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GENERAL-COVARIANT QUANTUM MECHANICS OF DIRAC PARTICLE IN CURVED SPACE-TIMES

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1 Introduction

In the present paper a self consistent and general-covariant analog of the standard *Non-Relativistic Quantum Mechanics* (NRQM) with relativistic corrections for a particle of spin 1/2, mass m and electric charge e is constructed. In other words, it is a semiclassical NRQM description (with relativistic corrections) of the interaction of the particle with a general external gravitational field treated as a general riemannian space-time V_4 with metric tensor $g_{\alpha\beta}(x)$, $x \in V_4$, $\alpha, \beta, \gamma, ... = 0, 1, 2, 3$. . An analogous construction for zero-spin particle has been presented in papers [1] and [2] . It might be called also the Foldy Wouthuysen representation of the Dirac theory in the general V_4 but here the lack of space-time symmetries and Cartesian systems of coordinates requires some reasonable principles for determination of operators of observables. the question that does not usually arise in the theory based on the minkowskian background M_4.

The construction can be done in a self consistent and general covariant form starting with a set of general-relativistic "First Relations" and considering asymptotics' of the latter in c^{-2}, c being the velocity of light. The Dirac equation in V_4 and corresponding conserved quadratic form for its solutions (a "scalar product") are, of course, the first among these relations. The others are known real quadratic functionals of the Dirac field Ψ of "mean" momentum. spatial curvilinear coordinates and spin of the particle, which as it will be shown originate asymptotically the NRQM matrix elements of the corresponding operators of observables. Just in this point our approach essentially differs also from that of papers [3], [4] where the observables are introduced. in fact, on the non-relativistic level .

As a preliminary, we introduce some notations used throughout the paper:

$h^{\alpha}_{(\beta)}(x)$ are *orthonormal tetrad vector fields* denoted by indices in parentheses; these indices may be raised and lowered by contraction with *the Minkowsky metric tensor* $\eta_{(\alpha\beta)} = \mathrm{diag}(1, -1, -1, -1,)$ (Remind that $h^{\alpha}_{(\beta)} h^{(\beta)}_{\gamma} = \delta^{\alpha}_{\gamma}$, $h^{(\alpha)}_{\beta} h^{\beta}_{(\gamma)} = \delta^{(\alpha)}_{(\gamma)}$.);

space-like tetrad vectors are denoted by latin indices $i, j, k, ... = 1, 2, 3$ (of course, in parentheses); in general, two or more adjacent indices in parentheses are enclosed in a common pair of parentheses. e.g. $\varepsilon_{(ijk)}$ is a completely antisymmetric symbol. $\varepsilon_{(123)} = 1$;

$\gamma^{(a)}$ and $\sigma^{(i)}$ are respectively the ordinary Dirac and Pauli matrices;

∇_α is the covariant derivative and $\tilde{\nabla}_\alpha \overset{def}{=} \nabla_\alpha - (ie/\hbar c)A_\alpha(x)$. i.e. the interaction with an external electromagnetic field is also considered for generality; another short notation is $\partial_{(a)} \overset{def}{=} h^\beta_{(a)}\partial_\beta$ and so for other sorts of derivatives;

$L^\alpha(A, B) \overset{def}{=} A^\beta\nabla_\beta B^\alpha - B^\beta\nabla_\beta A^\alpha$ is the Lie derivative of the vector field $B^\alpha(x)$ along $A^\alpha(x)$.

In parallel with the parameter of asypmtotic c^{-1} we shall also use the half-length of the Compton wave $\lambda = \hbar/2mc$ when it will be convenient.

Now we start with the following known structure:

- The general-relativistic Dirac equation for a bispinor field Ψ with the rest mass m of its quantum

$$i\gamma^\alpha(x)\tilde{\nabla}_\alpha\Psi + \frac{mc}{\hbar}\Psi = 0. \tag{1}$$

- The conserved, i.e. independent of the choice of a space-like 3-surface Σ, sesquilinear functional ($\overline{\Psi}$ is the ordinary Dirac conjugation of Ψ):

$$\{\Psi_1, \Psi_2\} = \int_\Sigma d\sigma^\alpha(x)\ \overline{\Psi_1}\ \gamma_\alpha(x)\Psi_2, \tag{2}$$

  where $d\sigma^\alpha(x)$ is the normal element of Σ.

- The real functional $P_V(\Psi;\Sigma)$ of the field Ψ, of a given vector field $V^\beta(x)$ (which might be, of course, one of the tetrad vectors) in V_4 and of Σ for the component of "*mean momentum of a quantum of the field*" Ψ along $V^\beta(x)$:

$$P_V(\Psi;\Sigma) = \int_\Sigma d\sigma^\alpha V^\beta T_{\alpha\beta} \tag{3}$$

  where $T_{\alpha\beta}$ is the (metric) energy-momentum tensor of the field Ψ.

- The three real functionals $Q^A(\Psi;\Sigma)$ of *"mean curvilinear coordinates of a quantum of the field* Ψ*"* on the hypersurface Σ

$$Q^A(\Psi;\Sigma) = \int_\Sigma d\sigma^\alpha q^A(x)\overline{\Psi}\gamma_\alpha\Psi \tag{4}$$

where $q^A(x)$ are three scalar functions denoted by indices $A, B, ... = 1, 2, 3$ (not the tetrad indices !) and having the following properties

$$d\sigma^\alpha \partial_\alpha q^A = 0, \qquad \text{rank } \|\partial_\alpha q^A\| = 3. \tag{5}$$

- The three real functionals $S_{(i)}(\Psi;\Sigma)$ of *"mean projections of spin"* on space-like tetrad vectors $h^\alpha_{(i)}$, $i, j, k, ... = 1, 2, 3$.

$$S_{(i)}(\Psi;\Sigma) = \frac{\hbar}{4}\int_\Sigma d\sigma^\alpha\ \overline{\Psi}\ (\gamma_\alpha\Sigma^{\beta\delta} + \Sigma^{\beta\delta}\gamma_\alpha)\varepsilon_{(ijk)}h^{(j)}_\beta h^{(k)}_\delta\Psi. \tag{6}$$

where $\Sigma^{\beta\delta} = (1/2)(\gamma^\beta\gamma^\delta - \gamma^\delta\gamma^\beta)$

So, our first aim is to deduce from Eq.(1) generally covariant analog of the Pauli equation for a two-component wave function $\psi(x)$ for which the sesquilinear form (2) would lead to the "ordinary" Hilbert space scalar product for $\psi(x)$ as wave functions, all the relations being asymptotic in c^{-2}.

$$\{\Psi_1, \Psi_2\} = (\psi_1, \psi_2) + O(c^{-(2n+2)}), \quad (\psi_1, \psi_2) \stackrel{def}{=} \int_\Sigma d\sigma\ \psi_1^\dagger\ \psi_2. \tag{7}$$

$\psi^\dagger$ being the hermitean conjugate of ψ. Due to the latter definition it is natural to assume that just the wave functions ψ have *the Born probability interpretation.*

An essential point is however that it is necessary in the NRQM to define also hermitean (with respect to the scalar product $(\cdot,\cdot)$) operators of observables that formalise physical measurements and thus a preparation of initial states (the projective postulate of the NRQM).

The invariants $P_V(\Psi;\Sigma)$, $Q^A(\Psi;\Sigma)$, $S_{(i)}(\Psi;\Sigma)$ may be considered as exact general-relativistic expressions for diagonal matrix elements of the corresponding NRQM operators of observables for the case when the field $\Psi(x)$ is the general-relativistic image of an NRQM wave function $\psi(x)$ (in the Schrodinger picture). This is evident enough in the case of P_V and $S_{(i)}$.

As concerns Q^1, it should be noted at first that owing to Eq.(5) the numerical values of the three functions $q^A(x)$ define a point on Σ and so they form a transformation to the definite curvilinear coordinates q^A on Σ from the general arithmetization of V_4 by coordinates x^α. So, values of $q^A(x)$ may be considered as numerical results (that are, of course, scalars, i.e. do not depend on the choice of coordinates $\{x^\alpha\}$ in V_4) of detection of a particle spatial position by a measurment of the curvilinear coordinates q^A. Formula (4) gives the simplest general-relativistic expression for the mean value of q^A and as is shown in [2] this expression gives rise to a natural generalisation of the known Newton Wigner operator of particle spatial Cartesian coordinates of the spinless particle. Equation(4) coincides also in its form with the definition of the (Cartesian) coordinate operator introduced and justified by Polubarinov [5] for the quantised field $\Psi(x)$ in M_4.

Having introduced the general-relativistic images of diagonal matrix elements $Z(\Psi;\Sigma)$ of an observable Z as a real quadratic functional one can uniquely define invariant images of non-diagonal matrix elements as sesquilnear hermitean functionals $Z(\Psi_1,\Psi_2;\Sigma)$ through the known procedure of *polarisation*:

$$
\begin{aligned}
4Z(\Psi_1,\Psi_2;\Sigma) &= Z(\Psi_1+\Psi_2;\Sigma) - Z(\Psi_1-\Psi_2;\Sigma) \\
&- iZ(\Psi_1+i\Psi_2;\Sigma) + iZ(\Psi_1-i\Psi_2;\Sigma). \qquad (8)
\end{aligned}
$$

The hermiticity of $Z(\Psi_1,\Psi_2;\Sigma)$ means that

$$Z(\Psi_1,\Psi_2;\Sigma) = \overline{Z(\Psi_2,\Psi_1;\Sigma)}. \qquad (9)$$

Further, the NRQM differential operator $\hat{Z}(x)$ (in the Schrodinger picture) that includes only derivatives along the 3-surface Σ can be defined by the relation, cf. Eq.(7),

$$Z(\Psi_1,\Psi_2;\Sigma) = (\psi_1, \hat{z}(x)\psi_2)_\Sigma + O(c^{-(2n+2)}) \qquad (10)$$

at least for some classes of Σ's. It is obvious that owing to the hermiticity of $Z(\Psi_1,\Psi_2;\Sigma)$ the operator $\hat{Z}(x)$ thus defined is (asymptotically) hermitean in the usual sense.

The paper is organised as follows. In Sec.2 the approximation in powers of c^{-2} to Eqs.(1, 2) will be considered. Sections 3, 5, 6, 7 are devoted to derivation of operators of momentum, spatial coordinates and spin in the corresponding approximation. In Sec.4 the unitary equivalence of the hamiltonian in the Pauli equation and the NRQM energy operator deduced from Eq.(3) is shown.

The consideration will be done on the heuristic level assuming that the necessary mathematical conditions are fulfilled.

From a physical point of view the notation $O(c^{-(2N+2)})$ is a simplified expression of the fact that dimensionless ratios of products of characteristic values of the fields involved and of their derivaties to c^{-2N-2} are small.

We keep also explicitly the Plank's constant $\hbar$ in mathematical expressions because it provides another type of approximation, the quasiclassical one, and matching these approximations could be interesting .

2 The Pauli equation

The problem of transition from the Dirac equation to its two-component approximation in powers of c^{-2} in an external electromagnetic field in the *Minkowsky space-time* M_4 has been considered in a general form by Stephani [6]. This approach was extended by Gorbatzevich [4] to the case of V_4. However, his approach was based on a postulative formulation of the NRQM structure for V_4 and only a NRQM hamiltonian was deduced from the Dirac equation and the observables were introduced on an analogy with the case of M_4. On the contrary , here the whole consruction is a natural approximation to the formulated general relativistic "First Relations" and so it is more consecutive and self-consistent. The results of the two approaches coincide for the form of the wave equations but not for the observables which include an essential part of information on the interaction of quantum particle with the external fields.

Now, to obtain the NRQM wave equation we rewrite firstly Eq.(1) in a more detailed form

$$i h^{\alpha}_{(\beta)} \gamma^{(\beta)} (\tilde{\partial}_{\alpha} - \Gamma_{\alpha}) \Psi + \frac{mc}{\hbar} \Psi = 0, \qquad (11)$$

where the bispinor connection Γ_α is defined through the relation

$$\nabla_\alpha \gamma_\beta - \Gamma_\alpha \gamma_\beta + \gamma_\beta \Gamma_\alpha = 0.$$

Using the representation of $\gamma_{(\alpha)}$ through the Pauli matrices $\sigma_(i)$

$$\gamma_{(0)} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \quad \gamma_{(i)} = \begin{pmatrix} \mathbf{0} & \sigma_{(i)} \\ -\sigma_{(i)} & \mathbf{0} \end{pmatrix}, \tag{12}$$

$\mathbf{I}$ being the 2×2 identity matrix one obtains

$$\Gamma_\beta = -\frac{i}{4}\Omega_\beta^{(ij)}\varepsilon_{(ijk)} \begin{pmatrix} \sigma^{(k)} & \mathbf{0} \\ \mathbf{0} & \sigma^{(k)} \end{pmatrix} - \frac{1}{2}\Omega_\beta^{(0i)} \begin{pmatrix} \mathbf{0} & \sigma_{(i)} \\ \sigma_{(i)} & \mathbf{0} \end{pmatrix}, \tag{13}$$

where

$$\Omega_\beta^{(ab)} \stackrel{def}{=} h^{(a)\mu}\nabla_\beta h_\mu^{(b)} \tag{14}$$

are the *Ricci rotation coefficients*.

Further, we substitute the following ansatz into Eq.(11):

$$\Psi = \exp(-\frac{2i}{\lambda}S(x)) \begin{pmatrix} \phi \\ -i\lambda G(x)\phi \end{pmatrix}, \tag{15}$$

where $G(x)$ is a differential 2×2-operator to be defined below and a (two-component) spinor ϕ and the real phase function $S(x)$ are assumed to satisfy to the conditions:

$$\tau^\alpha \partial_\alpha \arg\phi = o(1), \tag{16}$$

which means that the phase of ϕ changes in the direction of $\partial_\alpha S$ much more slow than $(mc/\hbar)S$, and

$$\partial_\alpha S\ \partial^\alpha S = 1, \tag{17}$$

which is the Hamilton-Jacobi equation for geodesics and so the vector field $\partial_\alpha S$ defines a *normal geodesic*, i.e. non rotating free-falling *frame of reference*. It should be remarked that this is the simplest but not necessary definition of S, and other time-like fields $\partial_\alpha S$ corresponding to non-geodesic frames of reference may be introduced.

Having been chosen $h_\alpha^{(0)} = \partial_\alpha S$ we obtain the following equation for ϕ as a result of the substitutions

$$i\hbar\tilde{T}\phi = -\frac{\hbar^2}{2m}\tilde{D}\ G\phi. \tag{18}$$

where the following notation is introduced:

$$\tilde{T} \stackrel{def}{=} \left(\tau^\alpha\tilde{\partial}_\alpha + \frac{1}{2}\nabla_\alpha\tau^\alpha\right) \cdot \mathbf{I}\ +\ \frac{i}{4}\Omega_\alpha^{(ij)}\varepsilon_{(ijk)}\sigma^{(k)}\tau^\alpha. \tag{19}$$

$$\tau_\alpha \stackrel{def}{=} c\partial_\alpha S \stackrel{def}{=} h_\alpha^{(0)} \qquad (\tau^\alpha \nabla_\alpha \tau_\beta = 0); \tag{20}$$

$$\tilde{D} \stackrel{def}{=} \sigma^{(i)} \left(h_{(i)}^\alpha \tilde{\partial}_\alpha + \frac{1}{2}\nabla_\alpha h_{(i)}^\alpha\right) - \frac{i}{4}\Omega_\alpha^{(ij)} \varepsilon_{(ijk)} h^{(k)\alpha} \cdot \mathbf{I} \tag{21}$$

and the operator G as an operator on solutions of Eq.(18) is determined by the nonlinear equation

$$G = \tilde{D} - \frac{i\lambda}{c} [\tilde{T} \ , \ G] + \lambda^2 G \cdot \tilde{D} \cdot G. \tag{22}$$

It should be realised that Eqs.(18),(19),(22) and many operator relations below have a valid mathematical sense only as asymptotical relations that is assumed to be clear in context.

Since τ^α is a geodesic vector field the operator $[\tilde{T} \ , \ G]$ and so the right-hand side of Eq.(18) contain only derivatives along the hypersurface $S(x) = const.$. (In the case of a non-geodesic Hamilton - Jacobi equation the derivatives along $\partial_\alpha S$ may be eliminated by application *of the evolution equation* (18).)

It is important also to realise that the left-hand term of Eq.(18)is considered to be of the order $O(1)$ in c^{-1} despite the formal presence of c in the definition of τ^α. This means that we pass to the non-relativistic and, in a philisophical sense, to the human measure of time.

Further, we have for solutions ϕ_1 and ϕ_2 of Eq.(18)

$$\{\Psi_1, \Psi_2\} = \int_S d\sigma(x) \ \phi_1^\dagger \ (\mathbf{I} + \frac{\lambda^2}{2} \ G^\dagger \cdot G) \ \varphi_2, \tag{23}$$

not Eq.(7). In Eq.(23) and further $G^\dagger$ denotes the hermitean conjugate of G in the conventional sense:

$$\int_S d\sigma(x) \ (G\phi_1)^\dagger \ \phi_2 = \int_S d\sigma(x) \ \phi_1^\dagger \ G^\dagger \phi_2.$$

However, we come easily to Eq.(7) transforming ϕ to

$$\psi = K^{-1}\phi \tag{24}$$

where the operator K satisfies the condition

$$K^\dagger \cdot (\mathbf{I} + \lambda^2 G^\dagger \cdot G) \cdot K = I, \tag{25}$$

that determines K up to an arbitrary right unitary transformation. Particularly, K may be chosen as an hermitean operator.

Rewriting Eq.(18) for ϕ in terms v one comes to

$$i\hbar\tilde{\mathcal{T}} v = K^{-1}\cdot(-\frac{\hbar^2}{2m}\ \tilde{D}\cdot G\cdot K - i\hbar[\tilde{\mathcal{T}}\ .K]\ )v \stackrel{def}{=} H \qquad (26)$$

which is the *Pauli equation* for V_4. It is easy to prove that the operator in the right-hand side is hermitean. We shall call it *the hamiltonian*, but it is not the operator of energy except some special cases though is connected with the latter by a unitary transformation, see Sec.4.

In the approximation up to $O(c^{-2})$ iteration of Eq.(22)

$$i\hbar\tilde{\mathcal{T}} v = -\frac{\hbar^2}{2m}(\tilde{D}^2 + \lambda^2\tilde{D}^4 + \frac{i\lambda}{2c}[\ [\tilde{\mathcal{T}}\ .\tilde{D}]\ ,\ \tilde{D}]\ + O(c^{-4}))v \qquad (27)$$

and

$$\begin{aligned}\tilde{D}^{\,2}\ &= \\ = (\tilde{\Delta}_S\ &+\ \frac{1}{4}R + \frac{1}{2c^2}(\tau^\alpha\nabla_\alpha\nabla_\beta\tau^\beta - \frac{1}{2}(\nabla_\alpha\tau^\alpha)^2) - \frac{1}{4}\nabla^\beta h^{(i)\alpha}\ \nabla_\alpha h_{(i),\beta})\cdot\mathbf{I} + \\ &+\ i\sigma_{(k)}\varepsilon^{(ijk)}(\frac{ie}{\hbar c}F_{(ij)} - \nabla_{(i)}h^\beta_{(j)}\tilde{\partial}_\beta - \frac{1}{2}\partial_{(i)}\nabla_\beta h^\beta_{(j)}) \qquad (28)\end{aligned}$$

where

$\tilde{\Delta}_S$ is the laplacian on the 3-surface S;

R is the scalar curvature;

$F^{(ij)}$ is the projection of the electromagnetic tensor $F_{\alpha\beta}$ on the spatial tetrad vectors, i.e. local magnetic field.

Here it is worth of attention that the zeroth order of the operator in the left-hand side of Eq.(27), i.e. of the hamiltonian, contains the term $\frac{1}{4}R$, i.e. just the term which arises in the quadrated (second order) equation for the bispinor field Ψ. If we had used this equation it were completely similar to the case of scalar field , see [1], where, if the term ξR is present in the the general-relativistic scalar wave equation, then it passes to the hamiltonian of the Schrodinger equation. However, the matter is that we have started here with the (first order) Dirac equation (1), which does not contain any curvature term.

As concerns the term $[\tilde{\mathcal{T}}\ .\tilde{D}]$, after simple transformations one has:

$$\begin{aligned}[\tilde{\mathcal{T}}\ .\tilde{D}]\ &=\ \sigma_{(i)}((\nabla^{(i)}\tau^\alpha)\tilde{\partial}_\alpha + \frac{1}{2}\nabla_\alpha(\nabla^{(i)}\tau^\alpha) - \frac{ie}{\hbar}F_{(0i)} - \frac{1}{2}\nabla_{(j)}(\tau^\alpha\Omega^{(ij)}_\alpha)) + \\ &+\ \frac{i}{2}\varepsilon_{(ijk)}\nabla_\alpha(h^{(k)[\alpha}\tau^{\mu]}\Omega^{(ij)}_\mu)\cdot\mathbf{I}. \qquad (29)\end{aligned}$$

the *indices in square brackets being antisymmetrised*.

Returning to Eq.(22) it is also worth to note that $[\tilde{\mathcal{T}}\,, G] = 0$ if

$$[\tilde{\mathcal{T}}\ ,\ \tilde{D}] = 0 \tag{30}$$

because the operator G is a series in powers of D. Then G may formally be written in a closed form:

$$G = \frac{\lambda^{-2}}{2}\Big(\mathbf{I} - (\mathbf{I} - 4\lambda^2\ \tilde{D}^2)^{\frac{1}{2}}\Big)\ \tilde{D}^{-1}. \tag{31}$$

It can be found from Eq.(29) that Eq.(30) takes place only when the following conditions are satisfied: $\nabla_\alpha \tau_\beta = 0$, i.e. the space-time is static, $\tau^\alpha F_{\alpha\beta} = 0$, i.e. there is no electric field in the frame of reference defined by the vector field τ^α, and the tetrad is chosen so that $\tau^\alpha \nabla_\alpha h^{(i)}_\beta = 0$

3 Operators of momentum

To determine the operator of projection of momentum on a given sufficiently smooth vector field $V^\alpha(x)$ we substitute the (metric) energy-momentum tensor for the field Ψ

$$T_{\mu\nu} = \frac{i\hbar}{2}(\overline{\Psi}\gamma_\nu\tilde{\nabla}_\mu\Psi - \overline{\tilde{\nabla}_\mu\Psi}\gamma_\nu\Psi) + \frac{i\hbar}{4}\nabla^\alpha(\overline{\Psi}\gamma_{[\nu}\gamma_\mu\gamma_{\alpha]}\Psi) \tag{32}$$

into Eq.(3). Having assumed that Ψ satisfies the Dirac equation (1) and subtracting a divergence of an antisymmetric tensor from the integrand one comes to the following expression:

$$\begin{aligned} P_V(\Psi;\Sigma) &= i\hbar\int_\Sigma d\sigma^\nu\ \Big(\overline{\Psi}\ \gamma_\nu(V^\mu\tilde{\nabla}_\mu + \frac{1}{2}\nabla_\mu V^\mu)\ \Psi\ + \\ &+\ \frac{1}{4}\nabla^\alpha V^\beta\ \overline{\Psi}\ \gamma_{[\nu}\gamma_\alpha\gamma_{\beta]}\ \Psi - \frac{1}{2}\nabla_{\{\mu}V_{\nu\}}\ \overline{\Psi}\ \gamma^\mu\ \Psi\Big) \end{aligned} \tag{33}$$

where *the indices in the braces are symmetrised.*

Of course, neither P_V, nor its polarisation can be represented as matrix elements of some differential operator in a space of solutions of Eq.(1) with respect to the product $\{.,.\}$, defined by Eq.(2), except the vector V^α is a Killing vector, i.e. $\nabla_{\{\mu}V_{\nu\}} = 0$. However, if we polarise $P_V(\Psi;\Sigma)$ and substitute the expressions from Eqs.(15),(24) into

$P_V(\Psi_1, \Psi_2; S)$ then after straightforward but rather lengthy calculations we come to a *scalar* hermitean operator $\hat{p}(x; V)$ of *the projection of momentum* on the vector field V^α through the (asymptotic) relation

$$P_V(\Psi_1, \Psi_2; S) = (\psi_1, \hat{p}(x; V)\psi_2) \tag{34}$$

where

$$\hat{p}(x; V) = K^\dagger \cdot \Big(p_S(x; V) + \lambda^2 G^\dagger \cdot \big(p_S(x; V) - 2mc^2 V^{(0)}\big) \cdot G - i\lambda^2\big(p_T(x; V) \cdot G - G^\dagger \cdot p_T(x; V)\big)\Big) \cdot K. \tag{35}$$

The hermitean (scalar!) differential operators $p_S(x; V)$ and $p_T(x; V)$ are defined by the following formulae:

$$p_T(x; V) \stackrel{def}{=} -i\hbar\big(V^{(0)}\tilde{D} + \frac{1}{2}\sigma^{(i)}\nabla_{(i)}V^{(0)}\big). \tag{36}$$

$$p_S(x; V) \stackrel{def}{=} i\hbar\big((V_S^\alpha\partial_\alpha + \frac{1}{2}\nabla_\alpha V_S^\alpha) \cdot \mathbf{I} - \frac{i}{4}L^{(j)}(V_S, h^{(i)})\ \varepsilon_{(ijk)}\sigma^{(k)}\big). \tag{37}$$

where

$$V_S^\alpha \stackrel{def}{=} h^\alpha_{(i)} h^{(i)}_\beta V^\beta, \tag{38}$$

i.e. it is the projection of V^α to the hypersurface S. Equation (35) together with Eqs.(36) - (38) looks rather cumbersome and is displayed here for generality.

It is simpler and more apparent to consider two particular choices of the vector field V. The first one is V lying in the hypersurface S, i.e. $V^\alpha = V_S{}^\alpha$ and consequently $V^{(0)} = 0$. The second one is when $V^\alpha = c^{-1}\tau^\alpha$, i.e. $V_S{}^\alpha = 0$. Now we pass to the first case, the latter will be considered in the next section.

Assuming that $V^{(0)} = 0$, $V_S{}^\alpha = V^\alpha$ and taking into account Eqs.(22),(25) one has

$$\hat{p}(x; V)\,|_{V=V_S} = K^\dagger \cdot \big(p_S(x; V) - \lambda^2 G^\dagger \cdot p_S(x; V) \cdot G\big) \cdot K \tag{39}$$

$$= p_S(x; V) + \big(\lambda^2[\tilde{D}\ ,\ [\tilde{D}\ ,\ p_S(x; V)]] + O(c^{-4}). \tag{40}$$

Now consider the commutator $[\hat{p}_S(x; V)\ ,\ \hat{p}_S(x; W)]$ when both vector fields V and W lie in S, i.e. $V = V_S$ and $W = W_S$:

$$[\hat{p}_S(x; V)\ ,\ \hat{p}_S(x; W)] = K^\dagger \cdot (\mathbf{I} + \lambda^2 G^\dagger \cdot G)^{-1} \cdot \Big([p_S(x; V)\ ,\ p_S(x; W)] + \lambda^2[G^\dagger \cdot G\ \ ,\ \ p_S(x; V - W)] \cdot \lambda^2 G^\dagger \cdot G\Big) \cdot K \tag{41}$$

$$= [p_S(x; V)\ ,\ p_S(x; W)] + O(c^{-2}). \tag{42}$$

We do not write out a rather long expression for $[p_S(x;V)\ ,\ p_S(x;W)]$ but mention that this commutator is equal to zero when

$$L(V,W)=0,\quad L(V,h^{(i)})=0,\quad L(W,h^{(i)})=0,\quad V^\alpha W^\beta F_{\alpha\beta}=0. \quad (43)$$

These conditions mean that V and W should form a coordinate basis on a two-dimensional submanifold of S, the tetrad vectors may be chosen to be constant on the submanifold (that is obviously a restriction on the geometry of V_4) and electromagnetic forces vanish on it. Then, one sees easily that also $[\hat{p}_S(x;V)\ ,\ \hat{p}_S(x;W)]=0$.

So, one comes easily to the conclusion that three independent and mutually commuting operators of momentum projections exist only in spatially flat Robertson - Walker space-times when the tetrad vectors form a normal coordinate basis. In more general space-times there is no three independent commuting components of momentum even in the limit $c^{-1}=0$ while projections of momentum of scalar particle on any coordinate basis on S mutually commute in this limit in the general V_4 [2].

4 The operator of energy and its connection with the hamiltonian

Let now $V^\alpha = c^{-1}\tau^\alpha$, i.e. $V^{(0)}=1$, $V_S=0$. Then, after obvious transformation, one has

$$\hat{p}_{\tau/c} = -\frac{\hbar^2}{2mc}K^\dagger\cdot\left(\tilde{D}\cdot G - G^\dagger\cdot\tilde{D} + G^\dagger\cdot G\right)\cdot K \quad (44)$$

Having substituted the expression Eq.(22) for G into the third term in the right-hand side of Eq.(44) and applying the relation (25) one obtains easily that

$$\begin{aligned} c\,\hat{p}_{\tau/c} &= H + K^{-1}\cdot(\ [i\hbar\tilde{T}\ ,\ K]\ + \\ &+\ \lambda^2(\mathbf{I}\ +\lambda^2 G^\dagger\cdot G)^{-1}\cdot G^\dagger\cdot[i\hbar\tilde{T}\ ,G]\cdot K) \\ &= H + O(c^{-2}). \end{aligned} \quad (45)$$

where H is the hamiltonian from the Pauli equation (26).

Moreover, since the operator K is defined by Eq.(25) up to a unitary transformation, the latter can be chosen so that the equality

$$c\,\hat{p}_{\tau/c} = H \tag{46}$$

is fulfilled exactly.

This assertion is evidently equivalent to that the linear differential equation for K

$$[i\hbar\tilde{\mathcal{T}}\ ,\ K] = \lambda^2(\mathbf{I} + \lambda^2 G^\dagger \cdot G)^{-1}\ \cdot G^\dagger \cdot [i\hbar\tilde{\mathcal{T}}\ . G] \cdot K \tag{47}$$

has solutions satisfying Eq.(25). In turn, this is the case if

$$[\tau^a\partial_a\ .\ K^\dagger \cdot (\mathbf{I} + \lambda^2 G^\dagger \cdot G) \cdot K\ ] = 0 \tag{48}$$

by virtue of Eqs.(25) and (46), i.e. the condition is conserved. Under condition (25) Eq.(46) is equivalent to

$$i\hbar[\tilde{\mathcal{T}}\ ,\ K^\dagger \cdot (\mathbf{I} + \lambda^2 G^\dagger \cdot G)\ K\ ] = 0. \tag{49}$$

The straightforward commutation with the use of Eq.(46) and its hermitean conjugate proves our assertion.

The unitary equivalence of the hamiltonian determined by dynamical equation (26) to the energy operator originating from the invariant functional (3) shows an intrinsic consistency of the approach. It establishes also a link between the NRQM Hamltonian H and the "hamiltonian" $P_{\tau/c}$ in the quantum theory of the field Ψ diagonalization of which gives rise to a quasiparticle interpretation of the latter theory, see, e.g. [7].

5 Operators of spatial coordinates

After polarization of $Q^A(\Psi;\Sigma)$ from Eq.(4) and using Eqs.(15),(24) one easily comes to the relation

$$Q^A(\Psi_1, \Psi_2; \Sigma) = (\psi_1, \hat{q}^A(x)\psi_2). \tag{50}$$

where

$$\hat{q}^A(x) \;=\; K^\dagger \cdot (q^A(x)\,\mathbf{I} \;+ \frac{\lambda^2}{2} G^\dagger \cdot q^A(x) \cdot G) \cdot K \tag{51}$$

$$= q^A(x)\,\mathbf{I} \;+\; K^{-1} \cdot [q^A(x),\ K] \;+\; \lambda^2 K^\dagger \cdot G^\dagger \cdot [q^A(x),\ G] \cdot K. \tag{52}$$

It is evident that the last two terms are at least of the order $O(c^{-2})$ and so the operator $\hat{q}^A$ is the multiplication by the function $q^A(x)$ at least up to this order. Having taken

$$K = K^\dagger = (\mathbf{I} + \lambda^2 G^\dagger \cdot G)^{-\frac{1}{2}}$$

as a solution of Eq.(25) we obtain

$$\begin{aligned}\hat{q}^A(x) &= q^A(x)\,\mathbf{I} + \frac{\lambda^2}{2}[\hat{D}\,.\,[\hat{D}\,.\,q^A]] + O(c^{-4}) \\ &= q^A(x)\,\mathbf{I} - \frac{\lambda^2}{2}(\partial^{(i)}\partial_{(i)}q^A + \sigma_{(k)}\varepsilon^{(ijk)}(\partial_{(i)}\partial_{(j)}q^A + \partial_{(j)}q^A\hat{D}_{(i)})) + \\ &\quad + O(c^{-4}). \qquad (53)\end{aligned}$$

As one sees from Eq.(53), the question on mutual commutativity of operators $\hat{q}^1$, $\hat{q}^2$, $\hat{q}^3$ and consequently on simultaneous measurability of spatial coordinates q^1, q^2, q^3 has an affirmative answer only in the zeroth order in c^{-2} and this fact cannot be changed by a unitary transformation of operator K. The condition $\partial^{(i)}\partial_{(i)}q^A = 0$ might be fulfilled in the sense that it is the condition of choosing harmonic coordinates q^A on the hypersurface S (i.e. $\Delta_S q^A(x) = 0$), if a Lorentz gauge condition $\nabla_\alpha h^\alpha_{(i)} = 0$ is imposed on the spatial part of the tetrad.

In the case of spin 0, see [2], the corresponding operators $\hat{q}^1, \hat{q}^2, \hat{q}^3$ commute up to $O(c^{-4})$. So the latter form a complete set of observables for scalar particle in a general V_4 in this approximation and it is sufficent for "practical" purposes of taking into account of einsteinian gravitation in the NRQM with the first nonvanishing relativistic correction, i.e. of the order of c^{-2}. As we have seen the situation is different for the Dirac particle and in V_4 one has to use the density matrix methods for such a purpose.

More detailed cosideration of the coordinate operator in V_4 and particularly its comparison with group-theoretical approaches to definition of the operator (when the latter are possible) will be presented elsewhere .

6 Operators of spin projections on the tetrad vectors

In a complete similarity with the coordinate operators one obtains for operators $\hat{S}(x)$ of projections of spin on tetrad space like vectors $h^{\alpha}_{(i)}$

$$\hat{S}_{(i)}(x) = \frac{\hbar}{2}(\sigma_{(i)} \cdot \mathbf{I} + \lambda^2\, G^{\dagger} \cdot \sigma_{(i)} \cdot G) \tag{54}$$

and again in the representation where $K = K^{\dagger}$ we obtain after simple calculations

$$\begin{aligned} \hat{S}^{(i)} &= \frac{\hbar}{2}\, K^{\dagger} \cdot \big(\sigma^{(i)}\, \mathbf{I} \;+\; \frac{\lambda}{2}\, G^{\dagger} \cdot \sigma^{(i)} \cdot G\big) \cdot K & (55) \\ &= \frac{\hbar}{2}\Big(\sigma^{(i)}\, \mathbf{I} \;+\; \frac{\lambda^2}{2}[\tilde{D}\,,\, [\tilde{D}\,,\, \sigma^{(i)}]]\Big) \;+\; O(c^{-4}) \\ &= \frac{\hbar}{2}\Big(\sigma^{(i)}\, \mathbf{I} \;+\; \lambda^2 (2\sigma^{(i)} \tilde{D}^{(j)} \cdot \tilde{D}_{(j)} \;-\sigma^{(j)}(\tilde{D}^{(i)} \cdot \tilde{D}_{(j)} + \tilde{D}^{(j)} \cdot \tilde{D}_{(i)}) \\ &+\; \varepsilon^{(ijk)}(2i\tilde{D}_{(j)} \cdot \tilde{D}_{(i)} + \frac{1}{4}\sigma_{(k)}\; \partial_{(j)}\Omega^{(pr)}_{\alpha}\varepsilon_{(prs)}h^{(s)\alpha}))\Big) \;+\; O(c^{-4}) & (56) \end{aligned}$$

The latter formula reflects the dependence of formal determination of spin on external field and on mode of motion and orientation of any measuring device if one takes into attention the relativistic origin of the NRQM. It seems worth to investigate in more detail and in connection with concrete although speculative measuring procedures.

7 Conclusion

So, it is shown that there is a consecutive way to deduce in a general V_4 the quantum-mechanical Schrodinger–Pauli evolution equation and operators of observables in a representation, in which the wave functions may be supposed to have Born's probabilistic interpretation and the operators are hermitean in the ordinary sense.

The results presented are ready for immediate use for taking into account of einsteinian gravitation in the semiclassical approximation of the NRQM in a (curvilinear) coordinate representation and in a free falling frame of reference. The latter is the simplest example of non-inertial frame of reference defined by its own non-trivial dynamics.

So we can say that the Quantum Mechanics with its formal physical measurement concept is extracted from the more general and abstract theory by fixing a classical hamiltonian dynamics that may apparently be considered as a dynamics of a measuring device and we might conclude that there are so many different Quantum Mechanics' as many classical hamiltonian dynamics' exist.

The generalisation of the geometrical basis of Quantum Theory to a Riemannian space-time background leads to a new information on the structure of observables which needs further determination. For example, the formulae of Sec. 6 seem to show the existence of something like the *Zitterbewegung* for spin projections.

Another interesting aspect of the approach considered here is an applcation of solutions (15) of the Dirac equation as an approximate basis for quantization of the field Ψ provided by a particle interpretation through the Born's interpretation of the NRQM wave functions φ. Again, this will lead to a correspondence between a second quantised field theory and a hamiltonian dynamics imposed on the "fast" phase function $S(x)$.

At last, if we introduce three momentum operators $\hat{p}_A$ taking coordinate basis defined by $q^A(x)$ as the vector fields $V^a(x)$, then the operators $\hat{p}_A, \hat{q}^A$ do not form a Heisenberg algebra together with $\mathbf{I}$ and not commute with $\hat{s}^{(i)}$ except the case $c^{-1} = 0$. The algebraic properties of these operators need a furhter investigation and comparison with results, for example, of geometric quantization.

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