94-249

объединенный
институт
ядерных
исследований
дубна

E2-94-249

R.M.Yamaleev

ELLIPTIC DEFORMED CLASSICAL MECHANICS IN THREE-DIMENSIONAL PHASE SPACE

Submitted to Proceedings of the Conference on Dynamical Systems and Chaos, Tokyo, May 23—27, 1994

1994

INTRODUCTION

The basic canonical structure of Hamilton dynamics is carried by a single canonical pair of variables of even— dimensional phase space where for a given space coordinates one has to put in correspondence the coordinates of momentum.In that sense the phase space of mechanics is characterized as two— dimensional space. Two—dimensionality of phase space is one of the basic statements of Hamilton dynamics. In 1973,Nambu [1] ,see also [2 – 6] ,suggested the natural way of extension of Hamilton equations and proposed the dynamical equations in a three—dimensional phase space. According to Nambu to formulate dynamic equations in three—dimensional phase space it is necessary to introduce two kinds of Hamiltonian. But the supposed formalism is quite general to use it in physical systems. To construct the theory of mechanics we need at any rate the notions of trajectory, velocity,force,work,energy or its analogies.Otherwise we need of physical and geometrical interpretations of dynamic variables of new mechanics.

In present paper we develope the way of introduction into three-dimensional phase space announced in [7].Our approach is based on the elliptic deformation of two dimensional phase space.This kind of deformation we call "elliptic" because oscillator of deformed mechanics is described by the Jacoby elliptic functions. The theory contains a parameter μ of energy dimensionality, so that for $\mu \to \infty$ the deformed dynamics is reduced to the Newton equations.

1. ELLIPTIC DEFORMATION OF NEWTON EQUATIONS

We postulate the next system of dynamic equations

$$\frac{dp}{dt} = -\frac{dV(x)}{dx}\frac{q}{\mu}, \tag{1.1}$$

$$\frac{dx}{dt} = \frac{p}{m}\frac{q}{\mu}, \tag{1.2}$$

$$\frac{dq}{dt} = -\frac{dV(x)}{dx}\frac{p}{m}. \tag{1.3}$$

Let us examine some properties of that system.The equations are formulated in three—dimensional space p, x, q. One may easily note the symmetry between p, m and q, μ in that system.

We need in the interpretation of the new dynamic variables p, x, q. For that purpose consider the limit $\mu \to \infty$. In that case $\frac{q}{\mu} \to 1$ and equations (1.1) – (1.2) are reduced to Newton equations of motion

$$\frac{dp}{dt} = -\frac{dV(x)}{dx},$$

$$\frac{dx}{dt} = \frac{p}{m}. \tag{1.4}$$

As regards (1.3) it is transformed to the equation for kinetic energy

$$\frac{dW}{dt} = -\frac{dV(x)}{dx}\frac{p}{m} \tag{1.5}$$

because the asympthotic expansion of q :

$$q = \mu + W + \frac{W_1}{\mu} + \cdots$$

Thus according the last equation the variable W can be interpretated as kinetic energy of the particle in the Newton mechanics.This reduction gives the next dimensions of the variables:

$dim(p) = dim(momentum), dim(x) = dim(length), dim(q) = dim(\mu) = dim(energy)$

The system (1.1) has two constants of motion.They are $H = \frac{p^2}{2m} + V(x)$, and $N = \frac{q^2}{2\mu} + V(x)$. By using these constants instead of (1.2) we get

$$\frac{dx}{dt} = \frac{2}{\sqrt{m\mu}}\sqrt{(H - V)(N - V)} \tag{1.6}$$

which straightway gives the integral

$$t - t_0 = \frac{\sqrt{m\mu}}{2}\int\frac{dx}{\sqrt{(H - V(x))(N - V(x))}}. \tag{1.7}$$

For the case of oscillator potential

$$V(x) = \frac{m\omega^2 x^2}{2},$$

integral (1.7) is reduced to the elliptic integral

$$t - t_0 = \int\frac{dy}{(1 - y^2)(1 - k^2 y^2)},$$

where we have put $y = \sqrt{\frac{m\omega^2}{2H}}x$ and $k^2 = \frac{H}{N}$. Hence the solution of deformed oscillator is represented by Jacoby elliptic function

$$x = \sqrt{\frac{2H}{m\omega^2}}sn(\omega(t - t_0)\sqrt{\frac{2N}{\mu}}, \sqrt{\frac{H}{N}}). \tag{1.8}$$

Thus in the deformed mechanics the trigonometric dependence of dynamical variables is deformed to Jacoby elliptic functions:

$$p = \sqrt{\frac{2H}{m\omega^2}}cos(\omega t) \rightarrow \sqrt{\frac{2H}{m\omega^2}}cn(\omega t\sqrt{\frac{2N}{\mu}}, \sqrt{\frac{H}{N}}),$$

$$x = \sqrt{\frac{2H}{m\omega^2}}sin(\omega t) \rightarrow \sqrt{\frac{2H}{m\omega^2}}sn(\omega t\sqrt{\frac{2N}{\mu}}, \sqrt{\frac{H}{N}}).$$

On making $k^2 = \frac{H}{N} \rightarrow 0$ we will obtain the ordinary oscillator model. Hence for small values of $\frac{H}{N}$ or for $H \ll N$ one may use the Newton equations(1.4). But for high energies,according to the present theory ,the system $(1.1)-(1.3)$ is more adequate.

If we make use of the functions $H = H(p,x)$ and $N = N(q,x)$ as Hamiltonians of the new system we can rewrite the system $(1.1) - (1.3)$ in quite general form

$$\begin{aligned} \frac{dp}{dt} &= \frac{dH}{dq}\frac{dN}{dx} - \frac{dH}{dx}\frac{dN}{dq} \\ \frac{dx}{dt} &= \frac{dH}{dp}\frac{dN}{dq} - \frac{dH}{dq}\frac{dN}{dp} \\ \frac{dq}{dt} &= \frac{dH}{dx}\frac{dN}{dp} - \frac{dH}{dp}\frac{dN}{dx}. \end{aligned} \tag{1.9}$$

It turns out nothing but the Nambu equations.

Due to the structure of equation (1.2) and the interpretation $x = x(t)$ as trajectory we are able to introduce the ordinary definitions of force F:

$$m\frac{d^2x}{dt^2} = F$$

and the work A performed by that force:

$$A = \int_1^2 F dx. \tag{1.10}$$

Let us calculate the integral of work equal to the change of kinetic energy of the particle in the potential field $V(x)$. We obtain

$$W = m\int_a^b \frac{d^2x}{dt^2}dx = \frac{1}{\mu}\int_a^b \frac{d(pq)}{dt}dx = E_{kin}(x=b) - E_{kin}(x=a),$$

where the value

$$E_{kin} = \frac{p^2q^2}{2m\mu^2} \tag{1.11}$$

we naturally interpret as kinetic energy. To find the potential part of the energy let us calculate the integral (1.10) using equations of motion .We get

$$\frac{1}{\mu}\int_a^b (q\frac{dp}{dt} + p\frac{dq}{dt})dx = -\frac{1}{\mu}\int_a^b \frac{\partial V}{\partial x}(\frac{p^2}{m} + \frac{q^2}{\mu})dx =$$

$$-\frac{1}{\mu}\int_a^b \frac{\partial V}{\partial x}(2(N+H)-4V)dx = 2\frac{1}{\mu}((N+H)(V(b)-V(a))-(V^2(b)-V^2(a))). \tag{1.12}$$

Thus the potential part of the energy is

$$E_{pot} = \frac{2}{\mu}((N+H)V(x) - V^2(x)),$$

where H, N are constants of motion.One finds the total energy by the sum of two parts:

$$E_{total} = E_{kin} + E_{pot} = \frac{2}{\mu}\frac{p^2q^2}{2m\mu^2} + ((N+H)V(x) - V^2(x)) =$$

$$\frac{2}{\mu}(\frac{p^2q^2}{2m\mu^2}+V(\frac{p^2}{2m}+\frac{q^2}{2\mu})+V^2)=$$

$$\frac{2}{\mu}(\frac{p^2}{2m}+V)(\frac{q^2}{2\mu}+V).$$

Hence the total energy is the product of Hamilton—Nambu functions:

$$E=\frac{2}{\mu}HN. \qquad (1.13)$$

Let us note that in the previous expressions the value $\pi = p\frac{q}{\mu}$ has played the role of momentum.Moreover π and E obey the general formular

$$v d\pi = dE. \qquad (1.14)$$

As distinguished from the usual case we could not define the explicit dependence $E = E(\pi)$, istead of that we got in (1.13) the explicit form of the function $E = E(p, q, x)$.

2. ELLIPTIC DEFORMATION OF RELATIVISTIC EQUATIONS OF MOTION

Since the deformation parameter k becomes essential for high value of energy, of necessity we have to put into correspondence of our deformation network with relativistic mechanics. To obtain equations of motion in that case we shall keep to the principle of symmetry beween the sets of values $\{H, p, m\}$ and $\{N, q, \mu\}$.It gives the next system of equations

$$\frac{dp}{dt}=-\frac{dV(x)}{dx}\frac{q}{\sqrt{q^2+\mu^2}}, \qquad (2.1)$$

$$\frac{dx}{dt}=\frac{pq}{\sqrt{(p^2+m^2)(q^2+\mu^2)}}, \qquad (2.2)$$

$$\frac{dq}{dt}=-\frac{dV(x)}{dx}\frac{p}{\sqrt{p^2+m^2}}. \qquad (2.3)$$

The system has two constants of motion

$$H=\sqrt{p^2+m^2}+V(x), N=\sqrt{q^2+\mu^2}+V(x). \qquad (2.4)$$

There is the principal difference of that system in comparision with non-relativistic system (1.1) – (1.3): the (2.1) – (2.3) one may reduce to the ordinary (non-deformed) relativistic equations on making use of the limit $\mu \to 0$. To find the total energy and momentum we shall use of the relation (1.14).Integrating this relation we get the expression for the work

$$A=\int\frac{d\pi}{dt}dx=\int v d\pi=\int dE=E_2-E_1, \qquad (2.5)$$

where E is the energy which we will define in the next form

$$E=\frac{1}{m}HN \text{or} E=\frac{1}{m}\sqrt{p^2+m^2}\sqrt{q^2+\mu^2}. \qquad (2.6)$$

From (2.2) we get

$$pq=mEv. \qquad (2.7)$$

On making use of the equations of motion (2.1), (2.3) we obtain

$$\frac{d\pi}{dt}=-\frac{\partial V}{\partial x}(\sqrt{(q^2+\mu^2)}+\sqrt{(p^2+m^2)}). \qquad (2.8)$$

Substituting this expression to (2.5) and using (2.4) calculate the integral (2.5) and find the potential part of the energy.We get

$$E_{pot}=\frac{1}{m}((H+N)V-V^2). \qquad (2.9)$$

Summing two parts of the energy we find the formular for total energy:

$$E = \frac{1}{m}(\sqrt{p^2+m^2}+V)(\sqrt{q^2+\mu^2}+V) \tag{2.10}$$

3. QUASICLASSICAL QUANTIZATION

In that section let us develope the quasiclassical quantization of the deformed oscillator model. According to the well—known Bohr quantization condition the energy levels of quantum oscillator obey

$$\frac{1}{h}\oint pdx = \pi(n+\frac{1}{2}), \tag{3.1}$$

where $p = \sqrt{2m(E_n - V)}$.

In deformed model the left side of this expression is deformed to the following integral

$$I(\mu) = \frac{1}{h\mu}\oint pqdx, \tag{3.2}$$

which is adiabatic invariant. On making use of the constants of motion N and H one gets the integral

$$I(\mu) = \frac{2\sqrt{m\mu}}{h\mu}\oint\sqrt{(H-V(x))(N-V(x))}dx. \tag{3.3}$$

Let us consider the case of oscillator potential $V(x) = \frac{m\omega^2x^2}{2}$. After corresponding labels we get table integral

$$I(\mu,a,b) = \alpha\oint\sqrt{(a^2-x^2)(b^2-x^2)}, \tag{3.4}$$

where $\alpha = \frac{\sqrt{m}m\omega^2}{h\sqrt{\mu}}$, $a^2 = \frac{2H}{m\omega^2}$, $b^2 = \frac{2N}{m\omega^2}$. The integrand function has four roots. According to these roots the interval of integration is composed on three parts.In result we get two kinds of integrals: real and imaginary. The real integral is equal to

$$\int_b^a\sqrt{(a^2-x^2)(b^2-x^2)} =$$
$$\frac{a}{3}((a^2+b^2)E(\frac{\pi}{2},k)-(a^2-b^2)F(\frac{\pi}{2},k))$$
$$k = \frac{b}{a}$$

The imaginary part of the integral is equal to

$$\int_b^a\sqrt{(a^2-x^2)(x^2-b^2)} =$$
$$\frac{a}{3}((a^2+b^2)E(\frac{\pi}{2},k')-2b^2F(\frac{\pi}{2},k'))$$
$$k' = \sqrt{\frac{a^2-b^2}{a^2}},$$

where we have used the next well—known elliptic integrals

$$F(\frac{\pi}{2},k) = a\int_b^a\frac{dy}{\sqrt{(a^2-y^2)(y^2-b^2)}}, aE(\frac{\pi}{2},k) = \int_b^a\frac{dy}{\sqrt{(a^2-y^2)(y^2-b^2)}},$$
$$k = \sqrt{\frac{a^2-b^2}{a^2}}.$$

Summing these results we postulate the next natural generalization of Bohr quantum condition

$$\frac{1}{h\mu}\oint pqdx = (n+\frac{1}{2})+i(m+\frac{1}{2}).$$

The quasiclassical wave function having doubly-periodic property is the elliptic function [8]

$$\Psi = \Psi(\frac{1}{h\mu} \oint pq dx).$$

It points that the quantization of deformed mechanics leads to the quantum mechanics with wave functions belonging to the elliptic functions.

REFERENCES

1. Y.Nambu,Phys.Rev.D7,No.8,(1973) 2405
2. F.B.Estabrook,Phys.Rev.D8,(1973)2740
3. F.Bayen and M.Flato ,Phys.Rev.D11,(1975)3049
4. M.Flato,A.Lichnerowicz,D.Sternheimer, J.Math.Phys.,17,(1976)1754
5. G.J.Ruggeri,Int.J.Theor.Phys.,12(1975)287; Lett.,Nuovo Cimento,17,(1976)169; Acta Cient.Venez.,32(1981)203
6. M.G.Sucre and A.J.Kalnay, Int.J.Theor.Phys.,12 (1975)149
7. R.M.Yamaleev,JINR Communications,P2-94-109, Dubna,1994; Abstracts of the conference on Dynamical Systems and Chaos,Tokyo 1994,May 23- 24,1994.
8. D.W.Masser,Elliptic functions and trancendence. Springer,1975.

Received bu Publishing Department on July 1, 1994.