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HARMONIC COORDINATES IN THE PROBLEM ON STATIC GRAVITATIONAL AND MASSLESS SCALAR FIELDS WITH A POINT SOURCE

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Fischer used the simplest generalization of the Schrödinfer (Klein — Gordon — Fock) equation describing the scalar field of a particle with the mass μ

$$\left(\nabla_\sigma \nabla^\sigma + \frac{\mu^2 c^2}{h^2}\right) U = -4\pi j \tag{1}$$

in the case when $\mu = 0$.

The corresponding material tensor for the Einstein equations $G_\mu^\nu = \frac{8\pi\,\kappa}{c^4} T_\mu^\nu$ has the form

$$T_\mu^\nu = \frac{1}{4\pi}\left(\nabla_\mu U \nabla^\nu U - \frac{1}{2}\delta_\mu^\nu \nabla_\sigma U \nabla^\sigma U\right). \tag{3}$$

(Formulae from (1) to (24) are the same as in the Fischer paper).

The static spherically symmetric metric was used:

$$ds^2 = e^{\nu(r)}(cdt)^2 - e^{\lambda(r)}dr^2 - r^2 d\Omega^2,$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2. \tag{5}$$

Outside the source of fields the Einstein equations give

$$-\frac{\lambda'}{r} + \frac{1}{r^2}(1 - e^\lambda) = -\frac{\kappa}{c^4}(U')^2, \tag{7a}$$

$$\frac{\nu'}{r} + \frac{1}{r^2}(1 - e^\lambda) = \frac{\kappa}{c^4}(U')^2, \tag{7b}$$

$$\frac{\nu''}{2} + \frac{\nu' - \lambda'}{2r} + \frac{\nu'}{4}(\nu' - \lambda') = -\frac{\kappa}{c^4}(U')^2, \tag{7c}$$

$$U'' + \left(\frac{2}{r} + \frac{\nu' - \lambda'}{2}\right) U' = 0, \tag{10}$$

the prime denotes $\frac{d}{dr}$. The last equation for the scalar field potential has the solution

$$U' = -\frac{G}{r^2}\exp\left(\frac{\lambda-\nu}{2}\right), \tag{11}$$

which tends to zero at spatial infinity. If one uses a convenient variable $Z = r\exp\frac{\nu-\lambda}{2}$, one obtains from equations (7,11)

$$(Z^2)' = 2r\,e^{\nu}, \quad Z' = e^{\frac{\nu+\lambda}{2}}, \tag{13,13a}$$

$$e^{\nu} = \frac{ZZ'}{r}, \quad e^{\lambda} = \frac{rZ'}{Z}, \quad U' = -\frac{G}{rZ}, \tag{15}$$

and the main equation for metric functions

$$rZ^2Z'' = \frac{\kappa G^2}{c^4}Z', \tag{16}$$

or

$$(rZ' - Z)' + \left(\frac{a^2}{Z}\right)' = 0, \quad a^2 \equiv \frac{\kappa G^2}{c^4}. \tag{18}$$

Upon integration, under the condition of correspondence with the Schwarzschild solution, i.e.

$$\frac{Z}{r} \to 1 - \frac{2\kappa m}{c^2 r} + \ldots \quad \text{when} \quad r \to \infty,$$

equation (18) gives

$$rZZ' = Z^2 + \frac{2\kappa m}{c^2}Z - a^2, \tag{19}$$

and upon the second integration

$$|Z - Z_0|^{1-p}|Z + Z_1|^{1+p} = r^2, \tag{20}$$

$$Z_{0,1} = \frac{1}{c^2}(\sqrt{\kappa^2m^2 + \kappa G^2} \mp \kappa m), \quad p = \frac{\kappa m}{\sqrt{\kappa^2m^2 + \kappa G^2}}. \tag{21}$$

On omitting symbols of absolute values (20) may be rewritten

$$\left(Z^2 + \frac{2\kappa m}{c^2}Z - a^2\right)\left(\frac{Z + Z_1}{Z - Z_0}\right)^p = r^2. \tag{22}$$

When $\kappa G^2 >> \kappa^2m^2$, Fischer used an approximate equation for Z in the form of a hyperbola

$$\overline{Z}^2 + \frac{2\kappa m}{c^2}\overline{Z} - a^2 = r^2, \tag{24}$$

$\overline{Z}$ means Z approximated.

Here some equations [2] follow which are absent in the Fischer paper:

$$\frac{Z}{r} = 1 - \frac{2\kappa m}{c^2 r} + \frac{\kappa G^2}{2c^4 r} + \frac{2}{3}\frac{\kappa m\kappa G^2}{c^6 r^3} + \ldots, \quad z \to \infty, \tag{25}$$

$$Z - Z_0 = \left(\frac{pc^2}{2\kappa m}\right)^{\frac{1+p}{1-p}} \cdot r^{\frac{2}{1-p}} - \frac{1+p}{1-p}\left(\frac{pc^2}{2\kappa m}\right)^{\frac{3+p}{1-p}} r^{\frac{4}{1-p}} + \ldots, \quad r \to 0, \tag{26}$$

$$e^{-\lambda} = \left(1 + \frac{r\nu'}{2}\right)^{-2}\left(1 + \frac{\kappa^2m^2 + \kappa G^2}{c^4r^2e^{\nu}}\right), \tag{27}$$

and [3]

$$rZ\nu' = \frac{2\kappa m}{c^2}; \quad e^{\nu\sqrt{1+\delta}} + \frac{2\kappa m}{c^2 r}e^{\frac{\nu}{2}(\sqrt{1+\delta}-1)} \cdot \sqrt{1+\delta} - 1 = 0, \ \delta \equiv \frac{\kappa G^2}{\kappa^2m^2}. \tag{28}$$

Results may be pictured in Fig.1.

Now we turn to harmonic coordinates using the method by Fock [4], who has investigated the Schwarzschild problem. From coordinates ct, r, θ, φ we are going to coordinates ct, $R(r)$, θ, φ, which correspond to the rectangular harmonic ones. Let us write

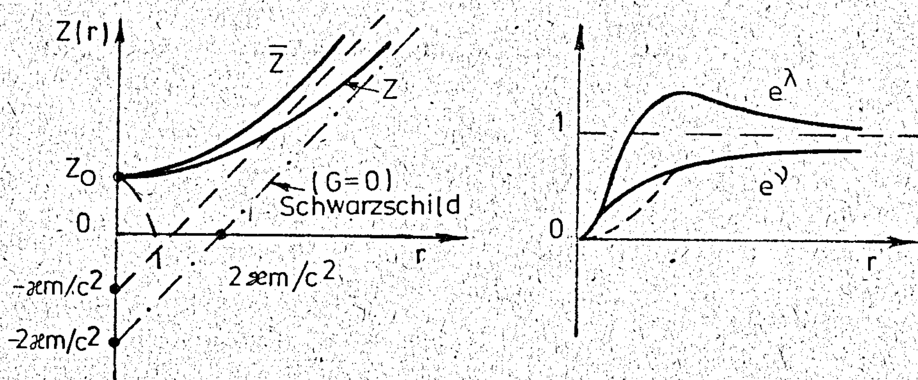

Fig.1

$$ds^2 = e^{\nu(r)}(cdt)^2 - e^{\lambda(r)}dr^2 - r^2 d\Omega^2 \equiv$$

$$\equiv V^2(r)(cdt)^2 - F^2(r)\left(\frac{dR}{dr}\right)^2 dr^2 - r^2 d\Omega^2 =$$

$$= V^2(r(R))(cdt)^2 - F^2(r(R))dR^2 - r^2(R)d\Omega^2. \quad (29)$$

Then the harmionic condition is [5]

$$\frac{1}{FV}\frac{d}{dR}\left(\frac{Vr^2(R)}{F}\right) - 2R = 0, \quad (30)$$

which, using r as an argument and $Z = r\exp\frac{\nu-\lambda}{2} = \frac{rV}{F}\left(\frac{dR}{dr}\right)^{-1}$, $Z_r' = \exp\frac{\nu+\lambda}{2}$, gives

$$\frac{d}{dr}\left(rZ\frac{dR}{dr}\right) = 2RZ', \quad (31)$$

or

$$(rZ^2R_r')_r' = (r^2R)_r' \cdot \frac{ZZ'}{r}. \quad (32)$$

Firstly we consider the case of a very large scalar constant $\kappa G^2 \gg \kappa^2 m^2$ and the Fischer hyperbola (24) approximation, for which

$$\bar{Z} = \sqrt{\kappa^2 m^2 + a^2 + r^2} - \kappa m, \quad (c = 1). \quad (33)$$

As $a^2 \gg \kappa^2 m^2$, it is possible to use a slightly cruder approximation and take

$$Z = \sqrt{a^2 + r^2}. \quad (34)$$

For it $ZZ'/r = 1$, and after integration of (32) we have the equation

$$rZ^2R' = r^2R + C_2,$$

the general solution of which is

$$R = \sqrt{a^2 + r^2}\left[\frac{C_2}{a^2}\left(\frac{1}{\sqrt{a^2+r^2}} - \frac{1}{2a}\ln\left|\frac{\sqrt{a^2+r^2}+a}{\sqrt{a^2+r^2}-a}\right|\right) + C_1\right]. \quad (35)$$

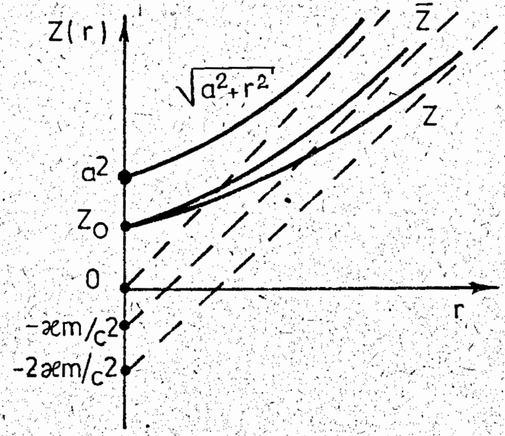

Fig.2

The condition of correspondence with the Newton theory $\lim R/r = 1$, $r \to \infty$, gives $C_1 = 1$. In analogy with the case of the Schwarzchild problem [6], the constant C_2 remains arbitrary. It may be defined when a general (internal and external) problem is treated.

If we take the simplest assumption $C_2 = 0$, like that by Lanczos and Fock, we obtain

$$R = \sqrt{\frac{\kappa G^2}{c^2} + r^2}. \quad (36)$$

It seems more interesting physically to be the case of a very small scalar constant $\kappa G^2 \ll \kappa^2 m^2$. Now we may use the Fischer solution asymptotics (25) and the Lanczos — Fock coordinate R for the Schwarzschild problem, namely, $R^{sch.} = r - \frac{\kappa m}{c^2}$ as the «zero» approximation. Then the harmonic condition (31) gives

$$R^{sc} = r - \frac{\kappa m}{c^2} + \frac{\kappa G^2}{2c^4 r} + \frac{5}{6}\frac{\kappa m \kappa G^2}{c^6 r^2} + \ldots, \quad r \to \infty. \quad (37)$$

(A consideration like that may be applied to a more general case, when $C_2^{sch} \neq 0$ and we have [6]

$$R^{sch.} = r - \frac{\kappa m}{c^2} + C_2^{sch}\frac{\kappa^2 m^2}{c^4 r^2} + \ldots, \quad r \to \infty).$$

Now, using (37), (15) and (25), we find expressions for metric functions (29):

$$r(R) = R + \frac{\kappa m}{c^2} - \frac{\kappa G^2}{2c^4 R} + \dots, \tag{38}$$

$$FV = \frac{dr}{dR} e^{\frac{\nu+\lambda}{2}} = \frac{dr}{dR} Z' = \frac{dr}{dR}\left(1 - \frac{\kappa G^2}{2c^4 r^2} + \dots\right) =$$

$$= \left(1 + \frac{\kappa G^2}{2c^4 R^2} + \dots\right)\left(1 - \frac{\kappa G^2}{2c^4 R^2} + \dots\right) = 1 + O\left(\frac{1}{R^3}\right), \tag{39}$$

$$\frac{F}{V} = \frac{dr}{dR} e^{\frac{\lambda-\nu}{2}} = \frac{dr}{dR}\frac{r}{Z} = 1 + \frac{2\kappa m}{c^2 R} + \frac{2\kappa^2 m^2}{c^4 R^2} + \dots \tag{40}$$

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