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$GL_q(N)$-COVARIANT BRAIDED DIFFERENTIAL BIALGEBRAS

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$GL_q(N)$-ковариантные дифференциальные брэйд-биалгебры

Мы исследуем возможность введения (брэйд-) коумножения для $GL_q(N)$-ковариантных дифференциальных комплексов на квантовых пространствах. Соответствующие *дифференциальные биалгебры* (и алгебры Хопфа) найдены нами для бозонной и фермионной квантовой гиперплоскости (с аддитивным коумножением) и для матричной брэйд-алгебры $BM_q(N)$, как с мультипликативным, так и с аддитивным коумножением. Последний случай при $N = 2$ имеет отношение к q-деформациям пространства Минковского и алгебры Пуанкаре.

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$GL_q(N)$-Covariant Braided Differential Bialgebras

We study the possibility to define the (braided) comultiplication for the $GL_q(N)$ -covariant differential complexes on some quantum spaces. We discover such *differential bialgebras* (and Hopf algebras) on the bosonic and fermionic quantum hyperplanes (with additive coproduct) and on the braided matrix algebra $BM_q(N)$ with both multiplicative and additive coproducts. The latter case is related (for $N = 2$) to q-Minkowski space and q-Poincare algebra.

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1. Throughout the recent development of differential calculus on quantum groups and quantum spaces, two principal and closely related concepts are readily seen. One of them, initiated by Woronowicz [1], is known as *bicovariant differential calculus* on quantum groups. Its characteristic feature is the covariance under the left and right "group shifts" Δ_L and Δ_R acting upon the differential complex in a consistent way. Brzezinski [2] has shown that this corresponds to existence of a *differential bialgebra*, i.e. a bialgebra structure with honest coproduct Δ on the whole algebra of coordinate functions and their differentials. This allows one to treat all the subject using the standard Hopf-algebra technique.

Another concept, introduced by Wess and Zumino [3] (see also [4]), proceeds from the requirement of covariance of the differential complex on a quantum space with respect to the coaction of some outer quantum group considered as a group of symmetry. In other words, the corresponding differential algebra must be a covariant comodule as well.

In the present letter, we want to unify both concepts by formulating the following set of conditions to be satisfied by q-deformed differential calculi:

α) associative algebra of generators and differential forms is respected by the (co)action of some quantum group;

β) external differentiation d obeys $d^2 = 0$ and the usual (graded) Leibnitz rule;

γ) differential algebra admits a (braided) coproduct of the form [2]

$$\Delta(a) = a_{(1)} \otimes a_{(2)}\,, \qquad \Delta(da) = da_{(1)} \otimes a_{(2)} + a_{(1)} \otimes da_{(2)}\,. \tag{1}$$

Following these criteria, we obtain several examples of $GL_q(N)$-covariant differential bialgebras: the braided matrix algebra $BM_q(N)$ (with additive and multiplicative coproducts) and also bosonic and fermionic quantum hyperplanes with additive coproduct. The first example seems to be of special interest because $BM_q(2)$ is presently considered as a candidate to the role of the q-Minkowski space [5]-[9].

2. To formulate and study the quantum-group-covariant differential calculus, the R-matrix formalism [10] proved to be extremely convenient. Let us first consider the case of braided matrix algebra $BM_q(N)$ with the generators $\{1, u^i_j\}$ (the latter form the $N\times N$-matrix u) and relations

$$R_{21}u_2R_{12}u_1 = u_1R_{21}u_2R_{12}\,, \tag{2}$$

where R is the $GL_q(N)$ R-matrix [10, 11]. The multiplication rule (2) is invariant under adjoint coaction of $GL_q(N)$,

$$u^i_j \to T^i_m S(T^n_j) \otimes u^m_n\,, \quad \text{or} \quad u \to TuT^{-1}\,, \tag{3}$$

where T^i_j obey the relations

$$R_{12}T_1T_2 = T_2T_1R_{12} \tag{4}$$

and commute with u^m_n. Eq.(2) ("reflection equations") first appeared in the course of investigations of 2-dimensional integrable models on a half-line (see [12] and references therein). Further it was studied by Majid [13] within the general framework of braided algebras.

From now on we prefer to use the notation $a \otimes 1 \equiv a, 1 \otimes a \equiv a'$ for any element a. The matrix notation will also be slightly modified [14] to simplify the relevant calculations:

$$P_{12}R_{12} \equiv \hat{R}_{12} \equiv R,\; R^{-1} \equiv \overline{R},\; u_1 \equiv u,\; u'_1 \equiv u'.$$

Thus, the Hecke condition for R reads

$$R - \overline{R} = q - q^{-1} \equiv \lambda\,, \tag{5}$$

whereas (2) becomes simply

$$RuRu = uRuR\,. \tag{6}$$

Differential complex on $BM_q(N)$ is defined by (6) and

$$R\,u\,R\,du = du\,R\,u\,\overline{R}\,, \tag{7}$$

$$R\,du\,R\,du = -du\,R\,du\,\overline{R} \tag{8}$$

(here and below we omit the wedge product symbol $\wedge$ in the multiplication of differential forms). Of course, one could perfectly well use

$$\overline{R}\,u\,R\,du = du\,R\,u\,R$$

instead of (7): these possibilities are absolutely parallel. We should also note that an agreement of (7) with (6) (via the Leibnitz rule), and some other formulas below, rely heavily on the Hecke condition (5) that is specific to the $GL_q(N)$ case.

Commutational relations (7),(8) have been found in the component form for $N = 2$ [6] in the context of the q-Poincare algebra, and then recast into the R-matrix form in [8]. Besides that, eq.(6) is known [13] to admit the multiplicative coproduct

$$\Delta(u^i_j) = u^i_k \otimes u^k_j\,, \quad \text{or} \quad \Delta(u) = u \otimes u \equiv u\,u'\,, \tag{9}$$

provided the nontrivial braiding relations

$$\overline{R}\,u'\,R\,u = u\,\overline{R}\,u'\,R \tag{10}$$

are used for commuting primed u-matrices with unprimed ones. Recall [15] that the braiding transformation $\Psi : A \otimes B \to B \otimes A$, where A and B are covariant comodules of a quantum group, is a map which commutes with the group coaction and, therefore, produces a covariant recipe for multiplying tensor products of generators:

$$(1 \otimes a)\,(b \otimes 1) \equiv a'\,b = \Psi(a \otimes b)\,.$$

For instance, eq.(10) is induced by the corresponding universal $\mathcal{R}$-matrix through the (somewhat symbolic) relation

$$\Psi(u' \otimes u) = < TuT^{-1} \otimes Tu'T^{-1}\,, \mathcal{R} > \,.$$

Now let us examine whether a map of the form (1) (see also [16, 17]),

$$\Delta(du) = du \otimes u + u \otimes du \equiv du\,u' + u\,du'\,, \tag{11}$$

together with (9) yields a proper coproduct for the whole algebra (6)-(8). Our statement is that it really does. Moreover, two different sets of the braiding relations can be used here equally well: one based on (10),

$$\begin{cases} \overline{R}\,u'\,R\,u = u\,\overline{R}\,u'\,R\,, \\ \overline{R}\,du'\,R\,u = u\,\overline{R}\,du'\,R\,, \\ \overline{R}\,u'\,R\,du = du\,\overline{R}\,u'\,R\,, \\ \overline{R}\,du'\,R\,du = -du\,\overline{R}\,du'\,R\,, \end{cases} \tag{12}$$

and the other,

$$\begin{cases} \overline{R}u'\,R\,u = u\,R\,u'\,R\,, \\ \overline{R}du'\,R\,u = u\,R\,du'\,R\,, \\ \overline{R}\,u'\,R\,du = du\,R\,u'\,R\,, \\ \overline{R}\,du'\,R\,du = -du\,R\,du'\,R\,. \end{cases} \tag{13}$$

The proof that (11) is an algebra homomorphism is straightforward. For illustration, we explicitly verify one of the required conditions using, say, the braiding (13):

$$R\,\Delta(u)\,R\,\Delta(du) = R\,u\,\underline{u'\,R\,du}\,u' + R\,u\,\underline{u'\,R\,u}\,du'$$

$$= \underline{R\,u\,R\,du}\,R\,\underline{u'\,R\,u'} + \underline{R\,u\,R\,u}\,\underline{R\,u'\,R\,du'} = du\,R\,\underline{u\,\overline{R}\,R\,R\,u'}\,R\,u'\,\overline{R}$$

$$+u\,R\,\underline{u\,R\,du'\,R}\,u'\,\overline{R} = du\,R\,\overline{R}\,u'\,R\,u\,u'\,\overline{R} + u\,R\,\overline{R}\,du'\,R\,u\,u'\,\overline{R}$$

$$= (du\,u' + u\,du')\,R\,u\,u'\,\overline{R} = \Delta(du)\,R\,\Delta(u)\,\overline{R}$$

(underlining indicates the parts to which the next operation is to be applied). Similar calculations for eq.(8) are in fact optional, because their result can be foreseen by differentiating the equality just obtained. Finally, we stress that the coassociativity of (9) and (11) is evident.

Note that the first equation in (13) has already been used as a braiding in [18] to make the algebra (6) a bialgebra with *additive* coproduct (see below).

For both versions of the braiding relations, (12) and (13), the differential complex (6)-(8) admits the coproduct (9),(11); so $BM_q(N)$ becomes a differential bialgebra. A counit is defined in an obvious way,

$$\varepsilon(1) = 1\,, \quad \varepsilon(u) = 1\,, \quad \varepsilon(du) = 0\,.$$

Moreover, the braided antipode can also be introduced in complete analogy with the differential $GL_q(N)$ case [17], thus making $BM_q(N)$ a differential Hopf algebra.

3. Now we proceed to another, additive, algebra map

$$\Delta(u) = u \otimes 1 + 1 \otimes u \equiv u + u'\,, \tag{14}$$

$$\Delta(du) = du \otimes 1 + 1 \otimes du \equiv du + du' \tag{15}$$

which proves to be a second coproduct on $BM_q(N)$. It has been found in [18] that (14) is compatible with (6), provided the braiding is defined by the first line in (13). Our result is that the whole differential complex (6)-(8) admits (14),(15) as a coproduct if we use one of the following four sets of the braiding relations:

$$\begin{cases} R\,u'\,R\,u = u\,R\,u'\,\overline{R}\,, \\ R\,u'\,R\,du = du\,R\,u'\,\overline{R} - \lambda\,u\,R\,du'\,, \\ R\,du'\,R\,u = u\,R\,du'\,R\,, \\ R\,du'\,R\,du = -du\,R\,du'\,R\,; \end{cases} \tag{16}$$

$$\begin{cases} \overline{R}\,u'\,R\,u = u\,R\,u'\,R\,, \\ R\,u'\,R\,du = du\,R\,u'\,R\,, \\ du'\,R\,u\,\overline{R} = R\,u\,R\,du' + \lambda\,du\,R\,u'\,, \\ R\,du'\,R\,du = -du\,R\,du'\,R\,; \end{cases} \tag{17}$$

the remaining two sets are obtained from (16) and (17) by changing the position of the prime $u \leftrightarrow u'$ (it corresponds to the inverse braiding transformation Ψ^{-1}).

In this case, it is also easy to define a counit ε and an antipode S,

$$\varepsilon(u) = \varepsilon(du) = 0\,, \quad S(u) = -u\,, \quad S(du) = -du\,,$$

thus completing the construction of the differential Hopf algebra (with additive coproduct) on $BM_q(N)$.

4. As it has been pointed out in [8], the braided matrix algebra $BM_q(2)$ can also be interpreted as a quantum hyperplane (q-Minkowski space) for the quantum Lorentz group $SO_q(3,1)$. The coordinate algebra of a quantum hyperplane is known to admit, in a quite general situation, an additive bialgebra structure [7]. So, a natural question arises: can one define (additive) differential bialgebras on the hyperplanes related to arbitrary Yang-Baxter R-matrices? We can answer this question affirmatively for R-matrices of the Hecke type (5), in particular, for the $GL_q(N)$-covariant differential complexes proposed by Wess and Zumino [3]:

$$\begin{aligned} R\,x_1\,x_2 &= c\,x_1\,x_2\,, \\ c\,R\,dx_1\,x_2 &= x_1\,dx_2\,, \\ c\,R\,dx_1\,dx_2 &= -dx_1\,dx_2 \end{aligned} \tag{18}$$

(c is equal to q for the bosonic and to $-q^{-1}$ for the fermionic hyperplanes). These commutation relations are invariant under the coaction of $GL_q(N)$

$$x^i \to T^i_j \otimes x^j\,, \; dx^i \to T^i_j \otimes dx^j\,, \quad \text{or} \quad x \to T\,x\,, \; dx \to T\,dx \tag{19}$$

(see [19] for the generalization to the case $dx \to T\,dx + dT\,x$), and admit the differential Hopf algebra structure with the coproduct

$$\Delta(x) = x + x'\,, \qquad \Delta(dx) = dx + dx' \tag{20}$$

and the counit and antipode given by

$$\varepsilon(x) = \varepsilon(dx) = 0\,, \quad S(x) = -x\,, \quad S(dx) = -dx\,,$$

if one of the following four sets of the braiding relations is implied:

$$\begin{cases} R\,x_1'\,x_2 = c^{-1}\,x_1\,x_2'\,, \\ R\,x_1'\,dx_2 = c\,dx_1\,x_2'\,, \\ R\,dx_1'\,x_2 = c^{-1}\,x_1\,dx_2' - \lambda\,dx_1\,x_2'\,, \\ R\,dx_1'\,dx_2 = -c\,dx_1\,dx_2'\,; \end{cases} \tag{21}$$

$$\begin{cases} \overline{R}\,x_1'\,x_2 = c\,x_1\,x_2'\,, \\ c^{-1}\,x_1'\,dx_2 = R\,dx_1\,x_2' + \lambda\,x_1\,dx_2'\,, \\ R\,dx_1'\,x_2 = c\,x_1\,dx_2'\,, \\ R\,dx_1'\,dx_2 = -c\,dx_1\,dx_2'\,. \end{cases} \tag{22}$$

Two other sets can be obtained from these relations by substitution $x \leftrightarrow x'$ and correspond to the inverse braiding.

We have to stress that all the braiding relations used in this paper are no more than the cross-multiplication rules for two copies (primed and unprimed) of the same differential algebra. Using the Yang-Baxter equation

$$RR'R = R'RR' \qquad (R' \equiv \hat{R}_{23})$$

one can show that these rules really define associative algebras with uniquely ordered monomials of generators. In other words, all the presented examples of the braiding transformation Ψ are proved to be consistent.

5. In this paper, we have investigated several examples of the $GL_q(N)$-covariant algebras which are known to be the braided Hopf algebras. We have shown that the $GL_q(N)$-covariant differential complexes on these algebras admit the braided differential Hopf algebra structure and the corresponding coproduct is defined by the formulas (1) proposed by Brzezinski [2] for the unbraided case. Moreover, it is not hard to demonstrate that all differential complexes investigated in this paper are bicovariant with respect to the left and right braided inner coactions Δ_L and Δ_R. All these observations lead us to the conjecture that the following variant of the Brzezinski theorem [2] is valid for the case of braided Hopf algebras (for the notation see [2]):

Theorem. Let $(\Gamma,\ d)$ be a braided bicovariant differential calculus over a braided Hopf algebra $\mathcal{B}$. Then $(\Gamma^\wedge,\ d)$ is a differential (exterior) braided Hopf algebra of $\mathcal{B}$. Converse statement is also correct.

We intend to return to the detailed consideration of this theorem in our next publication.

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