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TRACE ANOMALIES AND COCYCLES OF WEYL AND DIFFEOMORPHISM GROUPS

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1 Introduction

The conformal anomaly first obtained by M.Duff [1] plays the important role in the development of string theory. The existence of this anomaly is the main starting point of Polyakov's approach to quantization of two-dimensional gravity [2]. Trace, or Weyl anomaly exists also in all even dimensions, and continues to be an object of investigation. This article is devoted to the considerations of the following fields in that subject. First, we consider the construction of the analog of Liouville action in higher dimensions, and obtain some exact expressions, particularly the closed form of that action in four dimensions; next, we investigate in general form the well-known possibility of mapping the Weyl anomaly through local counterterms in effective action into the anomaly of diffeomorphisms group, and obtain the general prescription for that mapping (the resulting diffeomorphisms anomaly actually violates diffeomorphisms group only partially, maintaining the volume-preserving part of that group, ($2d$ case see in [3])); finally, we consider the general structure of trace anomaly in all dimensions and make claim that the structure, suggested by Deser and Shwimmer [4], is actually the consequence of Wess-Zumino consistency condition. Unlike the $2d$ case, at $d > 2$ the Weyl anomaly is not enough for construction of full effective action $W(g_{\mu\nu})$ of conformal matter field, because the parameters of local symmetry group:

$$Weyl \otimes Diff(2k)$$

do not cover all components of the metric. But one has a possibility of construction of the finite variation of effective action on Weyl rescaling of metric:

$$S(\sigma, g_{\mu\nu}) = W(e^{\sigma} g_{\mu\nu}) - W(g_{\mu\nu}) \,.$$

. This local action corresponds in $d = 2$ to Liouville action [2]. This action $S(\sigma, g_{\mu\nu})$ has a property of being 1-cocycle of Weyl group, and may be used for transition from $W(g_{\mu\nu})$ to Weyl invariant effective action $\tilde{W}(g_{\mu\nu})$, the finite variation of which under the diffeomorphism $x^{\mu} \to f^{\mu}(x)$ gives us the 1-cocycle of diffeomorphisms group $\tilde{S}(f^{\alpha}, g_{\mu\nu})$, ($2d$ case see in [3]).

The organization of the paper is as follows. In section 2 we consider the transformation of measure of conformal matter field under the Weyl, or $Diff(d)$ transformation of metric. We derive the cocycle property of $S(\sigma, g_{\mu\nu})$ or $\tilde{S}(f^{\alpha}, g_{\mu\nu})$ and define the local counterterms for transition from Weyl to $Diff(d)$ invariant effective action. In section 3 we consider the finite variation of $d = 4$ effective action, which we call $d = 4$ Liouville action. In section 4 we discuss the extension of this construction on higher even dimensions and investigate the connection between Wess-Zumino consistency condition [5] and structure of trace anomaly .

2 Cocycle and Effective action

Let's consider now the problem of the change of the measure in the functional integral for conformal matter field φ in external gravitational field under the Weyl transformation :

$$g_{\alpha\beta} \to e^{\sigma(x)} g_{\alpha\beta}; \qquad \varphi \to e^{\frac{2-d}{4}} \varphi. \tag{1}$$

The measure in the functional integral changes in the following way:

$$D_{e^{\sigma(x)}g}\varphi = D_g\varphi \exp S(\sigma; g_{\alpha\beta}). \tag{2}$$

This type of relations is very important, being the starting point of DDK calculation of the critical exponent of $2d$ gravity [6].

The action $S(\sigma; g)$ in (2) has to satisfy some conditions. First,in the case of infinitisimal transformation $\delta\sigma(x)$ it has to reproduce the trace anomaly:

$$S(\delta\sigma(x); g_{\alpha\beta}) = \int T^{\mu}_{\mu} \delta\sigma(x) \sqrt{g} d^{2k}x. \tag{3}$$

Second, $S(\sigma; g)$ has to satisfy the following property, which follows from the application of (2) to the composition of two Weyl transformations σ_1 and σ_2:

$$S(\sigma_1 + \sigma_2; g) = S(\sigma_1; e^{\sigma_2} g) + S(\sigma_2; g), \tag{4}$$

which means that $S(\sigma; g)$ is the 1-cocycle of the group of Weyl transformations. On the other hand, the action $S(\sigma; g)$ coincides with the finite variation of anomalous effective action, due to the property (3):

$$W(g) = \ln \int D\varphi \exp\{-S_w(\varphi; g)\}, \tag{5}$$

$$\delta_{\sigma} W(g) = \int T^{\mu}_{\mu} \delta\sigma(x) \sqrt{g} d^{2k}x, \tag{6}$$

where $S_w(\varphi; g)$ is classical Weyl and diffeomorphism invariant action for matter fields. In other words

$$S(\sigma, g) = W(e^{\sigma} g) - W(g) \tag{7}$$

and non-triviality of the cocycle $S(\sigma; g)$ follows from the fact that $W(g)$ is non-local, $Diff(2k)$-invariant functional on $g_{\alpha\beta}$.

Let's now consider the new non-local effective action:

$$\tilde{W}(g) = W(g) + S(\sigma; g)\Big|_{\sigma = -\frac{1}{k}\ln\sqrt{g}} \,, \tag{8}$$

where dimension of space-time is $d = 2k$. It's easy to see that due to the group relation (4) (cocyclic property) this action is Weyl invariant, but not diffeomorphism invariant. Let's now in analogy with (7) calculate the finite variation of $\tilde{W}(g)$ under diffeomorphism transformation:

$$\tilde{W}(f^*g) - \tilde{W}(g) = \tilde{S}(f;g). \tag{9}$$

It's easy to see that from (4),(8) and from

$$\ln\sqrt{f^*g(x)} = \ln\sqrt{g(f)} + \ln\Delta^f_x\,, \tag{10}$$

where

$$\Delta^f_x = \det\frac{\partial f^\alpha(x)}{\partial x^\beta}\,, \tag{11}$$

we obtain

$$\tilde{S}(f;g) = S(\sigma; g_{\alpha\beta}/(\sqrt{g})^{1/k})\Big|_{\sigma=\frac{1}{k}\ln\Delta^{f^{-1}}_x}. \tag{12}$$

This action has a cocyclic property for the $Diff(2k)$ group

$$\tilde{S}(f\circ h;g) = \tilde{S}(f;h^*g) + \tilde{S}(h;g)\,. \tag{13}$$

Therefore we can define the change of the measure for $Diff(2k)$ transformation in the case when we use Weyl invariant regularization:

$$D_{f^*g}\varphi = D_g\varphi\exp\tilde{S}(f;g_{\alpha\beta}), \tag{14}$$

where $\tilde{S}(f;g)$ is defined in (12) and has the property of being 1-cocycle of $Diff(2k)$ group.

3 Liouville action in $d = 4$

Let's now consider the application of our construction to the cases of 2 and 4 dimensions.

The trace anomaly in $d = 2k$ is some polynomial $A(R,k)$ degree k on Riemann tensor:

$$T^\alpha_\alpha = A(R,k). \tag{15}$$

In two dimensions A is simply

$$A(R,2) = \frac{c}{48\pi}R, \tag{16}$$

where R - $2d$ Euler characteristic's density. The four-dimensional anomaly has much more complicated form [1]

$$A = -\frac{1}{(2\pi)^2}[aC^2 + bE]\,, \tag{17}$$

here C is Weyl tensor, E is Euler characteristic's density . The coefficients a,b, are given by

$$\begin{aligned} a &= \frac{1}{480}(N_0 + 6N_{1/2} + 12N_1)\,, \\ b &= \frac{1}{1440}(N_0 + 11N_{1/2} + 62N_1), \end{aligned} \tag{18}$$

where $N_0, N_{1/2}$ and N_1 are the number of real spin 0, Dirac spin 1/2, and real spin 1 fields,respectively. If we look for the general solution of equation (4) with condition (3), we have to take $\sigma_2 = \sigma$ and $\sigma_1 = \delta\sigma$ and get the differential form of (4):

$$\delta S(\sigma;g) = S(\delta\sigma; e^\sigma g) = \int A(R(e^\sigma g;k))\delta\sigma\sqrt{g}d^{2k}x\,. \tag{19}$$

The explicit form of solution for two-dimensional case is famous Liouville action [2]

$$S_2(\sigma,g) = \frac{c}{48\pi}\int d^2x\sqrt{g}(\frac{1}{2}g^{\alpha\beta}\partial_\alpha\sigma\partial_\beta\sigma + R\sigma). \tag{20}$$

In four dimensions explicit form of S first has been found in [7]

$$\begin{aligned} S_4(\sigma,g) &= \int d^4x\sqrt{g}\frac{1}{(2\pi)^2}(b\{\frac{1}{8}[(\nabla_\alpha\sigma\nabla^\alpha\sigma)^2 + \frac{1}{2}\nabla_\alpha\sigma\nabla^\alpha\sigma\nabla^2\sigma \\ &\quad -(R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R)\nabla_\alpha\sigma\nabla_\beta\sigma]\} + \sigma A). \end{aligned} \tag{21}$$

This expression satisfies the cocyclic property (4) and can be used as phase function in the Weyl transformation of the measure (2). Therefore the expression (21) define the 1-cocycle of Weyl group in $d = 4$. Finally we can substitute (21) in (12) and obtain the $Diff(4)$ cocycle

$$\tilde{S}_4(f;g) = S_4(\sigma; g_{\alpha\beta}/(\sqrt{g})^{1/2})\Big|_{\sigma=\frac{1}{2}\ln\Delta^{f^{-1}}_x}. \tag{22}$$

The corresponding $Diff(2)$ cocycle was obtained in ref.[3]. In the next section we discuss the extension of this construction on higher even dimensions.

4 Conclusions and outlook

Let's discuss the connection between Wess-Zumino consistency condition [5] and structure of trace anomaly in any even dimensions. The Wess-Zumino condition means that the second Weyl variation of effective action $W(g)$ has to be symmetric.This means that variation of anomaly expression $A(R;k)$ on Ricci tensor is covariantly divergenceless:

$$\nabla^\alpha\frac{\delta A(R;k)}{\delta R^{\alpha\beta}} = 0. \tag{23}$$

The Euler characteristic satisfies this condition in any even dimension due to Bianchi identity. So we can suppose that in any dimension the general form of conformal anomaly is the Euler characteristcs density modulo Weyl invariant terms (like C^2 in $d = 4$)[4].The terms with lower order on R can be obtained as variations of local functionals and can be removed by adding local counterterms to an effective action. On the other hand, to construct the cocycles we need only the Weyl noninvariant part of anomaly which coincides with the Euler characteristcs density.Therefore we can use the known expression for Euler density:

$$E_{2k} = \frac{1}{2k!}\varepsilon^{\mu_1\nu_1\mu_2\nu_2\cdots\mu_k\nu_k}\varepsilon_{\alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_k\beta_k}R^{\alpha_1\beta_1}_{\mu_1\nu_1}R^{\alpha_2\beta_2}_{\mu_2\nu_2}\cdots R^{\alpha_k\beta_k}_{\mu_k\nu_k} \quad (24)$$

for construction of higher dimensions Weyl and $Diff(2k)$ cocycles. This work is now in progress and will be done in near future.

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