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RELATIVE MECHANICAL QUANTITIES IN SPACES WITH A TRANSPORT ALONG PATHS

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1. INTRODUCTION

The introduced in [1] transports along paths, which in particular can be linear [2], are applied in the present paper to defining certain mechanical quantities in spaces (manifolds), the tangent bundle of which is endowed with such a transport. Analogous problem has been considered in [3] but, in fact, in this work only linear transports along paths without self-intersections are used which is not generally necessary everywhere. We closely follow [3] without presupposing such restrictions.

All considerations in the present work are made in a (real) differentiable manifold M [4,5] whose tangent bundle $(T(M),\pi,M)$ is endowed with a transport along paths [1]. Here $T(M):=\bigcup_{x\in M}T_x(M)$, $T_x(M)$ being the tangent to the M space at $x\in M$ and $\pi:T(M)\longrightarrow M$ is such that if $V\in T_x(M)$, then $\pi(V):=x$.

By J and $\gamma:J\longrightarrow M$ are denoted, respectively, an arbitrary real interval and a path in M. If γ is of class C^1, its tangent vector is written as $\dot\gamma$.

The transport along paths in $(T(M),\pi,M)$ (cf. [1]) is a map $I:\gamma\longmapsto I^\gamma$, $I^\gamma:(s,t)\longmapsto I^\gamma_{s\to t}$, $s,t\in J$ being the transport along γ, where $I^\gamma_{s\to t}:T_{\gamma(s)}(M)\longrightarrow T_{\gamma(t)}(M)$, satisfy the equalities

$$I^\gamma_{t\to r}\circ I^\gamma_{s\to t}=I^\gamma_{s\to r},\quad r,s,t\in J, \tag{1.1}$$

$$I^\gamma_{s\to s}=id_{T_{\gamma(s)}(M)},\quad s\in J. \tag{1.2}$$

Here id_X is the identity map of the set X.

A linear transport (L-transport) along paths L in $(T(M),\pi,M)$ satisfies besides (1.1) and (1.2) also the equality (cf. [2])

$$L^{\gamma}_{s\to t}(\lambda U+\mu V)=\lambda L^{\gamma}_{s\to t}U+\mu L^{\gamma}_{s\to t}V,\quad s,t\in J,\quad U,V\in T_{\gamma(s)}(M). \qquad (1.3)$$

In Sect. 2, the concepts of relative velocity, deviation velocity and the corresponding to them accelerations between two point particles are introduced. Sect. 3 is devoted to the relative momentum of these particles. The central role in this investigation belongs by Sect. 4. In spaces, the tangent bundle of which is endowed with a metric and a transport along paths, the relative energy of two point particles is introduced and investigated. Certain connections between the mentioned concepts are studied and the notion of a proper (rest) energy is naturally obtained. A note on the zero-mass particles case is made. Sect. 5 illustrates the considered general concepts in the case of special theory of relativity. The paper ends with some concluding remarks in Sect.6.

2. RELATIVE VELOCITY AND RELATIVE ACCELERATION

Let there be given paths $x_a:J_a\longrightarrow M$, $a=1,2$ and $x:J\longrightarrow M$. Let there be fixed one-to-one maps $\tau_a:J\longrightarrow J_a$, $a=1,2$. (The maps τ_1 and τ_2 always exist because all real intervals are equipollent.) Let also be given the one parameter families of paths $\{\gamma_s:\ \gamma_s:J'_s\longrightarrow M,\ s\in J\}$ and $\{\eta_s:\ \eta_s:J''_s\longrightarrow M,\ s\in J\}$ having the properties $\gamma_s(r'_s):=:=x_1(\tau_1(s))=:\eta_s(t'_s)$, $\gamma_s(r''_s):=x_2(\tau_2(s))$ and $\eta_s(t''_s):=x(s)$ for some $r'_s,r''_s\in J'_s$ and $t'_s,t''_s\in J''_s$, $s\in J$.

Physically the paths x_1, x_2 and x are interpreted as trajectories (world lines) of, respectively, observed point particles 1 and 2 and a point observer observing them. The parameters $s\in J$, $s_1=\tau_1(s)$ and $s_2=\tau_2(s)$ are interpreted as proper times of the corresponding particles (cf. [8], sect. 2).

If the particles 1 and 2 are moving along the paths x_1 and x_2,

respectively, then their velocities are [6,7]

$$V_a:=\dot{x}_a,\quad a=1,2. \qquad (2.1)$$

The vectors V_1 and V_2 cannot be compared as they are defined at different points. To compare them, we put

$$(V_2)_1:=I^{\gamma_s}_{r''_s\to r'_s}V_2\in T_{x_1(\tau_1(s))}(M). \qquad (2.2)$$

As $(V_2)_1$ and V_1 are defined at one and the same point, the vector

$$\Delta V_{21}:=\Delta V_{21}(s;x):=I^{\eta_s}_{t'_s\to t''_s}((V_2)_1-V_1)=$$

$$=I^{\eta_s}_{t'_s\to t''_s}(I^{\gamma_s}_{r''_s\to r'_s}V_2-V_1)\in T_{x(s)}(M) \qquad (2.3)$$

is uniquely defined and represents their difference defined at x(s) with the help of I. This vector is called a *relative velocity* of the second observed particle with respect to the first one (as it is "seen" from the observer) at the point x(s).

This definition of a relative velocity is a natural generalization of the Newtonian concept for a relative velocity which can be simply defined as a difference of the 3-vectors representing the velocities of the corresponding particles.

Let the paths γ_s, $s\in J$ be of class C^1 and such that the maps

$$d^{\gamma}_s:J\longrightarrow T_{\gamma(s)}(M)=\pi^{-1}(\gamma(s)),\quad s\in J, \qquad (2.4a)$$

defined by

$$d^{\gamma}_s(t):=\int_s^t(I^{\gamma}_{u\to s}\dot{\gamma}(u))du,\quad s,t\in J, \qquad (2.4b)$$

be homeomorphisms from J into $d^{\gamma}_s(J)$ for every $s\in J$ (cf. [8], sect. 2).

According to [8], definition 2.3,the deviation vector of x_2

with respect to x_1 relatively to x at the point $x(s)$, $s\in J$, i.e. between the investigated particles, is

$$h_{21}:=h_{21}(s;x):=\left(I^{\eta_s}_{t'_s\longrightarrow t''_s}\circ d^{\gamma_s}_{r'_s}\right)(r''_s)=$$

$$=I^{\eta_s}_{t'_s\longrightarrow t''_s}\int_{r'_s}^{r''_s}(I^{\gamma_s}_{u\longrightarrow r'_s}\dot{\gamma}_s(u))du\ \in T_{x(s)}(M). \tag{2.5}$$

Let in the manifold M be given also a covariant differentiation ∇ and the deviation vector h_{21} (of x_2 with respect to x_1) have a C^1 dependence on s. Then there arises the concept for a *deviation velocity* V_{21} between the observed particles:

$$V_{21}:=\frac{D}{ds}\Big|_x h_{21}, \tag{2.6}$$

where $D/ds|_x:=\nabla_{\dot{x}}$ is the covariant differentiation along x and the deviation vector is given by (2.5). This velocity has a direct physical meaning because it can be measured. For example, if the observer defines somehow (e.g. by radiolocation) the relative position h_{21} of the observed particles, then he can find the deviation velocity from (2.6) in which s is now interpreted as observer's "proper time".

Generally speaking, the vectors ΔV_{21} and V_{21} do not coincide even in the Euclidean case (see [8], sect.4) in which we evidently have

$$\Delta V_{21}\Big|_{E^n}=V_2-V_1, \tag{2.7}$$

$$V_{21}\Big|_{E^n}=\frac{d}{ds}(x_2(\tau_2(s))-x_1(\tau_1(s)))=\frac{d\tau_2(s)}{ds}\cdot V_2-\frac{d\tau_1(s)}{ds}\cdot V_1. \tag{2.8}$$

Nevertheless, in the Newtonian mechanics, where we have an Euclidean world with an absolute simultaneity ($\tau_1=\tau_2=id_J$), these velocities coincide.

Let the manifold M be endowed with a transport of vectors along paths and a covariant differentiation. If x_1 and x_2 are C^2 paths, then the accelerations of the observed particles are

$$A_a:=\frac{D}{ds}\Big|_{x_a}V_a,\quad a=1,2 \tag{2.9}$$

and we can define in an analogous way the *relative acceleration* and the *deviation acceleration* between them and the observer, respectively, by the equalities

$$\Delta A_{21}:=I^{\eta_s}_{t'_s\longrightarrow t''_s}((A_2)_1-A_1),\quad (A_2)_1:=I^{\gamma_s}_{r''_s\longrightarrow r'_s}A_2, \tag{2.10}$$

$$A_{21}:=\frac{D}{ds}\Big|_x V_{21}=\left(\frac{D}{ds}\Big|_x\right)^2 h_{21}. \tag{2.11}$$

The treatment of ΔA_{21} and A_{21} is similar to the one of ΔV_{21} and V_{21}.

3. RELATIVE MOMENTUM

Let a point particle with a (rest) mass m be moving along the path $\gamma:J\longrightarrow M$. Then by definition (see [6], ch. III, §3) its momentum at the point $\gamma(s)$ is

$$p:=p(s):=\mu(s)\dot{\gamma}(s),\quad s\in J,$$

where $\mu:J\longrightarrow\mathbb{R}\backslash\{0\}$ is a scalar function with a dimension of mass. If $m\neq0$, then $\mu(s):=m$. If $m=0$, which is the case, e.g., with the photons, then the momentum p is considered as a primary defined quantity and μ is obtained from the above equation. It is important to be noted that in both the cases $\mu(s)\neq0$, $s\in J$. (The case $m=\mu(s)=0$ describes the vacuum but not a particle.)

So, the momenta of the observed particles are

$$p_a := p_a(s_a) := \mu_a(s_a) V_a, \quad s_a = \tau_a(s), \quad a=1;2, \quad s\in J, \tag{3.1}$$

where $\mu_a : J_a \longrightarrow \mathbb{R}\backslash\{0\}$, $a=1,2$ are scalar functions.

As the vector

$$(p_2)_1 := I^{\gamma_s}_{r''_s \longrightarrow r'_s} p_2 \in T_{x_1(\tau_1(s))}(M) \tag{3.2}$$

is in $T_{x_1(s_1)}(M)$, it can be compared with p_1. In accordance with this, (the vector of) the *relative momentum* of the second particle with respect to the first one as it is "seen" from the observer at $x(s)$ is defined by

$$\Delta p_{21} := \Delta p_{21}(s;x) := I^{\eta_s}_{t'_s \longrightarrow t''_s}((p_2)_1 - p_1) =$$

$$= I^{\eta_s}_{t'_s \longrightarrow t''_s}(I^{\gamma_s}_{r''_s \longrightarrow r'_s} p_2 - p_1) \in T_{x(s)}(M). \tag{3.3}$$

It is clear that in the Euclidean case (see [8], sect. 4) the relative momentum takes its well-known Newtonian form

$$\Delta p_{21}\big|_{E^n} = p_2 - p_1. \tag{3.4}$$

If the used above transport in $(T(M),\pi,M)$ is linear (see Sect. 1, eq. (1.3) or [2]), then due to (3.1)-(3.3) and (2.2)-(2.3) the following equalities are valid

$$(p_2)_1 = \mu_2(s_2)(V_2)_1, \tag{3.5}$$

$$\Delta p_{21} = \mu_2(s_2)\Delta V_{21} + [\mu_2(s_2)/\mu_1(s_1) - 1] I^{\eta_s}_{t'_s \longrightarrow t''_s} p_1. \tag{3.6}$$

4. RELATIVE ENERGY

Let in the tangent bundle $(T(M),\pi,M)$ be given a transport along

paths I and a real bundle metric g, i.e., [5] a map $g: x \longmapsto g_x$, $x\in M$, where the maps $g_x : T_x(M)\otimes T_x(M) \longrightarrow \mathbb{R}$ are bilinear, nondegenerate and symmetric. For brevity, the defined by g scalar products of $X,Y\in T_y(M)$, $y\in M$ will be denoted by a dot ($\cdot$), i.e. $X\cdot Y := g_y(X,Y)$. The scalar square of X will be written as $(X)^2$ for it has to be distinguished from the second component X^2 of X in some local basis (in the case when $\dim(M)>1$). As g is not supposed to be positively defined, $(X)^2$ can take any real values.

By definition the *relative energy* of the second particle with respect to the first one is called the (scalar) quantity

$$E_{21} := E_{21}(s) := \varepsilon((V_1(s_1))^2) p_{21}\cdot V_1(s_1) =$$

$$= \varepsilon((V_1(s_1))^2)(I^{\gamma_s}_{r''_s \longrightarrow r'_s} p_2(s_2))\cdot V_1(s_1), \tag{4.1}$$

where $\varepsilon(\lambda) := -1$ for $\lambda<0$ and $\varepsilon(\lambda) := +1$ for $\lambda\geq 0$. The introduction of the multiplier ε is due to the fact that if the particles coincide, i.e., if we apply (4.1) to one and the same particle, then the so obtained quantity has a meaning of a proper energy of that particle (see below) and according to the accepted opinion [6,7] it must be positive.

If there exists $s_0\in J$ such that $x_1(\tau_1(s_0)) = x_2(\tau_2(s_0))$, i.e. if at the "moment" $s=s_0$ the trajectories of the observed particles intersect each other, then from (4.1) and (1.2) we get

$$E_{21}(s_0) = \varepsilon((V_1(\tau_1(s_0)))^2) p_2(\tau_2(s_0))\cdot V_1(\tau_1(s_0)). \tag{4.2}$$

In the case of the space-time of general relativity, this expression coincides with the given in [6], ch. III, §6, eq. (23) definition for a relative energy which has the "bad" property that it is valid only for the "moment" $s=s_0$. So it does not allow the evolution of the relative energy in time to be studied. Evidently, our definition (4.1) is free from this deficiency.

Analogously to (4.1), the relative energy of the first particle with respect to the second one is

$$E_{12}:=E_{12}(s):=\varepsilon((V_2(s_2))^2)p_{12}\cdot V_2(s_2)=$$

$$=\varepsilon((V_2(s_2))^2)(I^{\gamma_s}_{r'_s\longrightarrow r''_s}p_1(s_1))\cdot V_2(s_2). \qquad (4.3)$$

If we use arbitrary transports along paths, then, generally, the quantities E_{12}, E_{21}, Δp_{21} and Δp_{12} are not connected somehow with each other. From the view point of the existence of a certain connection between them an essential role is played by the transports along paths which are consistent (at least along the paths γ_s and η_s, $s\in J$) with the fibred metric g, i.e. for which (cf. [9,10])

$$I^{\gamma}_{s\longrightarrow t}(U\cdot V)=(I^{\gamma}_{s\longrightarrow t}U)\cdot(I^{\gamma}_{s\longrightarrow t}V) \qquad (4.4)$$

for arbitrary $\gamma:J\longrightarrow M$, $s,t\in J$ and $U,V\in T_{\gamma(s)}(M)$. (For further considerations it is enough that this equality be valid only for $\gamma\in\{\gamma_s, \eta_s: s\in J\}$.)

If I and g are consistent, then the relative momentum (3.3) and the relative energy (4.1), as one can easily prove, are connected by the relation

$$E_{21}(s)=\varepsilon((V_1(s_1))^2)[\Delta p_{21}\cdot I^{\eta_s}_{t'_s\longrightarrow t''_s}V_1(s_1)+p_1\cdot V_1(s_1)]. \qquad (4.5)$$

If the transport along paths I is consistent with the operation multiplication with real numbers (see [10], example 3.2), i.e.

$$I^{\gamma}_{s\longrightarrow t}(\lambda U)=\lambda I^{\gamma}_{s\longrightarrow t}U,\ \lambda\in\mathbb{R},\ U\in T_{\gamma(s)}(M), \qquad (4.6)$$

then (3.5) holds and after its substitution into (4.1), one gets

$$E_{21}=\varepsilon((V_1(s_1))^2)\mu_2(s_2)(V_2)_1\cdot V_1(s_1). \qquad (4.7)$$

If the equalities (4.4) and (4.6) are simultaneously valid,

then with a direct verification we confine ourselves to that the relative energies E_{21} and E_{12} are connected by

$$\varepsilon((V_2(\tau_2(s)))^2)\mu_1(\tau_1(s))E_{21}(s)=\varepsilon((V_1(\tau_1(s)))^2)\mu_2(\tau_2(s))E_{12}(s). \qquad (4.8)$$

In particular, this equality is true for every L-transport consistent with the metric.

Let us apply definition (4.1) only to the first observed particle, for which it is enough to put in it $r'_s=r''_s$, $x_2=x_1$ and $\tau_2=\tau_1$, or equivalently to replace the subscript 2 with 1. Using (1.2), we see that the energy of this particle (with respect to itself) is

$$E_{11}(s)=\varepsilon((V_1(s_1))^2)p_1(s_1)\cdot V_1(s_1)=\mu_1(s_1)|(V_1(s_1))^2|=$$

$$=|(p_1(s_1))^2|/\mu_1(s_1), \qquad (4.9)$$

where $|\lambda|:=\varepsilon(\lambda)\lambda$ is the absolute value of $\lambda\in\mathbb{R}$.

The quantity E_{11} may be called a *proper* (or rest) *energy* of the considered particle. If $m_1>0$, then $\mu_1(s_1)=m_1$ and consequently $E_{11}\geq 0$. If $m_1(V_1(s_1))^2\neq 0$, then $E_{11}>0$ which corresponds to the most popular case of massive material particle.

If $m_1\neq 0$, then $\mu_1(s_1):=m_1$ and due to (4.9) the proper energy E_{11} is proportional to m_1, so E_{11} is a C^∞ function of m_1 for $m_1\in\mathbb{R}\backslash\{0\}$. From here comes the mind on E_{11} to be imposed the additional restriction for continuous dependence of m_1 at the point $m_1=0$, i.e. one may want

$$E_{11}=0 \text{ for } m_1=0, \qquad (4.10)$$

or, equivalently,

$$\lim_{m_1\to 0}E_{11}=0, \qquad (4.11)$$

which has far going physical corollaries. In fact, (4.9) shows the equivalence of (4.11) with

$$(V_1(s_1))^2=0 \text{ for } m_1=0, \tag{4.12}$$

or, which is all the same, with

$$(p_1(s_1))^2=0 \text{ for } m_1=0. \tag{4.13}$$

These relations are a direct generalization of the well-known fact from the special and general relativity that the massless particles are moving with the velocity of light, i.e. that their world lines lie on the light cone described by (4.12).

We want to note that without further assumptions $(V_1(s_1))^2=0$ does not imply $m_1=0$.

The energies E_{21} (or E_{12}) and E_{11} may be connected with the components of Δp_{21} (or Δp_{12}), p_{21} (or p_{12}) and p_1 in some local bases in the following way.

Let $(V_1)^2\neq 0$. Along x_1 we define a field of basis $\{\lambda_i\}$, i.e. the vectors $\lambda_i|_{\gamma(s)}\in T_{\gamma(s)}(M)$ form a basis in $T_{\gamma(s)}(M)$, such that $\lambda_1:=V_1\cdot|(V_1)^2|^{-1/2}$ and $\lambda_1\cdot\lambda_i=0$ for $i\neq 1$ (if $\dim(M)>1$). Here and henceforth the Latin indices run from 1 to $\dim(M)$. (In this case the concrete choice of λ_i for $i\neq 1$ is insignificant.) So $(\lambda_1)^2=\varepsilon((V_1)^2)$, due to which the component A^1 of any vector field $A=A^i\lambda_i$ along x_1 in $\{\lambda_i\}$ is

$$A^1=A\cdot\lambda_1/(\lambda_1)^2=\varepsilon((V_1)^2)(A\cdot V_1)|(V_1)^2|^{-1/2}. \tag{4.14}$$

Applying this equality to p_1, p_{21} and $\Delta\pi_{21}:=\Delta p_{21}|_{x=x_1}=(p_2)_1-p_1$, the last vector being the relative momentum of the second particle with respect to the first one as it is "seen" from the latter, and using (4.9) and (4.1), we find:

$$p_1^1=E_{11}|(V_1)^2|^{-1/2},\quad p_1^i=0 \text{ for } i\neq 1, \tag{4.15}$$

$$(p_2)_1^1=E_{21}|(V_1)^2|^{-1/2}, \tag{4.16}$$

$$\Delta\pi_{21}^1=(p_2)_1^1-p_1^1=(E_{21}-E_{11})|(V_1)^2|^{-1/2}. \tag{4.17}$$

Let $(V_1)^2\neq 0$ and g and I be consistent, i.e. (4.4) be valid. Defining along x a basis $\{l_i\}$ such that $l_1:=I^{\eta_s}_{t'_s\to t''_s}\lambda_1$ and $l_1\cdot l_i=0$ for $i\neq 1$, we see that the first component of Δp_{21} in $\{l_i\}$ is

$$\Delta p_{21}^1=\Delta p_{21}\cdot l_1/(l_1)^2=\Delta\pi_{21}\cdot\lambda_1/(\lambda_1)^2=\Delta\pi_{21}^1=(E_{21}-E_{11})|(V_1)^2|^{-1/2}. \tag{4.18}$$

If $(V_1)^2=0$, then (see (4.9)) $E_{11}=0$ and the invariant $(p_2)_1\cdot V_1=\Delta\pi_{21}\cdot V_1=E_{21}$ cannot be connected with some component of $(p_2)_1$ (or $\Delta\pi_{21}$) in a given local basis. In this case, we can say that the relative energy E_{21} is spread over all the components of $(p_2)_1$ (or $\Delta\pi_{21}$) and with basis transformations it cannot be connected with a single component of that vector.

If $(V_2)^2\neq 0$ and I and g are consistent, i.e. (4.4) holds, then defining along x_1 a basis $\{\lambda_{i'}\}$ such that $\lambda_{1'}:=I^{\gamma_s}_{r''_s\to r'_s}(V_2\cdot|(V_2)^2|^{-1/2})$ and $\lambda_{1'}\cdot\lambda_{i'}=0$ for $i\neq 1$, we find the first component of p_1 in $\{\lambda_{i'}\}$ as

$$p_1^{1'}=p_1\cdot\lambda_{1'}/(\lambda_1)^2=p_1\cdot(I^{\gamma_s}_{r''_s\to r'_s}(V_2\cdot|(V_2)^2|^{-1/2}))\cdot\varepsilon((V_2)^2)=$$
$$=\varepsilon((V_2)^2)(I^{\gamma_s}_{r'_s\to r''_s}p_1)\cdot V_2\cdot|(V_2)^2|^{-1/2}=E_{21}|(V_2)^2|^{-1/2}. \tag{4.19}$$

At the end of this section we want to stress the fact that all relative energies E_{11}, E_{21}, E_{12} and E_{22}, connected with the considered observed particles, are not arbitrary, as they are connected with the invariant $(\Delta\pi_{21})^2$ by

$$(\Delta\pi_{21})^2=\varepsilon((V_1)^2)\mu_1E_{11}+\varepsilon((V_2)^2)\mu_2E_{22}-2\varepsilon((V_1)^2)\mu_1E_{21}. \tag{4.20}$$

This follows from $(\Delta\pi_{21})^2=((p_2)_1)^2-2(p_2)_1\cdot p_1+(p_1)^2$ and the definitions of the corresponding energies. If the transport and the metric are consistent, then this equality can be written in a more symmetric form as

$$(\Delta p_{21})^2=(\Delta\pi_{21})^2=\varepsilon((V_1)^2)\mu_1E_{11}+\varepsilon((V_2)^2)\mu_2E_{22}-$$

$$-\varepsilon((V_1)^2)\mu_1E_{21}-\varepsilon((V_2)^2)\mu_2E_{12}, \qquad (4.21)$$

where we have used (4.8).

5. EXAMPLE: SPECIAL RELATIVITY

The purpose of this section is to find explicit forms of the introduced relative quantities in the concrete case of special relativity. (As a standard reference to the problems of this theory see, e.g., [6,7].)

Let there be given a standard (4-dimensional, flat, with signature (+---)) Minkowski's space-time M^4, in which as a concrete realization of the general transport along paths the parallel transport along them will be used. Let two point particles 1 and 2 with masses $m_1\neq0$ and $m_2\neq0$ be moving in M^4 with constant 3-velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively, with respect to a given frame of reference. Then, their world lines are $x_a(s_a)=(ct,t\mathbf{v}_a)+y_a$, a=1,2, where c is the velocity of light in vacuum, t is the time in the used frame, $s_a:=\tau_a(t):=t(1-\mathbf{v}_a^2/c^2)^{1/2}$, a=1,2 are the corresponding proper times and $y_1,y_2\in M^4$ are fixed.

According to (4.1), the 4-velocities [6] of the particles are

$$V_a=(c,\mathbf{v}_a)(1-\mathbf{v}_a^2/c^2)^{-1/2}, \quad a=1,2 \qquad (5.1)$$

and hence

$$(V_a)^2=(c^2-\mathbf{v}_a^2)((1-\mathbf{v}_a^2/c^2)^{-1/2})^2=c^2, \quad \varepsilon((V_a)^2)=+1, \quad a=1,2. \qquad (5.2)$$

Due to this by using (3.1)-(3.3), (4.1) and (4.9), we get:

$$p_a=m_a(c,\mathbf{v}_a)(1-\mathbf{v}_a^2/c^2)^{-1/2}, \quad a=1,2 \ (\mu_1=m_1, \ \mu_2=m_2), \qquad (5.3a)$$

$$(p_1)_2=p_1, \quad (p_2)_1=p_2, \quad \Delta p_{21}=\Delta\pi_{21}=p_2-p_1, \qquad (5.3b)$$

$$E_{21}=m_2c^2(1-\mathbf{v}_1\cdot\mathbf{v}_2/c^2)[(1-\mathbf{v}_1^2/c^2)(1-\mathbf{v}_2^2/c^2)]^{-1/2}, \qquad (5.4a)$$

$$E_{12}=m_1c^2(1-\mathbf{v}_1\cdot\mathbf{v}_2/c^2)[(1-\mathbf{v}_1^2/c^2)(1-\mathbf{v}_2^2/c^2)]^{-1/2}, \qquad (5.4b)$$

$$E_{11}=m_1c^2, \quad E_{22}=m_2c^2. \qquad (5.4c)$$

Evidently, E_{11} and E_{22} are the proper (rest) energies of the particles. If, e.g., $\mathbf{v}_1=0$, then $E_{21}=m_2c^2(1-\mathbf{v}_2^2/c^2)^{-1/2}=E_2$ is the energy of the second particle with respect to the used frame [6].

If $m_1\neq0$ and $m_2=0$, then in the above considerations one has to replace x_2 and s_2, respectively, by $x_2(s_2)=(ct,ct\mathbf{n}_2)+y_2$ and $s_2=t$, where $\mathbf{n}_2$ is a unit 3-vector ($\mathbf{n}_2^2=1$) showing the direction of movement of the second particle, i.e. $\mathbf{v}_2=c\mathbf{n}_2$, and, consequently

$$V_2=c(1,\mathbf{n}_2), \quad (V_2)^2=0, \quad \varepsilon((V_2)^2)=+1. \qquad (5.5)$$

If E_2 is the energy of the second particle with respect to the given frame, then its 4-momentum is [6]

$$p_2=(E_2/c,\mathbf{p}_2)=(E_2/c,(E_2/c)\mathbf{n}_2)=(E_2/c)(1,\mathbf{n}_2)=(E_2/c^2)V_2 \qquad (5.6)$$

and due to (3.1), we have

$$\mu_2=\mu_2(s_2)=E_2/c^2. \qquad (5.7)$$

In this case, (5.3b) is also true and (5.4) take the form:

$$E_{21}=E_2(1-\mathbf{v}_1\cdot\mathbf{n}_2/c)(1-\mathbf{v}_1^2/c^2)^{-1/2}, \qquad (5.8a)$$

$$E_{12}=m_1c^2(1-\mathbf{v}_1\cdot\mathbf{n}_2/c)(1-\mathbf{v}_1^2/c^2)^{-1/2},\qquad(5.8b)$$

$$E_{11}=m_1c^2,\ E_{22}=0,\qquad(5.8c)$$

the last of which is in accordance with (4.10).

Evidently, $E_{21}=E_2$ for $\mathbf{v}_1=0$, due to which (5.8a) expresses the usual Doppler effect in terms of energies of the corresponding particles [6]. In fact, if we have a moving with a 3-velocity $\mathbf{v}_1=\mathbf{v}$ source of massless particles (e.g. photons) with 3-velocities $\mathbf{v}_2=c\mathbf{n}$ and energy (with respect to the source) $E_{21}=E_0$, which are registered by immovable in this frame observer, we will find that the particles are with energy $E=E_2$, which due to (5.8a) is

$$E=E_0(1-\mathbf{v}\cdot\mathbf{n}/c)^{-1}(1-\mathbf{v}^2/c^2)^{1/2}.\qquad(5.9)$$

The corresponding formulae for $m_1=0$ and $m_2\neq0$ are obtained from the above ones by means of the change $1\to2\to1$ of the subscripts in them.

In the case when $m_1=m_2=0$, we have $x_a(s_a)=(ct,ct\mathbf{n}_a)+y_a$, $s_a=t$, $a=1,2$, so

$$\mathbf{v}_a=c\mathbf{n}_a,\ \mathbf{n}_a^2=1,\ V_a=c(1,\mathbf{n}_a),\ (V_a)^2=0,\ \varepsilon((V_a)^2)=+1,\qquad(5.10)$$

$$p_a=(E/c)(1,\mathbf{n}_a),\ \mu_a=\mu_a(t)=E_a/c^2,\ a=1,2,\qquad(5.11)$$

and the equations (5.3b) remain the same. Hence:

$$E_{21}=E_2(1-\mathbf{n}_1\cdot\mathbf{n}_2),\ E_{12}=E_1(1-\mathbf{n}_1\cdot\mathbf{n}_2),\qquad(5.12a)$$

$$E_{11}=E_{22}=0.\qquad(5.12b)$$

So, if $\mathbf{n}_1=\mathbf{n}_2$, then $E_{21}=E_{12}=0$ and vice versa.

At the end, we shall consider the concepts of relative velocity and deviation velocity in special relativity.

Let K be a fixed inertial frame of reference in which an arbitrary moving particle 2 has a 4-radius-vector $x_2(s_2)|_K=(ct,\mathbf{x}_2(t))$, $s_2=t(1-\mathbf{v}_2^2/c^2)^{1/2}$, $\mathbf{v}_2=\dot{\mathbf{x}}_2$, where t is the time in K. Let the inertial frame K′ be attached to the particle 1 having in K a world line $x_1(s_1)|_K=(ct,t\mathbf{v}_1)$, $\mathbf{v}_1=\text{const}$, $s_1=t(1-\mathbf{v}_1^2/c^2)^{1/2}$. The world line of the observer is completely arbitrary.

In the frame K, we have

$$V_a|_K=\frac{dx_a|_K}{ds_a}=(c,\mathbf{v}_a)(1-\mathbf{v}_a^2)^{-1/2},\ a=1,2,\qquad(5.13a)$$

and in K′, we get

$$V_1|_{K'}=(c,\mathbf{0}),\ V_2|_{K'}=(c,\mathbf{v}_2')(1-\mathbf{v}_2'^2)^{-1/2},\qquad(5.13b)$$

where $\mathbf{v}_2'$ is the 3-velocity of the particle 2 in K′ (i.e. with respect to the particle 1) in a sense of special relativity (see [6]).

Consequently, as we are working in a pseudo-Euclidean case, due to (2.6) the relative velocity is $\Delta V_{21}=V_2-V_1$. So, we get:

$$\Delta V_{21}|_K=(1-\mathbf{v}_2^2/c^2)^{-1/2}(c,\mathbf{v}_2)-(1-\mathbf{v}_1^2/c^2)^{-1/2}(c,\mathbf{v}_1),\qquad(5.14a)$$

$$\Delta V_{21}|_{K'}=(1-\mathbf{v}_2'^2/c^2)^{-1/2}(c,\mathbf{v}_2')-(c,\mathbf{0}).\qquad(5.14b)$$

Besides, in the pseudo-Euclidean case $h_{21}=x_2(s_2)-x_1(s_1)$, so that:

$$h_{21}|_K=(0,\mathbf{x}_2(t)-t\mathbf{v}_1),\qquad(5.15a)$$

$$h_{21}|_{K'}=(0,\mathbf{x}_2'),\qquad(5.15b)$$

where $\mathbf{x}_2'$ is obtained from $\mathbf{x}_2(t)$ by a Lorentz transformation describing the transition from K to K′ [6].

Due to (2.6) the deviation velocity is

$$V_{21}=dh_{21}/ds=(ds_1/ds)(dh_{21}/ds_1)=(dt/ds)(dh_{21}/dt),$$

where $s_1=t'$ is the time in K′ [6], from where, we get:

$$V_{21}\big|_{K}=\frac{dt}{ds}(0,\mathbf{v}_2-\mathbf{v}_1), \tag{5.16a}$$

$$V_{21}\big|_{K'}=\frac{ds_1}{ds}(0,\mathbf{v}'_2). \tag{5.16b}$$

So, if the observer coincides with the first particle, then $s_1=s$ and $V_{21}\big|_{K'}=(0,\mathbf{v}'_2)$. This shows that in fact the deviation velocity is a direct generalization of the relative velocity in a sense of special relativity.

6. COMMENTS

In this work many times was met the problem for comparing (defining the difference of) two defined at different points vectors. Below is presented a general scheme for the used in the present work method in a manifold M endowed with a transport along paths I in its tangent bundle.

Let $A_a\in T_{z_a}(M)$, $a=1,2$, $\gamma:J\longrightarrow M$ and $\gamma(s_a)=z_a$, $a=1,2$ for some $s_1,s_2\in J$ and $z_1,z_2\in M$. Let

$$(A_2)_1:=I^{\gamma}_{s_2\longrightarrow s_1}A_2,\quad (A_1)_2:=I^{\gamma}_{s_1\longrightarrow s_2}A_1. \tag{6.1}$$

Now instead of A_1 and A_2 one can compare the vectors A_1 and $(A_2)_1$, or equivalently the vectors A_2 and $(A_1)_2$. The corresponding differences (defined by I), by definition, are

$$\Delta A_{21}:=(A_2)_1-A_1,\quad \Delta A_{12}:=(A_1)_2-A_2. \tag{6.2}$$

Evidently, for a linear transport along paths L these two quantities are connected by

$$\Delta A_{12}=L^{\gamma}_{s_1\longrightarrow s_2}\Delta A_{21},\quad \Delta A_{21}=L^{\gamma}_{s_2\longrightarrow s_1}\Delta A_{12}. \tag{6.3}$$

In manifolds with a transport of vectors along paths and a co-

variant differentiation there arises a "mixed" acceleration $\frac{D}{ds}\big|_{x}\Delta V_{21}$, but there are not physical reasons that it plays some significant role.

Let X_a denote one of the vector fields of velocity, acceleration or momentum of the a-th, a=1,2 particle. In sections 2 and 3 we introduce the quantities

$$(X_2)_1:=I^{\gamma_s}_{r''_s\longrightarrow r'_s}X_2\in T_{x_1(\tau_1(s))}(M), \tag{6.4}$$

$$\Delta X_{21}:=\Delta X_{21}(s;x):=I^{\eta_s}_{t'_s\longrightarrow t''_s}((X_2)_1-X_1)=$$

$$=I^{\eta_s}_{t'_s\longrightarrow t''_s}(I^{\gamma_s}_{r''_s\longrightarrow r'_s}X_2-X_1)\in T_{x(s)}(M). \tag{6.5}$$

Analogously, if $\eta^*_s:J^*\longrightarrow M$, $s\in J$, $\eta^*_s(t^*_s):=x_2(s_2)$ and $\eta^*_s(t^{**}_s):=x(s)$ for some $t^*_s,t^{**}_s\in J^*_s$ and using the same paths γ_s, $s\in J$, we can define the quantities

$$(X_1)_2:=I^{\gamma_s}_{r'_s\longrightarrow r''_s}X_1\in T_{x_2(\tau_2(s))}(M), \tag{6.6}$$

$$\Delta X_{12}:=\Delta X_{12}(s;x):=I^{\eta^*_s}_{t^*_s\longrightarrow t^{**}_s}((X_1)_2-X_2)=$$

$$=I^{\eta^*_s}_{t^*_s\longrightarrow t^{**}_s}(I^{\gamma_s}_{r'_s\longrightarrow r''_s}X_1-X_2)\in T_{x(s)}(M), \tag{6.7}$$

the latter of which, in a case of linear transport along paths L, due to (1.1) and (1.3) is connected with ΔX_{21} by

$$\Delta X_{12}=-L^{\eta^*_s}_{t^*_s\longrightarrow t^{**}_s}\circ L^{\gamma_s}_{r''_s\longrightarrow r'_s}\circ I^{\eta_s}_{t''_s\longrightarrow t'_s}(\Delta X_{21}). \tag{6.8}$$

If L^{γ} does not depend on γ or if η^*_s is a product of γ_s and η_s and the equalities (2.6) and (2.7) of [1] are true, then according to [1], proposition 3.4 the last equality reduces to

$$\Delta X_{12}=-\Delta X_{21}. \tag{6.9}$$

For some purposes, in (4.1) one can put $\varepsilon(0)=-1$ instead of

$\varepsilon(0)=+1$. Our general results do not depend on that choice.

Opposite to (4.11), if we admit that $\lim_{m_2\to 0} E_{21}=0$, i.e. a continuous dependence of E_{21} on m_2, then we arrive at an explicit contradiction with the physical reality. Namely, if this is so, then due to $\mu_2(s_2)\neq 0$ from (4.1), we get $V_2(s_2)=0$ for $m_2=0$, which contradicts the fact that we are dealing with a material particle but not with the vacuum. Besides, the equality $E_{21}=0$ for $m_2=0$ means that any massless particle, e.g. a photon, has zero (relative) energy with respect to any other particle, something which, evidently, is not true.

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