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ON HIGHER DERIVATIVE GRAVITY IN TWO DIMENSIONS

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1 Introduction

Recently, much attention has been paid to the investigation of models of two-dimensional (2D) gravity. It is well known that the Einstein-Hilbert action in two dimensions coincides with the topological Euler number and, therefore, does not determine any dynamics for gravitational (metrical) degrees of freedom. Hence, one should consider some alternative dynamical descriptions of 2D gravity. One of the simplest models, mainly inspired by the string theory, is dilaton gravity [1], gravitational variables are the dilaton and metric fields $(\phi, g_{\mu\nu})$. In empty (without matter) space the classical equations of motion are exactly integrated [1] and the solution describes the 2D black hole. On the quantum level, it has been shown that this model is renormalizable [2]. The coupling with conformal matter is again exactly solvable classically and the solutions are configurations describing the formation of a black hole by collapsing matter [3].

An other way is to formulate the theory of 2D gravity in the framework of a consistent gauge approach. Independent variables are now vielbeins and the Lorentz connection $(e^a, \omega^a{}_b)$. The theory with Lagrangian quadratic in curvature R and torsion T [4] was shown to be exactly solvable [5]. One class of the solutions contains the de Sitter space-time with zero torsion. Other solutions are of the black hole type [5]. Generally, one can consider the Lagrangian to be an arbitrary (not-necessarily quadratic) function of curvature and torsion [6]. Such a theory has essentially the same type of classical solutions.

Describing the gravitational degrees of freedom on the 2D manifold M^2 only by the metric $(g_{\mu\nu})$ without introducing any additional variables, one considers the following action:

$$S = \int\limits_{M^2} d^2z \sqrt{-g} f(R). \tag{1.1}$$

where $f(R)$ is, in principle, arbitrary (non-linear) function of the scalar curvature R determined with respect to the 2D metric $g_{\mu\nu}$. Theories of such type were studied in higher dimensions [7] and in two dimensions [8,9]. Was observed that the theory (1.1) is equivalent to some type of scalar-tensor $(\phi, g_{\mu\nu})$ theory of gravity. Moreover, it was shown in [10] that (1.1) with Lagrangian $f = R \ln R$ describes the same black hole space-time as the string inspired 2D dilaton gravity.

One of the motivations for recent investigations of 2D gravity (mainly of the dilaton type) that it can be considered as a "toy" model to study the process of formation and subsequent evaporation of a black hole. It has been argued by Hawking [11] that such a process is not governed by the usual laws of quantum mechanics: rather, pure states evolve into mixed states. However, it is commonly believed that a successful quantization of gravity and matter will provide us with a consistent solution of this problem. Quantum corrections may completely change the gravitational equations and the corresponding space-time geometry at the Planck scales. This problem is hard to analyze in four space-time dimensions. However, in two dimensions one can attempt to attack this problem using the dilaton gravity theories as a toy model [3]. These toy models have an explicit semiclassical treatment of the back reaction of the Hawking radiation on the geometry of an evaporating black hole by including the one-loop Polyakov-Liouville term in the action (the review can be found in [12]). Unfortunately, the resulting equations are not exactly integrated and one can not obtain a definite answer. Therefore, one can try to find another theory of 2D gravity (among the alternatives) for which the relevant semiclassical equations would be analytically solvable.

The main goal of our paper is the study of this problem for 2D gravity described by an action of the form (1.1) along the lines of ref.[3]. We show that for $f = R\ln R$ the semiclassical field equations are exactly integrated and one can obtain a definite answer about the structure of space-time when the backreaction of the Hawking radiation on the black hole geometry is taken into account.

This paper is organized as follows. In the next two sections we investigate some aspects common for theories described by the action (1.1). In Sec.2 we demonstrate the integrability of classical field equations and find the exact solution. The one-loop renormalizability of the theory is analyzed in Sec.3. In the next two sections we mainly consider the case $f = R\ln R$. The coupling with conformal (scalar) matter is shown to be exactly solvable classically in Sec.4. The backreaction is taken into account in Sec.5.

2 Classical solution of the model

Under variation of the action (1.1) with respect to the metric $g_{\mu\nu}$ we obtain the following equations of motion:

$$\nabla_\mu\nabla_\nu[f'] = \frac{1}{2}g_{\mu\nu}\{f(R) - Rf'(R) + 2\Box(f')\} \tag{2.1}$$

where $f' \equiv \partial_R f(R)$ and $\Box = \nabla^\mu\nabla_\mu$.

At first sight, (2.1) is a system of differential equations of very high order with respect to derivatives. For example, if $f = R^2$, then (2.1) are equations of fourth order of metric $g_{\mu\nu}$ derivatives. However, we will see that it is not really so and the system (2.1) is rather easily solved.

Let us analyze at first possible solutions of (2.1) with the constant curvature $R = R_1 = const$. In this case we obtain that $f'(R) = const = f'|_{R=R_1}$ everywhere in M^2. Then, from (2.1) we get that such a solution exists if the function

$$V(R) = f(R) - Rf'(R) \tag{2.2}$$

is zero at the point $R = R_1$: $V(R_1) = 0$. If $V(R)$ becomes zero at P different points $R_i, i = 1,2,...,P$, then for given $f(R)$ there are P different solutions of (2.1) with constant curvature. An additional condition is that the function $f'(R)$ must be finite at $R = R_i$.

Assuming R to be a non-constant function on M^2, we consider a new variable $\phi = f'(R)$ provided that this equation is solved (at least locally) with respect to R:

$R = R(\phi)$. Denote $V(\phi) \equiv V(R(\phi))$. Then (2.1) is rewritten as an equation on the new field ϕ:

$$\nabla_\mu \nabla_\nu \phi = \frac{1}{2} g_{\mu\nu} \{V(\phi) + 2\Box\phi\} \tag{2.3}$$

We obtain from (2.3) that $\xi_\mu = \epsilon_\mu{}^\nu \partial_\nu \phi$ is the Killing vector [13]. Consequently, the field ϕ can be chosen as one of the coordinates on M^2. Then, metric reads

$$ds^2 = g(\phi)dt^2 - \frac{1}{g(\phi)} d\phi^2 \tag{2.4}$$

From (2.3) we get that

$$\Box\phi = -V(\phi) \tag{2.5}$$

For the metric (2.4) we have $\Box\phi = -g'(\phi)$ and eq.(2.5) reads

$$\partial_\phi g(\phi) = V(\phi) \tag{2.6}$$

The solution takes the form

$$g(\phi) = \lambda + \int^{\phi} V(\psi) d\psi \tag{2.7}$$

One can see that our model (1.1), (2.1) seems to be equivalent to some kind of 2D dilaton gravity with the dilaton field ϕ and potential $V(\phi)$.

Thus, surprisingly, our initial higher-derivative equations reduced to the first order equation (2.6) independently of the concrete form of the function $f(R)$. As a result, the solution (2.7) is determined only by one arbitrary integrating constant λ. The Killing vector $\xi_\mu = \epsilon_\mu{}^\nu \partial_\nu \phi$ has bifurcation at a point where $\xi^2 = -(\nabla\phi)^2$ equals zero. One can see from (2.4) that $g(\phi) = 0$ at this point and we have a horizon.

Since the scalar curvature for the metric (2.4) is equal to $R = -g''(\phi)$, one can easily check that curvature for the solution (2.7) really coincides (if $f''(R) \neq 0$) with $R = R(\phi)$ obtained by solving the equation $\phi = f'(R)$.

In the vicinity of points where $\phi'_R = f''(R) = 0$ the equation $\phi = f'(R)$ cannot be solved in a unique way. It is the only place where above solution is in-correct. Near a point like that the function $\phi(R)$ is as shown in Fig.1 : there are two values R_a, R_b which correspond to the same value ϕ. So there are two branches of solution of equation $\phi = f'(R)$. Let us consider this in more detail. Let $\phi'(R) = 0$ for some finite $R = R_0$. Note that only zero of odd order is interesting for us. In the vicinity of R_0 the function $\phi'(R)$ can be represented as follows

$$\phi'(R) = a(R - R_0)^{2k-1}, \; k = 1, 2, \dots, \; \textit{and} \;\; a > 0 \tag{2.8}$$

and hence we obtain

$$\phi(R) = \frac{a}{2k}(R - R_0)^{2k} + b \tag{2.9}$$

and

$$f(R) = \frac{a}{2k(2k+1)}(R - R_0)^{2k+1} + bR + c \tag{2.10}$$

There are two branches of the solution of eq.(2.9) with respect to R:

$$R = R_0 \pm [(\phi - b)\frac{2k}{a}]^{\frac{1}{2k}} \tag{2.11}$$

In the vicinity of the point $x \in M^2$ where $R(x) = R_0$ there are two regions: where $R > R_0$ and where $R < R_0$. Our solution (2.4), (2.7) is valid in any of these regions taken separately. Consider the region where $R > R_0$. Suppose for simplicity that $b = 0$, $a > 0$; then $\phi > 0$. Then, we get for the potential V:

$$V(\phi) = -\frac{2k}{2k+1}(\frac{2k}{a})^{\frac{1}{2k}}\phi^{1+1/2k} - R_0\phi + c \tag{2.12}$$

The corresponding metric (2.7) for $\phi > 0$ reads

$$g(\phi) = \lambda - \frac{R_0}{2}\phi^2 + c\phi - \frac{(2k)^2}{(2k+1)(4k+1)}(\frac{2k}{a})^{1/2k}\phi^{2+1/2k} \tag{2.13}$$

One can see that the metrical function $g(\phi)$ (2.13) has regular in $\phi = 0$ first and second derivatives

$$g'(0) = 0,\ g''(0) = -R_0$$

However, the non-analyticity of (2.13) in $\phi = 0$ manifests itself in that all the following derivatives are singular at this point:

$$g^{(p)}(0) = \pm\infty,\ p > 2$$

This singularity means, in particular, that invariant $(\nabla R)^2$ is singular at the point x where $R(x) = R_0$. It should be noted that singularities of this type were earlier observed in [14,15] for rather different theories.

In the region where $R < R_0$ we get

$$R = R_0 - (\frac{2k\phi}{a})^{1/2k}$$
$$V(\phi) = \frac{2k}{2k+1}(\frac{2k}{a})^{1/2k}\phi^{1+1/2k} - R_0\phi + c,\ \phi > 0 \tag{2.14}$$

Thus, the total space-time in the vicinity of the point $R = R_0$ ($\phi = 0$) is represented by gluing of two sheets (the coordinate $\phi > 0$ can be used to parameterize the points of both sheets in the neighborhood of $\phi = 0$). The total space-time is shown in Fig.2.

Really, the scalar curvature R itself can be used as one of the coordinates. It covers, in particular, the whole vicinity of the point R_0. Then, in the coordinates (t, R) the metric reads:

$$ds^2 = g(R)dt^2 - \frac{[f''(R)]^2}{g(R)}dR^2 \tag{2.15}$$

For $R \sim R_0$ we can put $g(r) \sim 1$, $f''(R) \sim a(R - R_0)^{2k}$ and hence the metric takes the form

$$ds^2 = dt^2 - a^2(R - R_0)^{4k} dR^2 \tag{2.16}$$

Let us now consider some examples.

Example 1.

$$f(R) = R \ln R \tag{2.17}$$

In this case $\varphi = \ln R + 1$, $R = e^{\varphi - 1}$. Hence $V(\varphi) = -R = -e^{\varphi - 1}$. Since the potential $V(R) = -R$ is zero in $R = 0$, it seems that one of the solutions is flat space-time. However, the function $f'(R) = \ln R + 1$ is not defined for $R = 0$. So if we come back to eq.(2.1), we observe that flat space-time is not really a solution of the field equations.

If R is a non-constant function on M^2, the solution is given by the metric (2.4) with

$$g(\varphi) = \lambda - R(\varphi) = \lambda - e^{\varphi - 1} \tag{2.18}$$

This solution coincides with that obtained in 2d dilaton gravity and describes asymptotically flat black hole space-time. The essential difference of the solution (2.18) from that we have in dilaton gravity is that it doesn't describe flat space-time for any integrating constant λ.

The Lagrangian (2.17) seems to be ill-defined at $R = 0$. However, we see that curvature $R(\phi)$ is everywhere positive and the point $R = 0$ really lies at the spatial infinity.

Example 2.

$$f(R) = aR^2 + bR + c \tag{2.19}$$

In this case $\phi = 2aR + b$, $R = \frac{1}{2a}(\varphi - b)$,i.e. $R(\varphi)$ is linear function. Then, we get $V(R) = -aR^2 + c = -\frac{1}{4a}(\phi - b)^2 + c$. If $c/a > 0$, then $V(R)$ is zero at the points $R = \pm\sqrt{c/a}$. Thus, there are two solutions with the constant curvature: $R = \pm\sqrt{c/a}$. If R is non-constant on M^2, then the solution is given by (2.4) with $g(\phi)$ in following form:

$$g(\phi) = \lambda + c\varphi - \frac{1}{12a}(\varphi - b)^3 \tag{2.20}$$

This function has extremums at the points $\varphi_{1,2} = b \pm 2a\sqrt{c/a}$ corresponding to the curvature $R = \pm\sqrt{c/a}$. Depending on the constant λ (if a, b, c are fixed), $g(\phi)$ can have one, two or three zeros. It is worth noting that the space-time described by the metric (2.20) is not asymptotically flat. $R = 0$ is reached at the point $\phi = b$ which stays on finite distance from any point $\phi \neq \pm\infty$. Thus, the points $\phi = \pm\infty$ lies at asymptotical infinity and the curvature is singular at this point. In this sense, the solution (2.20) is similar to that obtained in the 2D theory of gravity with torsion described by the action quadratic in curvature and torsion [5].

It should be noted that a behavior like that is rather typical of polynomial gravity (1.1). Indeed, let the function $f(R)$ near $R = 0$ look like as $f(R) = aR^{\alpha} + c$. Then, $\phi = \alpha a R^{\alpha-1}$, $R(\phi) = (\frac{\phi}{\alpha a})^{1/\alpha-1}$. One can see that for $\alpha > 1$ the point $R = 0$ corresponds to $\phi = 0$ and, consequently, lies at a finite distance from any point $\phi \neq 0$. It means that space-time is not asymptotically flat. The last is reached only if $\alpha < 1$: then $R \to 0$ means $\phi \to +\infty$. However, in this case the function $f(R)$ is not analytical in $R = 0$. It is the case in *Example1.*, where the solution is asymptotically flat. More generally, the solution is asymptotically flat if the function $f(R)$ satisfies the condition: $f'(R) \to \pm\infty$ if $R \to 0$. It is easy to see, however, that flat space-time is not a solution of field equations in this case, as we have seen in *Example1*.

3 One-loop renormalization

The complete quantization of the model (1.1) is a rather difficult problem. In this section, we just calculate one-loop counter-terms and check the renormalizability of the model in one loop and not considering these problems as the unitarity. We assume in this section that $f''(R) \neq 0$.

We use the background method. The metric $g_{\mu\nu}$ is written in the form: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is a classical background metric, $h_{\mu\nu}$ is a small quantum field. In the conformal gauge we have $h_{\mu\nu} = \sigma/2\bar{g}_{\mu\nu}$ and the theory reduces to quantization of only a conformal mode σ. Expanding the action (1.1) in powers of σ we obtain the quadratic in σ expression

$$S[g_{\mu\nu}] = S_{cl}[\bar{g}_{\mu\nu}] + S_q[\sigma]$$
$$S_q[\sigma] = \int\limits_{M^2} [f''(R)(\Box\sigma)^2 - 2Rf''(\sigma\Box\sigma) + (R^2 f'' + f - Rf')\sigma^2]\sqrt{\bar{g}}d^2z \quad (3.1)$$

where the curvature $R = R[\bar{g}]$ and the Laplacian $\Box = \frac{1}{\sqrt{\bar{g}}}\partial_\mu[\sqrt{\bar{g}}\bar{g}^{\mu\nu}\partial_\nu]$ are determined with respect to the background metric $\bar{g}_{\mu\nu}$. We see from (3.1) that $(1/f'')$ is effectively the loop expansion parameter for gravity.

The action $S_q[\sigma]$ can be written in the form

$$S_q[\sigma] = \int\limits_{M^2} \sqrt{\bar{g}}d^2z\sigma\hat{D}\sigma \quad (3.2)$$

where $\hat{D}$ is the fourth-order differential operator

$$\hat{D} = (\Box + X)f''(\Box + Y) \quad (3.3)$$

and the functions X, and Y satisfy the following equations:

$$X + Y = -2R,\ XY = R^2 f'' + f - Rf' \quad (3.4)$$

Calculating the functional integral over the conformal factor σ we can compute the infinite part of the one-loop effective action

$$I_\infty = \frac{1}{2}(\ln det\hat{D})_\infty - (\ln det\Delta_{gh})_\infty \tag{3.5}$$

where Δ_{gh} is the standard ghost operator corresponding to the conformal gauge.

By definition, for an elliptic $2r$ order differential operator Δ defined on a two-dimensional Riemannian manifold M^2 we get:

$$\ln det\Delta = -\int_\epsilon^{+\infty} \frac{dt}{t} Tre^{-t\Delta}, \ \epsilon \to +0 \tag{3.6}$$

The infinite part is given by ($L \to \infty$):

$$(\ln det\Delta)_\infty = -(B_0 L^2 + 2B_1 L + \frac{r}{2} B_2 \ln \frac{L^2}{\mu^2}) \tag{3.7}$$

where

$$(Tre^{-t\Delta})_{t\to 0} = \sum_{k=0}^{2} B_k t^{(k-2)/2r} + O(\sqrt{t}),$$
$$B_k = \int_{M^2} b_k(\Delta)\sqrt{g} d^2 z + \int_{\partial M^2} c_k(\Delta)\sqrt{\gamma} d\tau, \tag{3.8}$$

$b_k(\Delta)$ are the Seeley coefficients for the operator Δ ($b_{2p+1} = 0$). For simplicity we will assume that M^2 is a manifold without a boundary ($\partial M^2 = 0$) and will neglect all boundary effects.

Note that L^2 and L dependent terms are automatically absent in the dimensional or ζ-function regularization. So only the last term in (3.7) is of interest for us.

Now consider the Seeley coefficient for the elliptic fourth-order operator $\hat{D}$ (3.3) (see [16]). Suppose that for some Δ_4

$$\Delta_4 = \Delta_2 \Delta_2', \ det\Delta_4 = det\Delta_2 det\Delta_2' \tag{3.9}$$

Then, we get the corresponding expression for the Seeley coefficients [16]:

$$2B_2(\Delta_4) = B_2(\Delta_2) + B_2(\Delta_2'), \tag{3.10}$$

Since the operator $\hat{D}$ (3.3) has the structure (3.9) we obtain that

$$2B_2(\hat{D}) = B_2(\Box + X) + B_2(f''(\Box + Y)) \tag{3.11}$$

For the operator

$$\Delta_2 = \Box + X \tag{3.12}$$

the following result is well-known:

$$b_0(\Delta_2) = \frac{1}{4\pi}\ ;\ b_2(\Delta_2) = \frac{1}{4\pi}(1/6R + X) \tag{3.13}$$

To calculate the Seeley coefficients for the operator

$$\Delta_2' = f''(\Box + Y), \tag{3.14}$$

it is useful to observe that this operator can be transformed to (3.12) by introducing a new metric $\tilde{g}_{\mu\nu} = (f'')^{-1} g_{\mu\nu}$:

$$\Delta_2' = \Box_{\tilde{g}} + f''Y. \tag{3.15}$$

where $\Box_{\tilde{g}} = \frac{1}{\sqrt{\tilde{g}}}\partial_\mu[\sqrt{\tilde{g}}\tilde{g}^{\mu\nu}\partial_\nu]$. Now using the result (3.13) we obtain

$$B_2(\Delta_2') = \frac{1}{4\pi}\int\limits_{M^2}(1/6\tilde{R} + f''Y)\sqrt{\tilde{g}}d^2z \tag{3.16}$$

or in terms of the old metric $g_{\mu\nu}$ eq.(3.16) takes the form

$$B_2(\Delta_2') = \frac{1}{4\pi}\int\limits_{M^2}(1/6R + Y)\sqrt{g}d^2z \tag{3.17}$$

Thus, we obtain for $B_2(\hat{D})$:

$$B_2(\hat{D}) = \frac{1}{8\pi}\int\limits_{M^2}(1/3R + X + Y)\sqrt{g}d^2z \tag{3.18}$$

Using (3.4) we finally get

$$B_2(\hat{D}) = -\frac{1}{4\pi}\int\limits_{M^2} 5/6R\sqrt{g}d^2z \tag{3.19}$$

The Seeley coefficient for the ghost operator Δ_{gh} is well-known [17]

$$b_2(\Delta_{gh}) = \frac{1}{4\pi}(2/3R) \tag{3.20}$$

Taking into account (3.7), (3.19-20) we obtain from (3.5) that the corresponding one-loop counter-term

$$I_{ct} = a\int\limits_{M^2} R\sqrt{g}d^2z \tag{3.21}$$

is surprisingly non-dependent on the concrete function $f(R)$. So the model (1.1) seems to be one-loop renormalizable. Of course, if one takes the next loops this result could be changed and possibly not for any $f(R)$ the theory is renormalizable. However, we do not consider higher loops here.

4 Coupling with conformal matter

Let us consider interaction of higher derivative gravity (1.1) with 2d conformal matter described by the action

$$S_{mat} = \int\limits_{M^2} \frac{1}{2}(\nabla \psi)^2 \sqrt{g} d^2 z \tag{4.1}$$

Then, we get the complete system of equations of motion:

$$T_{\mu\nu} \equiv \nabla_\mu \nabla_\nu \phi - \frac{1}{2} g_{\mu\nu}[V(\phi) + 2\Box\phi] + \frac{1}{2}(\partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \partial_\alpha \psi \partial^\alpha \psi) = 0 \tag{4.2}$$

where $\phi = f'(R)$. The equation of motion for matter reads

$$\Box \psi = 0 \tag{4.3}$$

We will use the conformal gauge in which the components of metric: $g_{++} = g_{--} = 0$; $g_{+-} = \frac{1}{2} e^\sigma$. Then, eq.(4.2) takes the form

$$T_{\pm\pm} = \partial_\pm \partial_\pm \phi - \partial_\pm \sigma \partial_\pm \phi + \frac{1}{2} \partial_\pm \psi \partial_\pm \psi = 0 \tag{4.4}$$

$$T_{+-} = 0 \iff 4\partial_+ \partial_- \phi = -V(\phi) e^\sigma \tag{4.5}$$

Moreover, we have the self-consistency condition :

$$4\partial_+ \partial_- \sigma = R(\phi) e^\sigma \tag{4.6}$$

Equation (4.3) takes the form

$$\partial_+ \partial_- \psi = 0$$

and the solution reads

$$\psi = \psi_+(x^+) + \psi_-(x^-) \tag{4.7}$$

For the function $f(R)$ of general form eqs.(4.4-6) are extremely non-linear differential equations which are not exactly solved in general. However, in some particular cases, for concrete $f(R)$, this problem can be essentially simplified.

Let us consider the case when the following equation is valid:

$$\partial_+ \partial_- (\sigma - \phi) = 0 \tag{4.8}$$

It is the case when $f(R)$ satisfies the equation:

$$R + V(R) = R + f - Rf' = 0. \tag{4.9}$$

i.e., when $f(R) = R \ln R$. This case was considered in *Example 1*.

Then, we get

$$\sigma - \rho = w_+(x^+) + w_-(x_-) \tag{4.10}$$

On the other hand, one can see that $(\sigma + \rho)$ satisfies the Liouville equation

$$\partial_+\partial_-(\sigma + \rho) = \frac{1}{2\epsilon} e^{\sigma+\rho} \tag{4.11}$$

which has following general solution:

$$\sigma + \rho = \ln \frac{A'(x^+)B'(x^-)}{[1 - \frac{1}{4\epsilon}AB]^2} \equiv \beta(x^+, x^-), \tag{4.12}$$

where A and B are still unknown functions of x^+ and x^- respectively.

Thus, we obtain:

$$\sigma = \frac{1}{2}\beta + \frac{1}{2}w, \quad \rho = \frac{1}{2}\beta - \frac{1}{2}w \tag{4.13}$$

From this we see that

$$\partial_+^2\varphi - \partial_+\sigma\partial_+\rho = \frac{1}{2}(\partial_+^2\beta - 1/2(\partial_+\beta)^2) - \frac{1}{2}(\partial_+^2 w - 1/2(\partial_+ w)^2)$$

On the other hand, one can see the following identity:

$$(\partial_+^2\beta - 1/2(\partial_+\beta)^2) = \{A; x^+\}$$

where we introduced the Schwarzian derivative

$$\{\mathcal{F}; x\} = \frac{\partial_x^3\mathcal{F}}{\partial_x\mathcal{F}} - \frac{3}{2}\left(\frac{\partial_x^2\mathcal{F}}{\partial_x\mathcal{F}}\right)^2 \tag{4.14}$$

Then, we get for the $(++)$-component of equation (4.4):

$$\{A; x^+\} - (\partial_+^2 w_+ - 1/2(\partial_+ w_+)^2) + 2T_{++}^{\psi} = 0 \tag{4.15}$$

where $T_{++}^{\psi} = 1/2(\partial_+\psi_+)^2$. Similarly, we obtain for the $(--)$-component of eq.(4.4):

$$\{B; x^-\} - (\partial_-^2 w_- - 1/2(\partial_- w_-)^2) + 2T_{--}^{\psi} = 0 \tag{4.16}$$

Using the known property of the Schwarzian derivative (see for example [18]), one can see that eqs.(4.15)-(4.16) are invariant under $SL(2,R) \oplus SL(2,R)$ group transformations:

$$\begin{aligned} A &\to \frac{aA + b}{cA + d}, \quad ad - bc = 1 \\ B &\to \frac{mB + n}{kB + p}, \quad mp - kn = 1 \end{aligned} \tag{4.17}$$

Under the coordinate transformations $x^\pm \to y^\pm(x^\pm)$ we have:

$$\beta(x^+, x^-) \to \beta(y^+, y^-) - \ln\left(\frac{\partial y^+}{\partial x^+}\frac{\partial y^-}{\partial x^-}\right) \tag{4.18}$$

On the other hand, $w_\pm$ transforms as follows:

$$w_\pm(x^\pm) \to w_\pm(y^\pm) - \ln(\frac{\partial y^\pm}{\partial x^\pm}) \tag{4.19}$$

We can use this symmetry to put $w_\pm = 0$. Then, one obtains equations on functions A and B:

$$\begin{aligned} \{A; x^+\} &= -(\partial_+\psi_+)^2 \\ \{B; x^-\} &= -(\partial_-\psi_-)^2 \end{aligned} \tag{4.20}$$

When the matter is absent ($T_{\pm\pm} = 0$), the solution of equations

$$\{A; x^+\} = 0, \ \{B; x^-\} = 0 \tag{4.21}$$

is one of the following types. If $A'' = 0$, then

$$A = ax^+ + b \tag{4.22}$$

if $A'' \neq 0$, then

$$A = f - \frac{a}{x^+ + b} \tag{4.23}$$

Correspondingly, we get for $B(x^-)$:

$$B = mx^- + n \tag{4.24}$$

or

$$B = d - \frac{m}{x^- + n} \tag{4.25}$$

The metric takes the form:

$$ds^2 = \frac{[A'B']^{1/2}}{(1 - \frac{AB}{4e})} dx^+ dx^- \tag{4.26}$$

Shifting x^+, x^- on constants we get $b = n = 0$ in (4.22-25). Though A, B depend on the set of constants, the metric (4.26) depends only on one arbitrary constant. Let, for example, $A'' = B'' = 0$, then

$$ds^2 = \frac{c dx^+ dx^-}{(1 - \frac{c^2}{4e} x^+ x^-)} \tag{4.27}$$

where $c = \sqrt{am}$. In other cases, if A'' and B'' are not zero, the metric takes the form:

$$ds^2 = \frac{c dx^+ dx^-}{c^2 x^+ x^- - \frac{1}{4e}} \tag{4.28}$$

where $c = (1 - \frac{fd}{4e})(am)^{-1/2}$. The scalar curvature is given by the formula:

$$R = \frac{1}{e} \frac{(A'B')^{1/2}}{(1 - \frac{AB}{4e})} \tag{4.29}$$

It has singularity if $AB = 4\epsilon$, which one can also see from eqs.(4.27), (4.28). The points of horizon satisfy: $AB = 0$. The space-time (4.27-28) is of the same type as the black hole solution in the 2D dilaton gravity [1,3]. However, there is no such integrating constant for which the metric (4.27-28) is flat. The flat space-time is not a solution of field equations that has already been noted above. This is an essential difference between the string inspired 2D dilaton gravity [1] and higher derivative gravity (1.1) with $f = R\ln R$. Therefore, eqs.(4.4)-(4.6) do not describe the black hole formation from regular (flat) space-time due to the infalling matter as we had in the dilaton gravity [3]. The "bare" black hole is necessary. The infalling matter only deforms this initially singular space-time.

As an example let us now consider the falling of δ-like impulse of matter on the black hole. The matter energy-momentum tensor takes the form: $T^{\psi}_{++} = \lambda\delta(x^+ - x^+_0)$, $T^{\psi}_{--} = 0$ ($\lambda > 0$). It describes the δ-like impulse of matter propagating along the x^--direction. Suppose that the space-time for $x^+ < x^+_0$ is a solution of the field equations without matter such that $A = ax^+$, $B = mx^-$. For $x^+ > x^+_0$ the function $B(x^-)$ is the same while $A(x^+)$ is found from the equation:

$$\{A; x^+\} = -\lambda\delta(x^+ - x^+_0) \tag{4.30}$$

For $x^+ > x^+_0$ the function $A(x^+)$ is a solution of eq.(4.30) with the zero right-hand side

$$A = \frac{\alpha x^+ + \beta}{\kappa x^+ + \gamma}, \quad \alpha\gamma - \kappa\beta = 1 \tag{4.31}$$

where the constants $\alpha, \beta, \kappa, \gamma$ are found from the continuity condition of functions $A(x^+)$, $A'(x^+)$ and the gap condition for $A''(x^+)$ at the point $x^+ = x^+_0$. The last condition is easily obtained integrating (4.30) in the interval $(x^+_0 - \epsilon, x^+_0 + \epsilon)$ and then taking the limit $\epsilon \to 0$. As a result one obtains:

$$A''(x^+_0 + 0) - A''(x^+_0 - 0) = -\lambda A'(x^+_0) \tag{4.32}$$

From continuity of $A(x^+)$ and $A'(x^+)$ one gets:

$$ax^+_0 = \frac{\alpha x^+_0 + \beta}{\kappa x^+_0 + \gamma} \tag{4.33}$$

$$a = (\kappa x^+_0 + \gamma)^{-2} \tag{4.34}$$

and from the gap condition (4.32) we obtain

$$\frac{2\kappa}{(\kappa x^+_0 + \gamma)} = \lambda \tag{4.35}$$

These equations and $\alpha\gamma - \kappa\beta = 1$ are enough to find the form of $A(x^+)$ for $x^+ > x^+_0$:

$$A(x^+) = a\frac{x^+ + \frac{\lambda x^+_0}{2}(x^+ - x^+_0)}{1 + \frac{\lambda}{2}(x^+ - x^+_0)} \tag{4.36}$$

The metric for $x^+ < x_0^+$ takes the form

$$ds^2 = \frac{\sqrt{am}dx^+dx^-}{(1 - \frac{am}{4c}x^+x^-)} \tag{4.37}$$

and the corresponding curvature is the following

$$R = c^{-1}\frac{\sqrt{am}}{(1 - \frac{am}{4c}x^+x^-)} \tag{4.38}$$

This metric describes the black hole with horizon in $x^+x^- = 0$ and singularity at $x^+x^- = \frac{4c}{am}$.

We will assume that $x_0^+ > 0$, i.e. impulse falls from asymptotically flat region which lies right of horizon ($x^+ = 0$). Then, for $x^+ > x_0^+$ we obtain for the metric

$$ds^2 = \frac{\sqrt{am}}{[1 + \frac{\lambda}{2}(x^+ - x_0^+)]}\frac{dx^+dx^-}{[1 - \frac{am}{4c}(\frac{x^+ + \frac{\lambda x_0^+}{2}(x^+ - x_0^+)}{1 + \frac{\lambda}{2}(x^+ - x_0^+)})x^-]} \tag{4.39}$$

and the curvature

$$R = c^{-1}\sqrt{am}[1 + \frac{\lambda}{2}(x^+ - x_0^+) - \frac{am}{4c}(x^+ + \frac{\lambda x_0^+}{2}(x^+ - x_0^+))x^-]^{-1} \tag{4.40}$$

One can see from (4.40) that for $x^+ > x_0^+$ singularity lies on the curve:

$$x^- = \frac{4c}{am}\frac{(1 + \frac{\lambda}{2}(x^+ - x_0^+))}{(x^+ + \frac{\lambda x_0^+}{2}(x^+ - x_0^+))} \tag{4.41}$$

The derivative of the function (4.41):

$$\partial_+ x^- = -\frac{4c}{am}(x^+ + \frac{\lambda x_0^+}{2}(x^+ - x_0^+))^{-2}$$

is negative and we have that for $x^+ > x_0^+$ the function (4.41) is the monotonically decreasing one smoothly glued with $x^- = \frac{4c}{am}\frac{1}{x^+}$ at $x^+ = x_0^+$. Moreover, in the limit $x^+ \to \infty$ it limits to $x^- \to x_\infty^- = \frac{4c}{am}(x_0^+ + 2/\lambda)^{-1}$. The total space-time for all x^+ is shown in Fig.3. In the asymptotically flat region ($x^+ > 0$) it is similar to that we have for the 2D dilaton gravity case [3].

It should be noted that the function $f(R) = R\ln R$ is not a unique one for which equations (4.4-6) are exactly integrated. Indeed, we obtain from (4.5-6):

$$\begin{aligned} 4\partial_+\partial_-(\sigma - \phi) &= (R + V)e^\sigma \\ 4\partial_+\partial_-(\sigma + \phi) &= (R - V)e^\sigma \end{aligned} \tag{4.42}$$

These equations are reduced to the system of the Liouville equations if

$$R + V = ae^{-\phi}, \quad R - V = be^{\phi} \tag{4.43}$$

where a, b are constants. These conditions are equivalent to the system of differential equations on the function $f(R)$

$$R + f - Rf' = ae^{-f'}$$
$$R - f + Rf' = be^{f'} \tag{4.44}$$

One obtains immediately from this

$$R = 1/2[ae^{-f'} + be^{f'}] \tag{4.45}$$

Solving (4.45) with respect to f' one obtains:

$$f' = \ln \frac{R \pm \sqrt{R^2 - ab}}{b} \tag{4.46}$$

Integrating this we finally get:

$$f(R) = R \ln \frac{R \pm \sqrt{R^2 - ab}}{b} \mp \sqrt{R^2 - ab} \tag{4.47}$$

where both signs ($\pm$) are available if $(ab) > 0$. We do not consider here this kind of theory. Note only that for $a, b \neq 0$ it describes the asymptotically singular rather than asymptotically flat space-time.

5 Solution with backreaction

As we have discribed in the Introduction quantum corrections are usually assumed to remove the black hole singularity. One can try to analyze this problem semi-classically considering quantum gravity coupled to a large number N of free scalar fields. In the limit $h \to 0$ with Nh fixed, it is a system in which the leading order of a perturbative expansion is a quantum theory of matter in classical geometry. Integrating out the matter we have an effective action describing the backreaction of matter and Hawking radiation on the geometry, which we hope to treat classically [19]. In ref.[3] it was proposed to use this approach to study the problem in two dimensions for dilaton gravity. However, the resulting quantum-corrected field equations are not exactly solved [3,12,20] though some reasons observed in favor of that singularity are still present in this semiclassical theory. We apply here the approach of [3] for theory of gravity described by the action (1.1).

In two dimensions, integrating out the conformal scalar fields one gets the Polyakov-Liouville action:

$$S_{PL} = \frac{N}{96} \int d^2x_1 \sqrt{-g} \int d^2x_2 \sqrt{-g} R(x_1) \Box^{-1}(x_1, x_2) R(x_2) \tag{5.1}$$

here $\Box^{-1}$ denotes the Green function for the Laplacian. It should be noted that S_{PL} incorporates both the Hawking radiation and the effects of its backreaction on the geometry. We neglect here the contribution of the ghosts [21]. The full effective action

$$S_{eff} = S_{gr} + S_{mat} + S_{PL} \tag{5.2}$$

gives rise to the following system of equations (the metric is taken to be conformally flat $g_{-+} = \frac{1}{2}e^{\sigma}$):

$$\partial_\pm\partial_\pm\phi - \partial_\pm\sigma\partial_\pm\phi + 2c[\partial_\pm^2\sigma - \frac{1}{2}(\partial_\pm\sigma)^2 - t_\pm] + T^{\psi}_{\pm\pm} = 0 \tag{5.3}$$

$$4\partial_+\partial_-\phi = -e^{\sigma}(V(\phi) + 2cR) \tag{5.4}$$

where $c = \frac{N}{48}$ and $T^{\psi}_{\pm\pm} = \frac{1}{2}\partial_\pm\psi\partial_\pm\psi$. Equation (5.4) is obtained as variation of the action (5.2) with respect to g_{+-}. Since the scalar curvature is a known function of ϕ, we must add the condition of self-consistency:

$$4\partial_+\partial_-\sigma = R(\phi)e^{\sigma} \tag{5.5}$$

The scalar matter equation

$$\partial_+\partial_-\psi = 0$$

gives

$$\psi = \psi_+(x^+) + \psi_-(x^-)$$

For general function $f(R)$ these equations seem to be not exactly integrated. Therefore, we will consider in this section only the case

$$S_{gr} = \int R\ln R\sqrt{-g}d^2x$$

and show that for this type of gravitational action the system (5.3-5) is exactly solved. In this case $R(\phi) = e^{\phi-1}$, $V(\phi) = -R$. Equations (5.4), (5.5) take the form

$$4\partial_+\partial_-\phi = \frac{(1-2c)}{e}e^{\phi+\sigma} \tag{5.6}$$

$$4\partial_+\partial_-\sigma = \frac{1}{e}e^{\phi+\sigma} \tag{5.7}$$

Let $c \neq 1$.
Then, from (5.6-7) we obtain

$$\begin{aligned} &\partial_+\partial_-[(1-2c)\sigma - \phi] = 0 \\ &\partial_+\partial_-[\phi+\sigma] = \frac{(1-c)}{2e}e^{\phi+\sigma} \end{aligned} \tag{5.8}$$

These equations are easily solved

$$(1-2c)\sigma - \phi = w_+(x^+) + w_-(x^-) \equiv w$$
$$[\phi+\sigma] = \ln \frac{A'B'}{(1-\frac{(1-c)}{4c}AB)^2} \equiv \beta \qquad (5.9)$$

where $A = A(x^+)$, $B = B(x^-)$. Finally, we get for the conformal factor σ and field ϕ:

$$\sigma = \frac{1}{2(1-c)}(w+\beta); \; \phi = \frac{(1-2c)}{2(1-c)}\beta - \frac{1}{2(1-c)}w \qquad (5.10)$$

With respect to the coordinate changing $x^\pm \to y^\pm(x^\pm)$ we have

$$\beta(x^+,x^-) \to \ln \beta(y^+,y^-) + (\partial_+ y^+ \partial_- y^-)$$
$$w^\pm(x^\pm) \to w^\pm(y^\pm) - (1-2c)\ln(\partial_\pm y^\pm) \qquad (5.11)$$

Hence, the fields σ and ϕ transform as a usual conformal factor and a scalar field, respectively:

$$\sigma(x^+,x^-) \to \sigma(y^-,y^+) - \ln(\partial_- y^- \partial_+ y^+)\;, \phi(x^+,x^-) \to \phi(y^+,y^-) \qquad (5.12)$$

One can easily see that

$$\partial_\pm^2\phi - \partial_\pm\sigma\partial_\pm\phi + 2c[\partial_\pm^2\sigma - \frac{1}{2}(\partial_\pm\sigma)^2] =$$
$$\frac{1}{2(1-c)}[\partial_\pm^2\beta - \frac{1}{2}(\partial_\pm\beta)^2] + \frac{1}{2(1-c)}[\partial_\pm^2 w(-1+2c) + \frac{1}{2}(\partial_\pm w)^2] \quad (5.13)$$

As before, we have in terms of the Schwarzian derivative

$$\partial_+^2\beta - \frac{1}{2}(\partial_+\beta)^2 = \{A; x^+\}$$
$$\partial_-^2\beta - \frac{1}{2}(\partial_-\beta)^2 = \{B; x^-\} \qquad (5.14)$$

Let moreover $c \neq 1/2$, then we can use the symmetry (5.11) to put $w = 0$. Then, equations (5.3) take the form

$$\{A; x^+\} = -2(1-c)T^{\psi}_{++} + 4c(1-c)t_+(x^+)$$
$$\{B; x^-\} = -2(1-c)T^{\psi}_{--} + 4c(1-c)t_-(x^-) \qquad (5.15)$$

Eqs.(5.15) are ordinary differential equations with respect to A, B.

The metric and curvature, respectively, read:

$$ds^2 = e^{\frac{\beta}{2(1-c)}}dx^+dx^- = [\frac{A'B'}{(1-\frac{(1-c)}{4c}AB)^2}]^{\frac{1}{2(1-c)}}dx^+dx^- \qquad (5.16)$$

$$R = \frac{1}{c} e^{\frac{(1-2c)\rho}{2(1-c)}} = \frac{1}{c}\Big[\frac{A'B'}{(1-\frac{(1-c)}{4c}AB)^2}\Big]^{\frac{(1-2c)}{2(1-c)}} \tag{5.17}$$

If the right-hand side of eq.(5.15) is zero (i.e., matter is absent), then the solution of eq.(5.15) is already known (see (4.22-25)). Let, for example, it take the form (4.22), (4.24) with $b = n = 0$: $A = ax^+$, $B = mx^-$. Then, the metric and scalar curvature take the form

$$ds^2 = \Big[\frac{\sqrt{am}}{(1-\frac{(1-c)am}{4c}x^+x^-)}\Big]^{\frac{1}{1-c}} dx^+dx^- \tag{5.18}$$

$$R = \frac{1}{c}\Big[\frac{\sqrt{am}}{(1-\frac{(1-c)am}{4c}x^+x^-)}\Big]^{(\frac{1-2c}{1-c})} \tag{5.19}$$

If $c = 0$, we obtain the "classical" black hole space-time with space-like singularity at $x^+x^- = \frac{4c}{am}$ (we assume that $am > 0$).

For $c < 1/2$ or $c > 1$ we see from (5.19) that space-time still has singularity at $x^+x^- = \frac{4c}{am(1-c)}$. It is time-like for $c > 1$ and space-like for $c < 1/2$. As before, the points of horizon satisfy the condition $(\partial\phi)^2 = 0$, which for (5.18), (5.19) means that $x^+x^- = 0$. The diagram of this space-time is shown in Fig.4. The regions I and III are asymptotically flat.

Let $c > 1$ and consider the falling in this space-time of the matter impulse $T^{cl}_{++} = \lambda\delta(x^+ - x_0^+)$, $\lambda > 0$. We will neglect contribution of $t_\pm(x^\pm)$ into (5.15). We assume that impulse falls in the region I which is asymptotically flat, i.e. $x_0^+ < 0$. For $x^+ < x_0^+$ the space-time is described by the metric (5.18) and has curvature (5.19). For $x^+ > x_0^+$ the solution of eq.(5.15) is found in the same way as before (see the previous section). Moreover, the solution has the form similar to (4.36)

$$A(x^+) = a\frac{x^+ + \frac{\lambda(1-c)x_0^+}{2}(x^+ - x_0^+)}{1 + \frac{\lambda(1-c)}{2}(x^+ - x_0^+)}, \quad x^+ > x_0^+ \tag{5.20}$$

We obtain correspondingly for the metric

$$ds^2 = (am)^{\frac{-1}{2(c-1)}}\Big[1 + \frac{\lambda}{2}(1-c)(x^+ - x_0^+) - \\ \frac{am(1-c)}{4c}x^-(x^+ + \frac{\lambda}{2}(1-c)x_0^+(x^+ - x_0^+))\Big]^{\frac{1}{c-1}} dx^+dx^- \tag{5.21}$$

and scalar curvature

$$R = 1/c(am)^{\frac{2c-1}{2(c-1)}}\Big[1 + \frac{\lambda}{2}(1-c)(x^+ - x_0^+) - \\ \frac{am(1-c)}{4c}x^-(x^+ + \frac{\lambda}{2}(1-c)x_0^+(x^+ - x_0^+))\Big]^{-\frac{2c-1}{c-1}} \tag{5.22}$$

For $x^+ > x_0^+$ the singularity lies on the curve

$$x^- = \frac{4c}{am(1-c)} \frac{(1+\frac{\lambda(1-c)}{2}(x^+ - x_0^+))}{(x^+ + \frac{\lambda(1-c)x_0^+}{2}(x^+ - x_0^+))} \tag{5.23}$$

which is smoothly glued with the curve $x^- = \frac{4c}{am(1-c)}1/x^+$ at the point $x^+ = x_0^+$. Calculating a derivative of the function (5.23), we obtain that

$$\partial_+ x^- = -\frac{4c}{am(1-c)}(x^+ + \frac{\lambda(1-c)}{2}x_0^+(x^+ - x_0^+))^{-2} \tag{5.24}$$

i.e., the function (5.23) is monotonically increasing (remember that we consider the case $c > 1$). The function (5.23) takes an infinite value at $x_1^+ = (1 + \frac{2}{\lambda(1-c)x_0^+})x_0^+$, $x_0^+ < x_1^+ < 0$. It means that singularity in the region II is slightly shifted for $x^+ > x_0^+$, as is shown in Fig.5. On the other hand, the singularity in the region IV asymptotically tends to $x^-_\infty = \frac{2c\lambda}{am}(1 + \frac{\lambda(1-c)}{2}x_0^+)^{-1}$. The resulting space-time is shown in Fig.5. We see that for large N, $c > 1$, the singularity doesn't disappear but simply becomes time-like, which is similar to that we have for the 2D dilaton gravity [20].

Another case happens if c lies in the interval $1/2 < c < 1$. One can see that power in the expression for the curvature (5.19) becomes negative. Hence, the metric (5.18) describes space-time which is regular for any finite x^+ and x^-. In particular, it is the case for the points on the line $x^+x^- = \frac{4c}{am(1-c)}$ (or $AB = \frac{4c}{(1-c)}$). The curvature is zero though the metric g_{+-} takes an infinite value on this line. We obtain singularity if x^+ or x^- takes infinite value. It is convenient to change variables: $x^\pm = (y^\pm)^{-1}$. Then, the metric and curvature take the form

$$ds^2 = \frac{1}{(y^+y^-)^2}[\frac{\sqrt{am}}{1 - \frac{(1-c)am}{4cy^+y^-}}]^{\frac{1}{1-c}} dy^+ dy^-$$

$$R = 1/c[\frac{1}{\sqrt{am}}(1 - \frac{(1-c)am}{4cy^+y^-})]^{\frac{2c-1}{1-c}} \tag{5.25}$$

In the coordinates (y^+, y^-) the singularity lies on the light cone $y^+y^- = 0$. Asymptotically (for $y^+y^- \to \infty$) this space-time is of constant curvature.

The special case is $c = 1/2$.

One can see from (5.11) that $w^\pm(x^\pm)$ transform as usual scalar fields. Hence, one cannot put $w = 0$. Taking into account (5.13), (5.14) we obtain for eq.(5.3):

$$\{A; x^+\} + \frac{1}{2}(\partial_+ w)^2 + T^{\psi}_{++} - t_+ = 0$$

$$\{B; x^-\} + \frac{1}{2}(\partial_- w)^2 + T^{\psi}_{--} - t_- = 0 \tag{5.26}$$

The metric and curvature take the form:

$$
\begin{aligned}
ds^2 &= \frac{A'B'e^{w^+}e^{w_-}}{(1-\frac{AB}{8e})^2}dx^+dx^- \\
R &= 1/ee^{-w^+}e^{-w^-}
\end{aligned}
\tag{5.27}
$$

We may use the coordinate freedom to choose A and B as new coordinates: $u = A(x^+)$, $v = B(x^-)$. Since under $x^+ \to y^+(x^+)$ the Schwarzian derivative transforms as follows [18]:

$$
\{A; x^+\} - (\partial_+ y^+)^2\{A; y^+\} + \{y^+; x^+\} \tag{5.28}
$$

eqs.(5.26) are rewritten in the following form:

$$
\begin{aligned}
&\frac{1}{2}(\partial_u w)^2 + T^{\psi}_{uu} = t_u \\
&\frac{1}{2}(\partial_v w)^2 + T^{\psi}_{uu} = t_v
\end{aligned}
\tag{5.29}
$$

Note that inhomogeneous piece of law (5.28) cancels in (5.26) with the corresponding transformation of $t_\pm$, so that finally we come to expression (5.29). In the new coordinates we have

$$
\begin{aligned}
ds^2 &= \frac{e^{w^+(u)}e^{w_-(v)}}{(1-\frac{uv}{8c})^2}dudv \\
R &= 1/ee^{-w^+(u)}e^{-w^-(v)}
\end{aligned}
\tag{5.30}
$$

In general, the solution of eqs.(5.29) depends on the choice of boundary conditions,i.e., on appropriate functions t_u, t_v. These functions mean the flow of the Hawking radiation due to the falling matter with energy-momentum tensor $T^{\psi}_{\mu\nu}$. Physically, it seems to be reasonable to assume that the energy back radiated cannot be larger than the energy of the falling matter, i.e. $t_\pm \leq T^{\psi}_{\pm\pm}$. From (5.29) we obtain that unique possibility is the following: $T^{\psi}_{\pm\pm} = t_\pm$. Hence, one gets $w_\pm = const$ and, consequently, the total space-time is of the constant curvature: $R = e^{-(w+1)}$. Notice, that only for $c = 1/2$ there exists a constant curvature (de Sitter) solution of equations (5.3)-(5.5).

The other special case is $c = 1$.

Then, as one can see from (5.6) and (5.7) we obtain

$$
\partial_+\partial_-(\sigma + \phi) = 0 \tag{5.31}
$$

This equation has the solution

$$
\sigma + \phi = w = w^+(x^+) + w^-(x^-) \tag{5.32}
$$

Inserting this into (5.7) we get an equation on conformal factor σ:

$$
\partial_+\partial_-\sigma = \frac{1}{4e}e^w = \frac{1}{4e}e^{w^-}e^{w^+} \tag{5.33}
$$

which has the solution:

$$\sigma = \frac{1}{4e}\int^{x^+} e^{w^+(z^+)}dz^+ \int^{x^-} e^{w^-(z^-)}dz^- + \alpha(x^+) + \beta(x^-) \tag{5.34}$$

We use the coordinate freedom to put $\alpha(x^+) = 0$, $\beta(x^-) = 0$. One can see that $\partial_\pm\sigma$ satisfy the following equation

$$\partial_\pm^2\sigma = \partial_\pm w_\pm \partial_\pm \sigma \tag{5.35}$$

Taking this into account and putting (5.32), (5.34) into (5.3) we obtain that $w_\pm$ satisfy the following equations:

$$\partial_\pm^2 w_\pm = -(T^{\psi}_{\pm\pm} - 2t_\pm) \tag{5.36}$$

The general solution of (5.36) takes the form

$$w_\pm(x^\pm) = -\int^{x^\pm} du \int^u (T^{\psi}_{\pm\pm} - 2t_\pm)dz \tag{5.37}$$

If matter doesn't contribute (i.e. the right-hand side of (5.36) is zero), then

$$w^+ = ax^+ + b,\ w^- = mx^- + d \tag{5.38}$$

where a, b, m, d are constants. Below we consider the case $a, m > 0$.

Let us now consider the δ-like matter contribution ($t_\pm$ are putted to zero) $T^{\psi}_{++} = \lambda\delta(x^+ - x_0^+)$, $T^{\psi}_{--} = 0$ ($\lambda > 0$). Then, $w^- = mx^- + d$ for all x^+ and w^+ takes the form

$$\begin{aligned} w^+ &= ax^+ + b, \quad if \quad x^+ < x_0^+ \\ &= (a-\lambda)x^+ + b + \lambda x_0^+, \quad if \quad x^+ > x_0^+ \end{aligned} \tag{5.39}$$

Choosing the integrating constants in (5.34) to be zero we have correspondingly for σ:

$$\begin{aligned} \sigma &= \frac{1}{4eam}e^{mx^-+d}e^{ax^++b}, \quad if \quad x^+ < x_0^+ \\ &= \frac{1}{4em(a-\lambda)}e^{mx^-+d}[e^{(a-\lambda)x^++b+\lambda x_0^+} - \frac{\lambda}{a}e^{ax_0^++b}], \quad if \quad x^+ > x_0^+ \end{aligned} \tag{5.40}$$

We see that the metric $g_{+-} = \frac{1}{2}e^\sigma$ is everywhere positive and regular for any finite x^+, x^-.

It is worth observing that the scalar curvature $R = 1/ee^{w-\sigma}$ can be written in the form:

$$R = \frac{1}{e}\chi e^{-\alpha\chi} \tag{5.41}$$

where we introduced the function λ ($\lambda > 0$) taking the form

$$
\begin{aligned}
\lambda &= e^{(b+d)} e^{(mx^- + ax^+)}, \quad if \;\; x^+ < x_0^+ \\
&= e^{(b+d)} e^{(mx^- + (a-\lambda)x^+ + \lambda x_0^+)}, \quad if \;\; x^+ > x_0^+
\end{aligned}
\tag{5.42}
$$

and function α is $\alpha = \frac{1}{4eam}$ for $x^+ < x_0^+$ and

$$
\alpha = \frac{1}{4em(a-\lambda)}[1 - \frac{\lambda}{a} e^{(\lambda - a)(x^+ - x_0^+)}]
\tag{5.43}
$$

for $x^+ > x_0^+$. One can see that α is positive both for $\lambda < a$ and $\lambda > a$. Moreover it takes positive finite value in the limit $\lambda \to a$:

$$
\alpha \to \frac{1}{4em}[\frac{1}{a} + x^+ - x_0^+], \quad if \quad \lambda \to a
\tag{5.44}
$$

Hence, we obtain that the function α is positive for all x^+ and the curvature R (5.41) is finite for all x^+, x^-. We obtain the asymptotically flat space-time which is free from singularity and horizons.

Thus, the solution of equations (5.3)-(5.5) for $c = 1$ describes everywhere regular space-time. This case gives us good example when the quantum corrections (taken into account in the form of the Polyakov-Liouville term in the action (5.2)) can really remove the space-time singularity of the classical (black hole) solution. It should be noted that this result essentially depends on the quantum state or, equivalently, on the choice of appropriate boundary conditions (functions $t_\pm$). In the case under consideration the choice was to get asymptotically flat space-time.

Some remarks are in order. As we have seen in Section 3, (eq.(3.1)) for action (1.1) the value $(1/f'')$ is effectively loop expansion parameter for gravity. The semiclassical approximation (5.2) is valid under the condition: $|1/f''| << N$. For $f = R \ln R$ we have $(f'')^{-1} = R$. Consequently, we obtain the condition: $|R| << N$. This condition is rather natural and it means that semiclassical approximation works far from the points where the curvature infinitely grows. It is seen from the above consideration that for fixed very large N ($c >> 1$) there necessary exists a region near the space-time singularity where this condition is not valid and hence the semiclassical approximation failed. However, we can see that for $N = 48$ ($c = 1$) the curvature R (5.41) is bounded and has a maximum value: $R_{max} = (e^2\alpha)^{-1}$. We have that $\alpha = (4eam)^{-1}$ where (am) is an integrating constant. Thus, we obtain that a semiclassical approximation is valid for $c = 1$ everywhere in the space-time if $(am) << 12e$. The last condition can always be held by an appropriate choice of the integrating constant (am). For $N = 24$ ($c = 1/2$) the curvature R was shown to be constant: $R = e^{-(w+1)}$. By an appropriate choice of the constant w one can control the condition $R << N$, so the semiclassical approximation is correct also in this case.

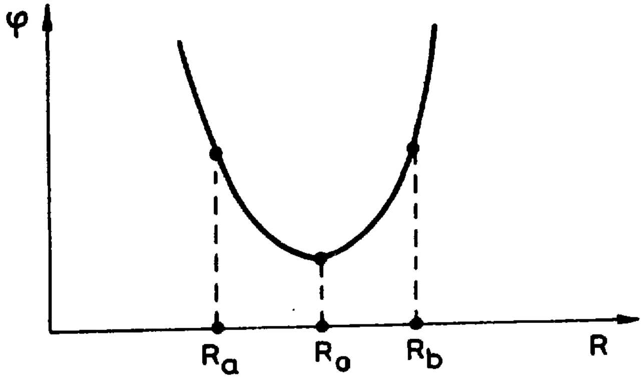

Fig.1: The shape of the function $\phi(R)$ in the vicinity of point $R = R_0$ where $\phi'(R) = f''(R) = 0$. There are two values R_a, R_b of R which correspond to the same value ϕ. So the inverse function $R(\phi)$ has two branches.

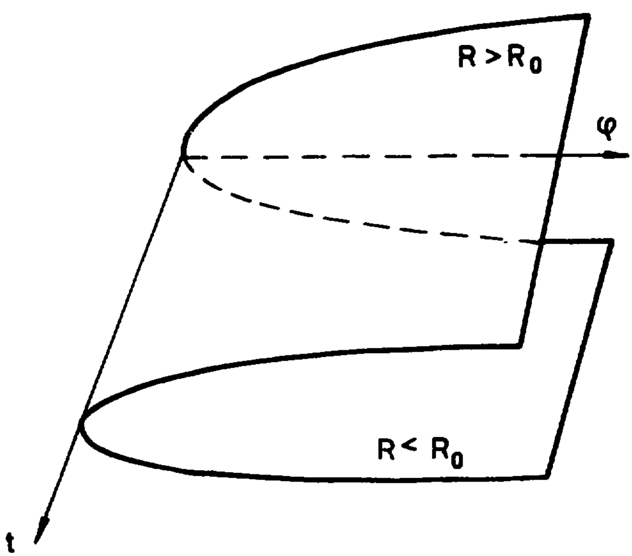

Fig.2: The space-time near the time-like line $R = R_0$ ($\phi = 0$) where $\phi'(R) = f''(R) = 0$. It consists of two sheets glued along the line $R = R_0$.

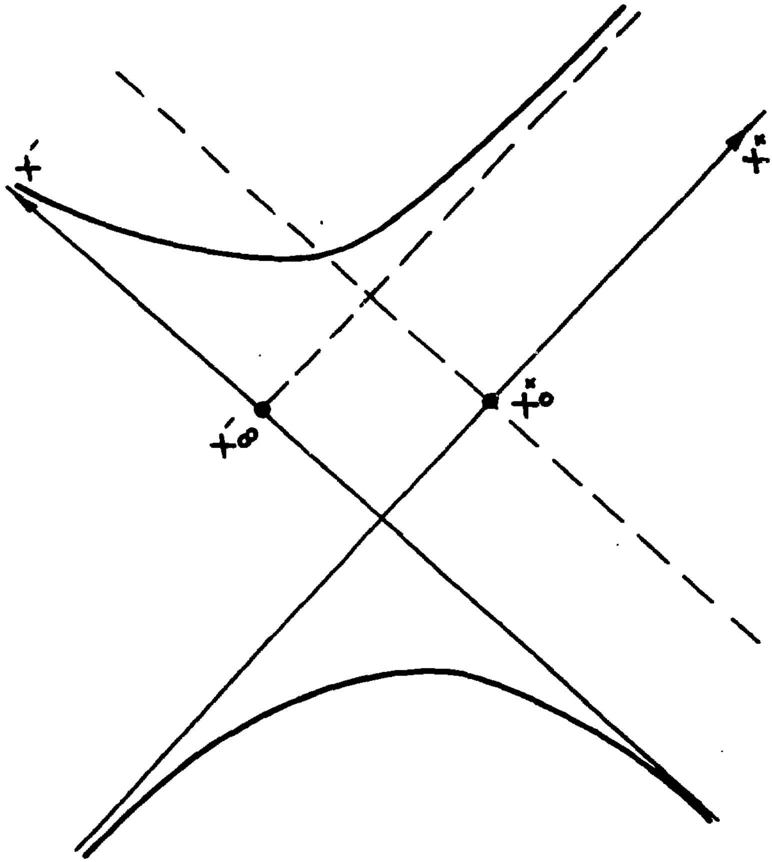

Fig.3: *The space-time obtained by the falling of δ-like impulse of matter at $x^+ = x_0^+$ on the black hole. For $x^+ > x_0^+$ the singularity is slightly deformed and asymptotically reaches the new horizon at $x^- = x_\infty^-$.*

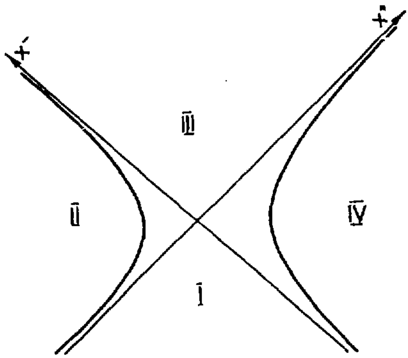

Fig.4: The black hole space-time deformed by quantum backreaction for $c > 1$. The singularity now is time-like and points of horizon satisfy the condition $x^+x^- = 0$. The regions I and III are asymptotically flat.

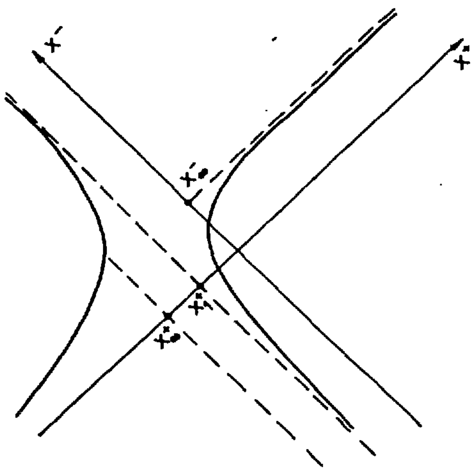

Fig.5: The space-time obtained by the falling of δ-like impulse of matter at $x^+ = x_0^+$ on the black hole for $c > 1$. The singularity for $x^+ > x_0^+$ is slightly shifted and asymptotically tends to new horizon at $x^+ = x_1^+$ and $x^- = x_\infty^-$.

6 Discussion

In resume, we have obtained that the preliminary hope that one-loop quantum corrections remove the classical black hole singularity . not realized for very large N ($c >> 1$). The space-time singularity is still present in the general solution for the action (5.2) in this regime. However, something interesting happens when N takes some finite (not very large) values. We have shown that for $N = 48$ ($c = 1$) the solution of the system (5.2) describes geodesically complete space-time regular everywhere. The corresponding scalar curvature (5.41) takes only finite values. In the other case, when $N = 24$ ($c = 1/2$) the semiclassical action (5.2) describes the (de Sitter) space-time of constant curvature. This space-time is also obviously free from singularities. Remember that N is the number of scalar fields or, more generally, N is the number of sorts of particles in a matter multiplet.

We conclude with some remarks in the order of discussion. It seems to be reasonable to consider the requirement of space-time regularity as some kind of principle: " *The space-time singularities must be absent in the complete quantum theory of gravity and matter* ". Then, our semiclassical analysis can be interpreted as that this "regularity principle" is not valid in general. But it leads to some restrictions on the particle contents of the theory. In the case under consideration, it constraints the number of matter fields N. There are some well known principles in modern physics which bound the particles spectrum: for example, the requirement of anomalies cancellation. Therefore, it would not be very surprising if the black hole physics gives us one more. In this paper, we have considered the two-dimensional case. However, the same situation can take place in four dimensions [15].

Of course, our study is just semiclassical and cannot be considered as a strict proof. The analysis in the framework of the complete quantum theory is necessary. However, the above consideration seems to be an argument in favor of the hypothesis on the relation between absence of the space-time singularities and particle spectrum of the theory.

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References

[1] G.Mandal, A.Sengupta and S.Wadia, Mod.Phys.Lett. **A6** (1991), 1685; E.Witten, Phys.Rev. **D44** (1991), 314;

[2] J.G.Russo, A.A.Tseytlin, Nucl.Phys. **B382** (1992), 259.

[3] C.G.Callan, S.B.Giddings, J.A.Harvey and A.Strominger, Phys.Rev. **D45** (1992), R1005;

[4] M.O.Katanaev, I.V.Volovich, Ann.Phys. **v.197** (1990), 1; M.O.Katanaev, J.Math.Phys. **v.34** (1993), 700. W.Kummer, D.J.Schwarz, Phys.Rev. **D45** (1992), 3628; W.Kummer, D.J.Schwarz, Nucl.Phys. **B382** (1992), 171;

[5] S.Solodukhin, JETP Lett., **57** (1993), 329; Phys.Lett. **B319** (1993), 87.

[6] E.W.Mielke, F.Gronwald, Yu.N.Obukhov, R.Tresguerres and F.W.Hehl, Phys.Rev. **D 48** (1993), 3648; I.V.Volovich, Mod.Phys.Lett. **A8** (1993), 1827.

[7] J.D.Barrow, A.C.Ottewill, J.Phys. **A16** (1983), 2757; S.Cotsakis, G.Flessas, Phys.Rev. **D48** (1993, 3577; S.Mignemi, D.L.Wiltshire, Phys.Rev. **D46** (1992), 1475; V.H.Hamity, D.E.Barraco, Gen.Rel.Grav. **25** (1993), 461;

[8] H.Kawai, R.Nakayama, *"Quantum R^2 gravity in Two Dimensions"* , KEK preprint 92-212.

[9] T.T.Burwick, *"All-Order Quantum Gravity in Two Dimensions"*, SLAC-PUB-6274 (1993).

[10] V.P.Frolov, Phys.Rev. **D46** (1992), 5383.

[11] S.W.Hawking, Phys.Rev. **D14** (1976), 2460.

[12] J.A.Harvey and A.Strominger, *"Quantum Aspects of Black Holes"*, Enrico Fermi Institute Preprint (1992), hep-th/9209055; S.B.Giddings, *"Toy Model for Black Hole Evaporation"*, UCSBTH-92-36, hep-th/9209113.

[13] T.Banks, M.O.'Loughlin, Nucl.Phys. **B362** (1991), 649.

[14] S.P.Trivedi, Phys.Rev. **D47** (1993), 4233; A.Strominger and S.P.Trivedi, Phys.Rev. **D48** (1993), 5778.

[15] D.I.Kazakov, S.N.Solodukhin, *"On Quantum Deformation Of The Schwarzschild Solution"*, Preprint JINR, E2-93-371 (1993).

[16] E.S.Fradkin, A.A.Tseytlin, Nucl.Phys. **B201** (1982), 469.

[17] E.S.Fradkin, A.A.Tseytlin, Ann.Phys. **143** (1982), 413.

[18] A.H.Chamseddine, M.Reuter, Nucl.Phys. **B317** (1982), 757.

[19] E.Tomboulis, Phys.Lett. **B70** (1977), 361.

[20] T.Banks, A.Dabholkar, M.R.Douglas and M.O.'Loughlin, Phys.Rev. **D45** (1992), 3607; S.W.Hawking and J.M.Stewart, Nucl.Phys. **B400** (1993), 393

[21] A.Strominger, Phys.Rev. **D46** (1992), 4396.

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