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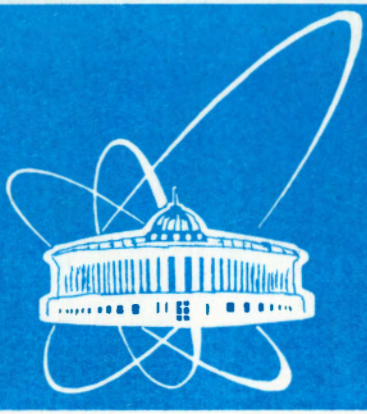

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FIRST ORDER DEVIATION EQUATIONS IN SPACES WITH A TRANSPORT ALONG PATHS

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1. INTRODUCTION

The work deals with the derivation of first order (with respect to a corresponding parameter) equations satisfied by the introduced in [1] relative mechanical quantities in spaces (manifolds) whose tangent bundle is endowed with a transport along paths, which in the present paper is supposed to be linear [2]. These equations, in fact, were found in [3] in a component form, i.e., in a fixed local (and coordinate) basis. Besides, in [3] implicitly were used linear transports along paths without self-intersections. The present investigation closely follows the ideas of [3], but to make a difference of it, here are used a coordinate independent language (where it is possible) and linear transports along arbitrary paths.

As the mentioned equations have a form and physical interpretation similar to that of the equation of geodesic deviation (of first order for the infinitesimal deviation vector) [4,5], they are called deviation equations (for the corresponding quantities).

Section 2 contains certain approximate results concerning linear transports along paths in vector bundles. In sections 3 through 7 the first order deviation equations are derived for, respectively, the deviation vector, relative velocity, relative momentum, relative acceleration and the relative energy of two arbitrary moving point particles. Also connections between these quantities are found.

Below, for reference purposes, some definitions and constructions are presented.

All considerations in the present work, with an exception of a

part of Sect. 2, are made in a (real) differentiable manifold M [6,7] whose tangent bundle $(T(M),\pi,M)$ is endowed with a linear transport (L-transport) along paths [2] and a covariant differentiation (linear connection) ∇ [6,7]. Here $T(M):=\bigcup_{x\in M} T_x(M)$, $T_x(M)$ being the tangent to M space at $x\in M$ and $\pi:T(M)\longrightarrow M$ is such that $\pi(V):=x$ for $V\in T_x(M)$.

By J and $\gamma:J\longrightarrow M$ we denote, respectively, an arbitrary real interval and a path in M. If γ is of class C^1, its tangent vector is written as $\dot\gamma$.

The linear transport (L-transport) along paths in $(T(M),\pi,M)$ (cf. [2]) is a map $L:\gamma\longmapsto L^\gamma$, $L^\gamma:(s,t)\longmapsto L^\gamma_{s\longrightarrow t}$, $s,t\in J$ being the L-transport along γ, where $L^\gamma_{s\longrightarrow t}:T_{\gamma(s)}(M)\longrightarrow T_{\gamma(t)}(M)$, satisfy the equalities

$$L^\gamma_{t\longrightarrow r}\circ L^\gamma_{s\longrightarrow t}=L^\gamma_{s\longrightarrow r},\quad r,s,t\in J, \tag{1.1}$$

$$L^\gamma_{s\longrightarrow s}=id_{T_{\gamma(s)}(M)},\quad s\in J, \tag{1.2}$$

$$L^\gamma_{s\longrightarrow t}(\lambda U+\mu V)=\lambda L^\gamma_{s\longrightarrow t}U+\mu L^\gamma_{s\longrightarrow t}V,\quad s,t\in J,\ U,V\in T_{\gamma(s)}(M). \tag{1.3}$$

Here id_X is the identity map of the set X.

If X, Y, and Z are vector fields on M [7], then the tensors (operators) of torsion T and curvature R of the covariant differentiation ∇, respectively, are [7]

$$T(X,Y):=\nabla_X Y-\nabla_Y X-[X,Y], \tag{1.4}$$

$$R(X,Y)Z:=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{[X,Y]}Z, \tag{1.5}$$

where [X,Y] is the commutator of X and Y.

The covariant differentiation (derivative) along the C^1 path $\gamma:J\longrightarrow M$, i.e., $\nabla_{\dot\gamma}$, will be denoted by $D/ds|_\gamma$, $s\in J$, or simply by D/ds if there is no risk of misunderstanding.

Let there be given paths $x_a:J\longrightarrow M$, $a=1,2$ and a one-parameter family of paths $\{\gamma_s:J'\longrightarrow M,\ s\in J\}$ such that $\gamma_s(r'):=x_1(s)$ and $\gamma_s(r''):=x_2(s)$ for some $r',r''\in J'$. The tangent vectors to the paths $\gamma_s:r\longmapsto\gamma_s(r)$ and $\gamma_{\cdot}(r):s\longmapsto\gamma_s(r)$, $s\in J$, $r\in J'$ are denoted, respectively by $\dot\gamma_s$ and $\gamma'(r)$.

The differentiation $D/ds|_{x_1}$ will for, brevity, be written as D/ds.

The deviation vector of x_2 with respect to x_1 at $x_1(s)$ (cf. [8], eq. (2.5)) is

$$h_{21}=h_{21}|_{\gamma_s(r')}=h(s;x_1)=\int_{r'}^{r''}(L^{\gamma_s}_{u\longrightarrow r},\dot\gamma_s(u))du. \tag{1.6}$$

Let the paths x_1 and x_2 be world lines of the point particles, respectively, 1 and 2. Their velocities [4], the relative velocity (of 2 with respect to 1; cf. [1]), and the corresponding to them accelerations, respectively, are:

$$V_1=\dot x_1,\quad V_2=\dot x_2, \tag{1.7a}$$

$$\Delta V_{21}=L^{\gamma_s}_{r''\longrightarrow r},V_2-V_1, \tag{1.7b}$$

$$A_a=\frac{D}{ds}\Big|_{x_1}V_a,\quad a=1,2, \tag{1.8a}$$

$$\Delta A_{21}=L^{\gamma_s}_{r''\longrightarrow r},A_2-A_1. \tag{1.8b}$$

The momenta of the considered particles are [4,1]

$$p_a=\mu_a V_a,\quad a=1,2,$$

where $\mu_a:J\longrightarrow R\setminus\{0\}$, $a=1,2$ are (nonvanishing) scalar functions (identified with the corresponding proper masses if the latter are nonzero; cf.[4]).

The relative momentum of the second particle with respect to the first one is (cf. [1], sect. 3)

$$\Delta p_{21} = L^{\gamma_s}_{r'' \longrightarrow r'} p_2 - p_1. \tag{1.9}$$

2. SOME APPROXIMATE RESULTS FOR LINEAR TRANSPORTS ALONG PATHS

In this section, approximate results concerning linear transports (L-transports) along paths in vector bundles will be obtained. They will be used later on in the present work. For details of the theory of L-transports the reader is referred to [2], only a few facts of which will be cited below.

Let L^{γ} be an L-transport along the path $\gamma: J \longrightarrow B$ in the vector bundle (E,π,B) (see [2]; cf. Sect. 1) and $\{e_i(s): i=1,\dots,\dim(\pi^{-1}(x)),\ x\in B\}$ be a bases in $\pi^{-1}(\gamma(s))$, $s\in J$. Let in the basis $\{e_i(r)\}$, $r=s,t$ the transport $L^{\gamma}_{s\longrightarrow t}$ along γ from s to t, $s,t\in J$ be described by the matrices $H(t,s;\gamma):=\|H^i_{\ j}(t,s;\gamma)\|$. So (cf. [2], sect. 2, eq. (2.10)) if $u_s=u^i_s e_i(s)\in\pi^{-1}(\gamma(s))$, $s\in J$, where here and hereafter the Latin indices run from 1 to $\dim(\pi^{-1}(x))$, $x\in B$ and summation from 1 to $\dim(\pi^{-1}(x))$, $x\in B$ over repeated on different levels indices is assumed. Then

$$L^{\gamma}_{s\longrightarrow t} u_s = H^i_{\ j}(t,s;\gamma) u^j_s e_i(t). \tag{2.1}$$

If γ and H are C^{N+1} functions, N being an integer, then due to $H(s,s;\gamma)=\mathbb{I}$, with $\mathbb{I}$ the unit matrix, (see [2], eq. (2.12)), the expansions:

$$H(t,s;\gamma)=\mathbb{I}+\sum_{m=1}^{N}\frac{1}{m!} H_{i_1\dots i_m}(s;\gamma)(\gamma^{i_1}(t)-\gamma^{i_1}(s))\cdots(\gamma^{i_m}(t)-\gamma^{i_m}(s))+$$

$$+O((t-s)^{N+1}), \tag{2.2a}$$

$$H(t,s;\gamma)=\mathbb{I}+\sum_{m=1}^{N}\frac{1}{m!} {}^{m}H(s;\gamma)(t-s)^m+O((t-s)^{N+1}) \tag{2.2b}$$

are valid, where

$$H_{i_1\dots i_m}(s;\gamma):=\left.\frac{\partial^m H(t,s;\gamma)}{\partial\gamma^{i_m}(t)\cdots\partial\gamma^{i_1}(t)}\right|_{t=s}, \tag{2.3a}$$

$${}^{m}H(s;\gamma):=\left.\frac{\partial^m H(t,s;\gamma)}{\partial t^m}\right|_{t=s}. \tag{2.3b}$$

The matrices (2.3b) can easily be expressed through the matrices (2.3a), e.g. we have:

$${}^{1}H(s;\gamma)=H_i(s;\gamma)\dot\gamma^i(s), \tag{2.4a}$$

$${}^{2}H(s;\gamma)=H_{ij}(s;\gamma)\dot\gamma^i(s)\dot\gamma^j(s)+H_i(s;\gamma)\frac{d\dot\gamma^i(s)}{ds}. \tag{2.4b}$$

If in (2.1) we substitute $H(t,s;\gamma)$ with its N-th approximation with respect to $t-s$, which due to (2.2) is

$${}^{(N)}H(t,s;\gamma)=\mathbb{I}+\sum_{m=1}^{N}\frac{1}{m!} H_{i_1\dots i_m}(s;\gamma)(\gamma^{i_1}(t)-\gamma^{i_1}(s))\cdots(\gamma^{i_m}(t)-\gamma^{i_m}(s))=$$

$$=\mathbb{I}+\sum_{m=1}^{N}\frac{1}{m!} {}^{m}H(s;\gamma)(t-s)^m, \tag{2.5}$$

where the second equality is with a precision of $O((t-s)^{N+1})$, we get a map ${}^{(N)}L^{\gamma}_{s\longrightarrow t}:\pi^{-1}(\gamma(s))\longrightarrow\pi^{-1}(\gamma(t))$ defined in $\{e_i\}$ by

$${}^{(N)}L^{\gamma}_{s\longrightarrow t} u_s := {}^{(N)}H^i_{\ j}(t,s;\gamma) u^j_s e_i(s). \tag{2.6}$$

Evidently, ${}^{(N)}L^{\gamma}$ is the N-th approximation to L^{γ} (in $\{e_i\}$), i.e., they coincide up to terms of (N+1)-th order with respect to the difference $t-s$.

Further in this paper we shall work only with the zero-th (N=0) and first (N=1) approximations to L^{γ}, which according to (2.5) are described by

$$^{(0)}H(t,s;\gamma)=\mathbb{I}, \tag{2.7a}$$

$$^{(1)}H(t,s;\gamma)=\mathbb{I}+H_1(s;\gamma)(\gamma^1(t)-\gamma^1(s))=$$

$$=\mathbb{I}+{}^1H(s;\gamma)(t-s)=\mathbb{I}+H_1(s;\gamma)\dot{\gamma}^1(s)(t-s). \tag{2.7b}$$

(The second and third equalities in (2.7b) are up to $O((t-s)^2)$.)

Let us note that the zero-th approximation (2.7a) is constant and, hence, it depends neither on s, t and γ, nor on the used L-transport.

From here on in this work we will consider only the case of the tangent bundle to a differentiable manifold M, i.e. it is supposed that $(E,\pi,B)=(T(M),\pi,M)$.

It is important to be emphasized that in the case $(E,\pi,B)=(T(M),\pi,M)$ the components of the involved in (2.7b) matrices $(-H_1(s;\gamma))$, i.e. $[-H^i_{.jk}(s;\gamma)]=-[\partial H^i_{.j}(t,s;\gamma)/\partial\gamma^k(t)]|_{t=s}$, are coefficients of an affine connection along γ, i.e. under a change of the basis $e_1(s)$ they transform like usual coefficients of an affine connection [10]. This proposition is a simple corollary of (2.3a) for m=1 and the circumstance that $H^i_{.j}(s;\gamma)$ in this case are components of a two-point tensor from $T_{\gamma(t)}\otimes T^*_{\gamma(s)}$ (see [2], sect. 2). We shall note here, without a proof, that the so arising connection with the coefficients $[-H^i_{.jk}(s;\gamma)]$ along γ is flat, i.e., its curvature tensor is zero.

If in the tangent bundle $(T(M),\pi,M)$ an affine connection with the coefficients $\{\Gamma^i_{.jk}(x)\}$ (see e.g. [7]) is given, then it is easy to calculate that for the parallel transport defined by the connection

$$H^i_{.jk}(s;\gamma;\|)=-\Gamma^i_{.jk}(\gamma(s)) \tag{2.8}$$

is valid, where an additional argument $\|$ indicates that the calcu-

lations are made for the pointed parallel transport. According to the above said, this equality means that connection along any path γ induced by the parallel transport coincides with the restriction of the affine connection on the path γ generating this transport.

Let there be given paths $x_a: J\longrightarrow M$, a=1,2 and a one-parameter family of paths $\{\gamma_s: J'\longrightarrow M,\ s\in J\}$ such that $\gamma_s(r'):=x_1(s)$ and $\gamma_s(r''):=x_2(s)$ for some $r',r''\in J'$.

Let in M be given an affine connection ∇ with local coefficients $\Gamma^i_{.jk}$ [7], B be a C^1 vector field on $\{\gamma_s(r):\ r\in J',\ s\in J\}$ and

$$\Delta B_{21}:=L^{\gamma_s}_{r''\longrightarrow r'}B_{x_2(s)}-B_{x_1(s)}\in T_{x_1(s)}(M). \tag{2.9}$$

Applying to the first term of this definition (2.1) and (2.2b) for N=1 and taking into account (2.4a), $x_1(s)=\gamma_s(r')$ and $x_2(s)=\gamma_s(r'')$, we find after some simple calculations

$$\Delta B_{21}=\left(\frac{D^H}{dr}\Big|_{\gamma_s}B\right)_{\gamma_s(r')}(r''-r')+O((r''-r')^2)=\left(\frac{D}{dr}\Big|_{\gamma_s}B\right)_{\gamma_s(r')}(r''-r')+$$

$$+S(B,\zeta_{21})|_{\gamma_s(r')}+O((r''-r')^2), \tag{2.10}$$

where $D/dr|_{\gamma_s}:=\nabla_{\dot{\gamma}_s}$ is the covariant derivative along $\gamma_s: J'\longrightarrow M$ generated by ∇, $D^H/dr|_{\gamma_s}:=\nabla_{\dot{\gamma}_s}|_{\Gamma^i_{.jk}=-H^i_{.jk}}$ is the covariant derivative along γ_s generated by the connection with local coefficients $-H^i_{.jk}$ and S is a tensor field of the type (1,2) whose components in any local basis are

$$S^i_{.jk}|_{\gamma_s(r)}:=-H^i_{.jk}(r;\gamma_s)-\Gamma^i_{.jk}(\gamma_s(r)),\ r\in J'',\ s\in J, \tag{2.11}$$

i.e. $(S(B,\zeta_{21})|_{\gamma(r')})^i=S^i_{.jk}|_{\gamma(r')}B^j_{\gamma_s(r')}\zeta^k_{21}|_{\gamma(r')}$. Here, as in (2.10),

$$\zeta_{21}|_{\gamma_s(r')}=(r''-r')\dot{\gamma}_s(r') \tag{2.12}$$

is the infinitesimal deviation vector at $\gamma_s(r')$ (cf. [8], eq.

(2.10)) which due to (2.7b) (see also [8], eq. (2.9)) is connected with the deviation vector (1.6) of x_2 with respect to x_1 through the equality

$$h_{21}=\zeta_{21}+O((r''-r')^2). \tag{2.13}$$

In this equality, as well as till the end of this work, we will work only with a precision of terms up to $O((r''-r')^2)$.

3. EQUATION FOR THE DEVIATION VECTOR

In fact, the equation, mentioned in the title of this section, has been derived in [8], example 3.2 and it is expressed by equation (3.27) of [8] which, in our case equivalently, can be written as

$$\frac{D^2\zeta_{21}}{ds^2}=R(V_1,\zeta_{21})V_1 + T(V_1,D\zeta_{21}/ds) + \frac{DT}{ds}(V_1,\zeta_{21}) + T(F_s,\zeta_{21}) + \\ + \frac{DF_s(r)}{dr}\Big|_{r=r'}(r''-r') + O((r''-r')^2), \tag{3.1}$$

where V_1 is the velocity of the first particle, the force field F_s is defined by

$$F_s(r):=\nabla_{\gamma,\gamma'}\Big|_{\gamma_s(r)}=\Big(\frac{D}{ds}\Big|_{\gamma_{\cdot}(r)\gamma'}\Big)_{\gamma_s(r)} \tag{3.2}$$

and all quantities are taken at the point $\gamma_s(r')$.

We want to emphasize two features of equation (3.1). Firstly, it is independent of the concrete choice of the used transport along paths L, and also (up to terms $O((r''-r')^2)$) of the family $\{\gamma_s\}$. And secondly, the derivation of this equation in [3] shows that in it the correction $O((r''-r')^2)$ is strictly equal to zero.

Eq. (3.1) is a generalization, in the case of arbitrary paths

in spaces with torsion, of the classical equation of geodesic deviation [4,5].

4. EQUATION FOR THE RELATIVE VELOCITY

If along x_1 and x_2 particles 1 and 2 are moving, respectively, then their velocities are given by (1.7a). Fist of all we want to find the connection between their relative velocity (1.7b) and the first order deviation velocity $D\zeta_{21}/ds$ which due to (2.13) is connected with the deviation velocity Dh_{21}/ds through the equality

$$Dh_{21}/ds=D\zeta_{21}/ds+O((r''-r')^2). \tag{4.1}$$

Taking into account $x_1(s)=\gamma_s(r')$ and $x_2(s)=\gamma_s(r'')$, we find $\zeta^1_{21}(s)=x^1_2(s)-x^1_1(s)+O((r''-r')^2)$ and $V^1_2(s)-V^1_1(s)=\frac{\partial^2\gamma^1_s(r)}{\partial r\partial s}\Big|_{r=r'}\times(r''-r')+$ $+O((r''-r')^2)$. Differentiating covariantly the former of these equations with respect to s along x_1 and using the latter one, we get

$$\frac{D\zeta_{21}}{ds}=\Big(\frac{D}{ds}\Big|_{\gamma_s\gamma'_s}\Big)\Big|_{r=r'}(r''-r')+T(V_1,\zeta_{21})\Big|_{x_1(s)}+O((r''-r')^2). \tag{4.2}$$

On the other hand, from (2.10) for $B_{\gamma_s(r)}=\gamma'_s(r)$, (2.9), and (1.7) we see that

$$\Delta V_{21}=\Big(\frac{D}{dr}\Big|_{\gamma_s\gamma'_s}\Big)\Big|_{r=r'}(r''-r')+S(V_1,\zeta_{21})\Big|_{x_1(s)}+O((r''-r')^2). \tag{4.3}$$

Comparing this equality with (4.2), we find the following relation between the relative velocity ΔV_{21} and the first order deviation velocity $D\zeta_{21}/ds$:

$$\frac{D\zeta_{21}}{ds}=\Delta V_{21}+(T(V_1,\zeta_{21})-S(V_1,\zeta_{21}))\Big|_{x_1(s)}+O((r''-r')^2). \tag{4.4}$$

From here we can make the conclusion that up to second order terms the deviation velocity describes the "general relative velo-

city" of particle 2 with respect to particle 1. It is caused by the (nongravitational) interaction between the particles and all properties of the manifold M (curvature, torsion, transport along paths). The relative velocity (1.7b) is caused by the (nongravitational) interaction of the particles and the transport along paths used.

Substituting (4.4) into the left-hand side of (3.1) and performing some evident transformations, we get the *deviation equation for the relative velocity* ΔV_{21} in the form

$$\frac{D\Delta V_{21}}{ds}=R(V_1,\zeta_{21})V_1+\frac{D}{ds}[S(V_1,\zeta_{21})]+\frac{DF_s(r)}{dr}\Big|_{r=r'}(r''-r')+O((r''-r')^2), \qquad (4.5)$$

where all quantities are evaluated at $x_1(s)$.

This equation describes up to second order terms the change of the relative velocity of the second particle with respect to the first one along the world line of the latter.

5. EQUATION FOR THE RELATIVE MOMENTUM

In our case (see Sect. 1) due to [1], eq. (3.6), the relative momentum of the particles Δp_{21} defined by (1.9), is

$$\Delta p_{21}=\mu_2(s)\Delta V_{21}+[\mu_2(s)/\mu_1(s)-1]p_1, \qquad (5.1)$$

which may be obtained also as an approximate result from (2.9), (4.3) and (2.10) for $B_{\gamma_s(r)}=\mu(s,r)\gamma_s'(r)$ with μ being a C^1 function such that $\mu(s,r')=\mu_1(s)$ and $\mu(s,r'')=\mu_2(s)$. Differentiating this equality covariantly along x_1, we get

$$\frac{D\Delta p_{21}}{ds}=\mu_2(s)\frac{D\Delta V_{21}}{ds}+\frac{D\mu_2(s)}{ds}\Delta V_{21}+\frac{D}{ds}[(\mu_2(s)/\mu_1(s)-1)p_1(s)]$$

and substituting here (4.5), we obtain

$$\frac{D\Delta p_{21}}{ds}=\frac{\mu_2(s)}{(\mu_1(s))^2}R(p_1,\zeta_{21})p_1+\mu_2(s)\frac{D}{ds}\left[\frac{1}{\mu_1(s)}S(p_1,\zeta_{21})\right]+\frac{D\mu_2(s)}{ds}\Delta V_{21}+$$

$$+\frac{D}{ds}\left[\left(\frac{\mu_2(s)}{\mu_1(s)}-1\right)p_1(s)\right]+\mu_2(s)\frac{DF_s(r)}{dr}\Big|_{r=r'}(r''-r')+O((r''-r')^2). \qquad (5.2)$$

Here, if it is necessary, ΔV_{21} may be substituted with the obtained for it expressions from (5.1) or (4.3).

This is the *first order deviation equation for the relative momentum* of the second particle with respect to the first one. It describes the evolution of Δp_{21} along the trajectory of the first particle. The physical interpretation of (5.2) will be considered below in section 7.

6. EQUATION FOR THE RELATIVE ACCELERATION

From the definitions (1.8a) and (3.2) we find the following representation for the accelerations of the particles studied:

$$A_1=F_s(r'),\ A_2=F_s(r''). \qquad (6.1)$$

This is very natural as from (3.2) and the physical interpretation of the deviation equation (3.1) (see [8]) it is clear that $F_s(r)$ has a sense of a (nongravitational) force per unit mass acting on a particle situated at the point $\gamma_s(r)$; i.e., $F_s(r)$ is the acceleration of that particle.

If in (2.9) and (2.10) we let $B_{\gamma_s(r)}=F_s(r)$, we find the relative acceleration (1.8b) of the particles in the form

$$\Delta A_{21}=\Delta F_{21}=\frac{D}{dr}\Big|_{\gamma_s}F_s(r)\Big|_{r=r'}(r''-r')+S(A_1,\zeta_{21})\Big|_{x_1(s)}+O((r''-r')^2). \qquad (6.2)$$

The first order deviation acceleration between the considered particles is $D^2\zeta_{21}/ds^2$ and according to (2.13) is connected with

the deviation acceleration D^2h_{21}/ds^2 by

$$D^2h_{21}/ds^2=D^2\zeta_{21}/ds^2+O((r''-r')^2). \tag{6.3}$$

Expressing $DF_s(r)/ds|_{r=r'}(r''-r')$ from (6.2) and substituting the so-obtained result into (3.1), we get the following relation between the first order deviation acceleration $D^2\zeta_{21}/ds^2$ and the relative acceleration ΔA_{21}:

$$\frac{D^2\zeta_{21}}{ds^2}=\Delta A_{21}+R(V_1,\zeta_{21})V_1+T(A_1,\zeta_{21})-S(A_1,\zeta_{21})+T(V_1,D\zeta_{21}/ds)+$$

$$+\frac{DT}{ds}(V_1,\zeta_{21})+O((r''-r')^2). \tag{6.4}$$

Here, for some purposes, it is convenient to replace the first order deviation velocity $D\zeta_{21}/ds$ with the right-hand side of (4.4).

The last equation shows that up to second order terms the deviation acceleration is caused by the (nongravitational) interaction between the particles and the properties of the space M. The cause for the relative acceleration is only the (nongravitational) interaction between the particles and the L-transport along paths used.

If we express $DF_s(r)/ds|_{r=r'}(r''-r')$ from (4.5) and substitute the result into (6.2), we shall find the following relation between the relative velocity ΔV_{21} and the relative acceleration ΔA_{21}:

$$\Delta A_{21}=\frac{D\Delta V_{21}}{ds}-R(V_1,\zeta_{21})V_1+S(V_1,\frac{D\zeta_{21}}{ds})+\frac{DS}{ds}(V_1,\zeta_{21})+O((r''-r')^2). \tag{6.5}$$

In accordance with the physical interpretation of the involved in this equation quantities it can be called a *first order deviation equation for the relative acceleration.*

7. EQUATION FOR THE RELATIVE ENERGY

Before the derivation of the equation mentioned in the title, we will write the deviation equation for the relative momentum (5.2) in a form which is a direct analog of the second Newton's law, i.e., as an equation of motion for the considered case.

Let $\mu: J\times J\longrightarrow R$ be a C^1 function, $\mu(s,r')=\mu_1(s)$, $\mu(s,r'')=\mu_2(s)$ and $K(s,r):=\mu(s,r)F_s(r)$ (see (3.2)). The quantity $K(s,r)$ has a meaning of a (nongravitational) force acting on a particle with momentum $\mu(s,r)\gamma_s'(r)$ situated at $\gamma_s(r)$. Putting $B_{\gamma_s(r)}=K(s,r)$ into (2.10) and taking into account (6.1) and

$$\mu_2(s)-\mu_1(s)=\mu(s,r'')-\mu(s,r')=\frac{\partial\mu(s,r)}{\partial r}\Big|_{r=r'}(r''-r')+O((r''-r')^2),$$

we get

$$\Delta K_{21}=(\mu_2(s)-\mu_1(s))A_1+\mu_1(s)\frac{DF_s(r)}{dr}\Big|_{r=r'}(r''-r')+\mu_1(s)S(A_1,\zeta_{21})+$$

$$+O((r''-r')^2)=(\mu_2(s)-\mu_1(s))A_1+\mu_2(s)\frac{DF_s(r)}{dr}\Big|_{r=r'}(r''-r')+$$

$$+\mu_2(s)S(A_1,\zeta_{21})+O((r''-r')^2). \tag{7.1}$$

Physically ΔK_{21} is the (covariant) difference between the forces acting on the particles studed.

Expressing the term $\mu_2(s)DF_s(r)/dr|_{r=r'}(r''-r')$ from (7.1) and substituting the result into (5.2), we obtain

$$\frac{Dp_{21}}{ds}=\frac{\mu_2(s)}{(\mu_1(s))^2}R(p_1,\zeta_{21})p_1+\frac{\mu_2(s)}{\mu_1(s)}\Big[S(p_1,D\zeta_{21}/ds)+\frac{DS}{ds}(p_1,\zeta_{21})\Big]+$$

$$+\frac{d\mu_2(s)}{ds}\Delta V_{21}+\frac{1}{\mu_1(s)}\frac{d(\mu_2(s)-\mu_1(s))}{ds}p_1+\Delta K_{21}+O((r''-r')^2). \tag{7.2}$$

This is the first order deviation equation for the relative momentum in the form of an equation of motion.

Let in the tangent bundle $(T(M),\pi,M)$ be given also a real bundle metric g (besides the transport along paths I and the covariant differentiation ∇), i.e., [6] a map $g: x \longmapsto g_x$, $x \in M$, where the maps $g_x : T_x(M) \otimes T_x(M) \longrightarrow R$ are bilinear, nondegenerate and symmetric. For brevity, the scalar products of $X, Y \in T_y(M)$, $y \in M$ defined by g will be denoted by a dot ($\cdot$), i.e. $X \cdot Y := g_y(X,Y)$. The scalar square of X will be written as $(X)^2$ for it has to be distinguished from the second component X^2 of X in some local basis (in a case when $\dim(M)>1$). As g is not supposed to be positively defined, $(X)^2$ can take any real values.

Then, according to [1], eq. (4.1), in the considered case the relative energy of the second particle with respect to the first one is

$$E_{21} = \varepsilon((V_1(s))^2)(I_{r'' \longrightarrow r'}^{\gamma_s} p_2(s)) \cdot V_1(s) =$$

$$= \varepsilon((V_1(s))^2)(\Delta p_{21} \cdot V_1 + p_1 \cdot V_1), \qquad (7.3)$$

where we have used (1.9).

Differentiating (7.3) with respect to s along x_1 and substituting the obtained result into (7.2), we find:

$$\frac{dE_{21}}{ds} = \varepsilon((V_1)^2)\left\{\frac{\mu_2(s)}{(\mu_1(s))^3}[R(p_1,\zeta_{21})p_1]\cdot p_1 + \frac{\mu_2(s)}{(\mu_1(s))^2}p_1 \cdot \left[S(p_1, \frac{D\zeta_{21}}{ds}) + \right.\right.$$

$$\left. + \frac{DS}{ds}(p_1,\zeta_{21})\right] + \frac{d\mu_2(s)}{ds}V_1 \cdot \Delta V_{21} + \frac{1}{(\mu_1(s))^2}\frac{d(\mu_2(s)-\mu_1(s))}{ds}p_1 \cdot p_1 + V_1 \cdot \Delta K_{21} +$$

$$\left. + \frac{Dg}{ds}(\Delta p_{21}, V_1) + \Delta p_{21} \cdot A_1 + \frac{d\mu_1(s)}{ds}V_1 \cdot V_1 + \mu_1(s)(2V_1 \cdot A_1 + \frac{Dg}{ds}(V_1,V_1))\right\} +$$

$$+ O((r''-r')^2), \qquad (7.4)$$

where all quantities are taken at the point $x_1(s)$.

This is the *first order deviation equation for the relative energy*. It has a meaning of an equation for the energy balance and can be considered as a generalization of the energy conservation equation in the situation studied.

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