(x_n)) >, \tag{4.1}$$

where the classical theory is defined by the classical action $S(\phi)$ —the functional on the classical fields $\phi(x)$ corresponding to the fields operators $\hat{\phi}(x)$.

The Green functions can be collected together in the generating functional $Z(J)$ (partition function) — a formal power series on the "classical sources" $J(x)$:

$$Z(J) = \sum_{N=0}^{\infty} \frac{1}{N!} \int G(x_1, \dots, x_N) J(x_1) \dots J(x_N) dx_1 \dots dx_N. \tag{4.2}$$

In the case where $S(\phi)$ is the action of free theory

$$S(\phi) = S(\phi)_{free} = \int \left(\frac{1}{2} \phi(x) K(\partial) \phi(x) \right) dx \tag{4.3}$$

where $K(\partial)$ is some invertible differential operator (for example $K = \partial^2$) the functional $Z(J)$ can be easily calculated:

$$Z(J)_{free} = e^{\int J(x) \Delta(x-y) J(y) dx dy}, \tag{4.4}$$

where $\Delta(x-y)$ is two-point Green function of the free theory which is obtained by inverting operator $K(\partial)$

$$K(\partial) \Delta(x, y) = \delta(x - y).$$

In the case of full interacted theory where

$$S(\phi) = S(\phi)_{free} + S(\phi)_{int}$$

the functional $Z(J)$ is given by the following formal expression

$$Z(J) = e^{S_{int}(\frac{\delta}{\delta J})} Z_{free} = e^{S_{int}(\frac{\delta}{\delta J})} e^{\int J(x) \Delta(x-y) J(y) dx dy}. \tag{4.5}$$

It is well known Gell-Mann and Low formula which leads to the perturbative expansion of the Greens functions in terms of Feynman graphs [4]. To every monom in a power series expansion of (4.5) by J correspond Feynman graphs connected or disconnected.

PFLC–Theorem. In the functional $\log Z(J)$ give contribution only connected Feynman graphs.

(The analogous statement is in Statistical Physics where to $\log Z$ correspond the free energy of the system and in the Probability Theory where to $\log Z(J)$ correspond semyinvariant [5]).

The standard proofs of this Theorem are based on the recursive procedure of the Feynman graphs investigation.

We discuss PFLC–Theorem using the Theorem 1.

We rewrite (4.5) symbolically

$$Z = e^{T} e^{K}. \tag{4.6}$$

To consider $\hat{T}$ and K on an equal footing we rewrite (4.6):

$$Z = \Pi e^{\hat{T}} e^{K} \tag{4.7}$$

where $\Pi\hat{A} = \hat{A}1$ and $\hat{K}$ is the operator of the multiplication on K (compare with Example 2 of the Section 1).

To every element a of the associative CH-algebra $(A_\Pi(\hat{T},\hat{K}),\Pi)$ generated by $\hat{T}$, $\hat{K}$ and projection operator Π correspond Feynman graphs connected or disconnected. (For example to the element $c = \Pi a \Pi b$ correspond disconnected Feynman graphs if Πa and Πb are not trivial elements of the $A_\Pi(\hat{T},\hat{K})$.) From the Theorem 1 it follows that the PFLC-Theorem is reduced to the algebraic statement: *to the elements of the Lie CH-algebra $(\mathcal{G}_\Pi(\hat{T},\hat{K}),\Pi)$ correspond the connected Feynman graphs.* This statement is right as far as to $\hat{T}$ and $\hat{K}$ correspond connected Feynman graphs (which takes place for field theory standard lagrangians). One can show it noting that to the commutator operation in the $(\mathcal{G}_\Pi(\hat{T},\hat{K})$ corresponds the gluing of the corresponding Feynman subgraphs.

5. Discussions

To find algebraic reformulation of the PFLC-Theorem we considered associative algebras provided with additional operation by means of linear operator Π which obeys to the conditions (1.2) and the Lie algebras corresponding to them (CH-algebras). We show that these algebras have the properties similar to usual ones (Theorem 2). In particular one can formulate Campbell-Hausdorff like statement (Theorem 1).

It is interesting to study nontrivial examples of CH-algebras. Following to V.M. Buchstaber we consider

Example 3 [6]. Let A be associative algebra whith unity which is the left B-module (B is subalgebra in A) and A admits the expansion $A = B + \sum_{k\geq 1} B \cdot a_k$ in a way that $a_k a_l = \sum_{q\geq 1} b_{kl}^q a_q$ where $b_{kl}^q \in B$. Then it is easy to see that $\Pi: A \to B; a = \sum_{k\geq 0} b_k a_k \to b_0$ ($a_0 = 1$) provides A with CH-structure. (Compare with 3.9–3.12.)

One can show that every associative CH-algebra (A,Π) can be represented in this way using that $C = \ker\Pi$ is the subalgebra which is as well as A the left module over the subalgebra $B = \mathrm{Im}\Pi$ (as it follows from (1.2)).

The interesting example of this construction is

Example 4 [6]. Let X be the Hopf algebra with comultiplication δ and augmentation $\varepsilon: X \to k$. (Compare with **Remark** after **Lemma**.) Let an algebra $\mathcal{M}$ be left X-module such that $\forall x \in X$ and $\forall u, v \in M$ $x(uv) = \sum_i x_i^1(u) \cdot x_i^2(v)$ if $\delta x = \sum_i x_i^1 \otimes x_i^2$. (Milnor module.) The algebra $A = \mathcal{M}X$ of the linear combinations ($A \ni a = \sum u_k x_k$, $u_k \in \mathcal{M}, x_k \in X$) is so called Novicov's **O–double** [3]. (The multiplication is defined by $(ux)\cdot(vy) = u\sum_i x_i^1(v)x_i^2 y$ where $\delta x = \sum_i x_i^1 \cdot x_i^2$.) If $\{e_\alpha\}$ is basis in X such that $\varepsilon e_0 = 1$ and $\varepsilon e_\alpha = 0$ for $\alpha \neq 0$ then the linear operator Π defined on O-double $\mathcal{M}X$ by the condition $\Pi(\sum_\alpha u_\alpha e_\alpha) = u_0 e_0$ provides $\mathcal{M}X$ with natural CH structure.

The CH-algebra $(\mathcal{M}X,\Pi)$ is on one hand the natural generalization of the algebra described in Example 2 where X is the algebra of differentiations with constant coefficients and $\mathcal{M}$ is the algebra of the functions. On other hand this algebra arises in the different

applications. The model example for O double is the algebra $A^U = \Lambda X$ of cohomological operations in complex cobordism theory [7] where X is so called "Landweber-Novikov" algebra [7,8], Λ is the U cobordism ring "for the point". The algebra A^U is related to the differential operators on some infinite-dimensional Lie group.—It was shown in [9] that if $Diff(\mathbf{R})$ is the group of the diffeomorphisms of the real line $\mathbf{R}$ and G its subgroup: $G = \{Diff \ni f : f(0) = 0,\ f' = 0\}$ then X is isomorphic to the universal enveloping algebra $\mathcal{U}(\mathcal{G}(G))$ of the Lie algebra $\mathcal{G}$ of the group G. Λ is the ring of polinoms on this group. We see that A^U returns us again to the example 2.

It is interesting to study in details these and other examples of CH-algebras.

On other hand it is interesting to study how much the statements of the Theorems 1 and 2 depend on the conditions (1.2) on the linear operator Π.

The considerations of the 4-th section lead us in fact to redefinition of the connectivity conditions as the conditions of belonging to specially constructed Lie algebra. It can be interesting to analyze in details the relations between this definition and usual ones.

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