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CONSTRAINTS, COMPLEX STRUCTURES AND QUANTIZATION

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I. INTRODUCTION

Constrained systems are often considered in physics and therefore their quantization deserves particular attention. To quantize such systems one usually uses the method of canonical quantization. The method of geometric quantization of Kostant and Souriau[1–5] is a generalization of the standard canonical quantization on the curved phase manifolds M. Geometric quantization of the constrained systems have been considered in Refs.6–12. In these papers it has been supposed that the G-invariant polarization F exists on the symplectic manifold (M,ω). But the condition of G-invariance of polarization F does not always take place and little is known about the quantization of constrained systems when the G-invariant polarizations of M is absent.

The existence of polarization imposes the restrictions on the geometry of the manifold M. In particular, the existence of a complex polarization of M is equivalent to the existence of a complex structure J on M which is covariantly constant with respect to a symplectic connection. We shall consider the groups G which do not preserve any complex structure J on the manifold M, but preserve some family of complex structures. Of course, this imposes strong restrictions on the geometry of the manifold M. After quantization of the manifold M we shall obtain the family of the Fock spaces H_J parametrized by the complex structures J and what's more the group G maps the Fock spaces with different complex structures each into another. That is why for defining of the Fock space of G-invariant states we should have the method of "covariant" identification of the (projective) Fock spaces corresponding to the different complex structures. In this paper we shall describe a rather wide class of symplectic manifolds on which a family of complex structures exists and discuss the quantization of such manifolds.

II. GEOMETRIC QUANTIZATION

A classical (mechanical) system is given by its phase space, i.e. a symplectic manifold M with a 2-form ω. In the formalism of geometric quantization it is necessary to introduce

i) a prequantization bundle L over M;

ii) a polarization F of M;

iii) a metaplectic structure on M.

We define the prequantization bundle L over M as a complex line bundle with the connection ∇ (associated with the connection form) compatible with the Hermitian structure $<,>$ in fibres, the curvature 2-form F_∇ of which ($\frac{1}{i}F_\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$) coincides with the symplectic 2-form ω on M.

Theorem 2.1 : *The prequantization bundle L over (M,ω) exists if and only if the cohomology class of ω is integral.*

For proof see Refs. 1–5.

Take for the Hilbert space of prequantization the space $\mathcal{L}^2(M,L)$ defined as the completion of the space of smooth sections of L over M with compact supports with respect to the inner product

$$(s,t) = \int_M < s,t > \omega^n,$$

where $<,>$ is given by the Hermitian structure of L. Then we define the Kostant-Souriau prequantization $r : C^\infty(M) \to End\mathcal{L}^2(M, L)$ by setting

$$r(f) = f - i\nabla_{X_f}$$

where a vector field X_f is defined by the formula $X_f \rfloor \omega = -df$ and $r(f)$ is Hermitian if f is R-valued. It is not hard to see that

$$[r(f), r(h)] = -ir(\{f, h\}).$$

Thus, if we define operators

$$s(f) = ir(f) = \nabla_{X_f} + if, \tag{2.1}$$

acting in the space $\mathcal{L}^2(M, L)$, then we have

$$[s(f), s(h)] = s(\{f, h\}).$$

The introduced Hilbert space $\mathcal{L}^2(M, L)$ is too large to represent the phase space (M, ω) and we need a polarization.

Let TM be a tangent bundle over M and $T^C M = TM \otimes C$ its complexification. We call by a polarization of (M, ω) a subbundle $F \subset T^C M$ such that

i) a fibre $F_x \subset T^C_x M$ is a Lagrangian subspace in $T^C_x M$ for all $x \in M$, i.e. the restriction of ω to F_x vanishes and $dim F_x = n$;

ii) a space of sections of the bundle F is closed under the Lie bracket.

If $X \to \bar{X}$ is a complex conjugation then a subbundle $\bar{F}$ will also be a polarization. The polarization F is called the complex polarization if $\bar{F} \cap F = 0$, i.e. $T^C_x M = \bar{F}_x \oplus F_x$ for any $x \in M$.

If M is a symplectic manifold with a complex structure J then a canonical Kähler polarization F is defined by

$$F_x = \left\{ Y_x \in T^C_x M : J_x Y_x = -iY_x, \quad x \in M \right\}. \tag{2.2}$$

Kähler polarization is called positive (see Ref.13) if the metric g on M defined by the formula

$$g(X, Y) = \omega(X, JY), \quad X, Y \subset TM,$$

is positive definite.

Let F be a polarization $F \subset T^C M$ of a symplectic manifold (M, ω). Then we can introduce the space of quantization

$$H_F = \left\{ \psi \in \mathcal{L}^2(M, L) : \quad \nabla_X \psi = 0, \ \forall X \in \Gamma(M, F) \right\}, \tag{2.3}$$

where by $\Gamma(W, V)$ we denote the space of sections of a bundle V over W. For the Kähler polarization F of (M, ω) anambiguously defined by the complex structure J we will denote the space of quantization by H_J.

The introduction of a metaplectic structure on M is equivalent to the extension of the structure group of the bundle TM from the symplectic group $Sp(2n, R)$ to the metaplectic group $Mp(2n, R)$ which is the connected double covering of $Sp(2n, R)$.

Theorem 2.2 : *A metaplectic structure on M exists if and only if the 1st Chern class of M is even.*

• For proof see Refs. 1–5.

For a symplectic manifold (M, ω) of dimension $2n$ let $L(M)$ be the set of pairs (x, F_x), where $x \in M$ and F is a nonnegative polarization.[13,5] This manifold has the structure of a bundle over M with projection $(x, F_x) \to x$. We define also the manifold $L^+(M)$ of positive Kähler polarizations of M as the bundle $\pi : L^+(M) \to M$ of positive almost Hermitian structures on M associated with the principal bundle of symplectic frame of M. Denote by $\mathcal{R}(M, Sp(2n, R))$ the principal $Sp(2n, R)$-bundle $\mathcal{R} \to M$ of symplectic frame on M and by S the symmetric space $Sp(2n, R)/U(n)$. Then $L^+(M)$ is defined as a bundle

$$L^+(M) = \mathcal{R} \times_{U(n)} S \tag{2.4}$$

associated with the principal bundle $\mathcal{R}$. It means that the fibre $\pi^{-1}(x)$ of $L^+(M) \to M$ over a point $x \in M$ coincides with the space $S_x = Sp_x(2n, R)/U_x(n)$ of positive Hermitian structures on T_xM. Sections of π are identified with almost complex structures on M.

III. SYMPLECTIC AND COMPLEX STRUCTURES ON R^{2n} AND $L^+(R^{2n})$

Let us consider a vector space R^{2n} of dimension $2n$ with a canonical symplectic structure

$$\omega = \frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu, \quad (\omega_{\mu\nu}) = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \tag{3.1}$$

where $\mu, \nu, ... = 1, ..., 2n$. With the help of ω for two arbitrary functions f and h one can define a Poisson bracket $\{f, h\} = \omega^{\mu\nu}\partial_\mu f \partial_\nu h$, where $\omega^{\mu\lambda}\omega_{\lambda\nu} = \delta^\mu_\nu$. On R^{2n} we introduce a compatible with ω complex structure $J = (J^\mu_\nu)$, i.e. an endomorphism $J \in Sp(2n, R)$ of the space (R^{2n}, ω) such that $J^2 = -1$. Compatibility of ω and J means that

$$\omega(JX, JY) = \omega(X, Y) \Longleftrightarrow \omega_{\lambda\sigma} J^\lambda_\mu J^\sigma_\nu = \omega_{\mu\nu}$$

for any vector fields X and Y. Such ω and J define a Kähler structure on R^{2n} and the 2-form ω is of type $(1, 1)$ in the complex structure J. We shall consider translationally invariant complex structures, so they are defined by constant components J^μ_ν.

On R^{2n} we introduce the metric

$$g = \omega J \Longleftrightarrow g_{\mu\nu} = \omega_{\mu\lambda} J^\lambda_\nu$$

and the Hermitian metric

$$h = g + i\omega \Longleftrightarrow h_{\mu\nu} = g_{\mu\nu} + i\omega_{\mu\nu}.$$

We suppose that the metric g is positive definite, i.e. J is positive. It is well known that the collection of all such J, that are compatible with ω, is parametrized by the symmetric space $S = Sp(2n, R)/U(n)$. It may be shown (see, e.g., Ref.14) that

$$S = Sp(2n, R) \cap sp(2n, R) = \{J \in Sp(2n, R) : J^\lambda_\mu \omega_{\lambda\nu} = J^\lambda_\nu \omega_{\lambda\mu}\}. \tag{3.2}$$

Remind that $sp(2n, R) = \{A \in gl(2n, R) : {}^tA\omega + \omega A = 0\}$, where t is transposition.

The space S is parametrized by the symmetric complex $n \times n$ matrix $\tau = \tau_1 + i\tau_2$, $\det\tau_2 \neq 0$, because the general form of the complex structure matrix J, satisfying all above-formulated conditions, is[14]

$$J = pJ^0p^{-1} = \begin{pmatrix} \tau_1\tau_2^{-1} & -\tau_1\tau_2^{-1}\tau_1 - \tau_2 \\ \tau_2^{-1} & -\tau_2^{-1}\tau_1 \end{pmatrix}, \tag{3.3}$$

$$J^0 = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix}, \quad p = \begin{pmatrix} \tau_2 & \tau_1 \\ 0 & 1_n \end{pmatrix}, \quad p^{-1} = \begin{pmatrix} \tau_2^{-1} & -\tau_2^{-1}\tau_1 \\ 0 & 1_n \end{pmatrix}.$$

Using the matrices $J = (J^\mu_\nu)$ of the complex structure, one may identify R^{2n} with C^n introducing the operators

$$P = \frac{1}{2}(1 - iJ),\ \bar{P} = \frac{1}{2}(1 + iJ),\ P + \bar{P} = 1,\ P\bar{P} = 0 \iff$$

$$P^\mu_\nu = \frac{1}{2}(\delta^\mu_\nu - iJ^\mu_\nu),\ \bar{P}^\mu_\nu = \frac{1}{2}(\delta^\mu_\nu + iJ^\mu_\nu),\ P^\mu_\nu + \bar{P}^\mu_\nu = \delta^\mu_\nu,\ P^\mu_\lambda \bar{P}^\lambda_\nu = 0, \tag{3.4}$$

that project onto the $(1,0)$ and $(0,1)$ parts of a vector. This means that each vector $X \in R^{2n}$ may be decomposed into the holomorphic $X^{(1,0)}$ and antiholomorphic $X^{(0,1)}$ with respect to J parts:

$$X = X^{(1,0)} \oplus X^{(0,1)}, \quad X^{(1,0)} \equiv \frac{1}{2}(1 - iJ)X, \quad X^{(0,1)} \equiv \frac{1}{2}(1 + iJ)X,$$

$$JX^{(1,0)} = iX^{(1,0)}, \qquad JX^{(0,1)} = -iX^{(0,1)}. \tag{3.5}$$

Any tensor on R^{2n} may be decomposed in the same way.

According to the description from Sec.II, we define the space $L^+(R^{2n})$ of all positive Kähler polarizations of (R^{2n},ω) and $L^+(R^{2n})$ will be the product manifold $R^{2n} \times S$. Space $L^+(R^{2n})$ is the trivial bundle:

$$\pi : L^+(R^{2n}) \to R^{2n} \tag{3.6}$$

of all positive Kähler structures on R^{2n}. Points $z \in L^+(R^{2n})$ are the pairs $z = (x, J)$ where $x \in R^{2n}$, $J \in S = Sp(2n, R)/U(n)$ and $\pi(x, J) = x$. The fibre $\pi^{-1}(x)$ in any point $x \in R^{2n}$ is the space S of Kähler structures on R^{2n} defined above. Sections of the bundle (3.6) are the spaces (R^{2n}, ω, J) with fixed complex structures $J \in S$.

As in Riemannian case we can provide $L^+(R^{2n})$ with a natural complex structure $\mathcal{J}$. In fact, let us consider the natural splitting of the tangent bundle $TL^+(R^{2n})$ into a direct sum

$$TL^+(R^{2n}) = R^{2n} \oplus T(S) \tag{3.7}$$

of horizontal and vertical subbundles of $TL^+(R^{2n})$. Complex structure J on R^{2n} has been described before. The fibre $T_J(S)$ in $z = (x, J) \in L^+(R^{2n}) = R^{2n} \times S$ is tangent to S at J, so it has a natural complex structure J_S (see, e.g., Ref.14). It may be defined as follows. The condition $J^2 = -1$ implies that $J^\lambda_\mu(dJ^\nu_\lambda) + (dJ^\lambda_\mu)J^\nu_\lambda = 0 \Rightarrow J^\nu_\lambda J^\sigma_\mu dJ^\lambda_\sigma = dJ^\nu_\mu$. This means[15] that the non-zero projections of dJ^σ_λ are $P^\mu_\sigma dJ^\sigma_\nu$ and $\bar{P}^\mu_\sigma dJ^\sigma_\nu$. We introduce on S a complex structure $J_S = (J^{\mu\lambda}_{\nu\sigma})$ (where the indices $(^\mu_\nu)$ are considered as the upper ones and $(^\lambda_\sigma)$ as the lower ones) with components $J^{\mu\lambda}_{\nu\sigma} = J^\mu_\sigma \delta^\lambda_\nu$. It is easy to see that

$$J^{\mu\alpha}_{\nu\beta} J^{\beta\lambda}_{\alpha\sigma} = -\delta^{\mu\lambda}_{\nu\sigma} \Leftrightarrow J_S^2 = -1, \tag{3.8a}$$

$$J^{\mu\alpha}_{\nu\beta} P^{\beta}_{\sigma} dJ^{\sigma}_{\alpha} = i P^{\mu}_{\sigma} dJ^{\sigma}_{\nu}, \quad J^{\mu\alpha}_{\nu\beta} \bar{P}^{\beta}_{\sigma} dJ^{\sigma}_{\alpha} = -i \bar{P}^{\mu}_{\sigma} dJ^{\sigma}_{\nu}, \tag{3.8b}$$

where $\delta^{\mu\lambda}_{\nu\sigma} := \delta^{\mu}_{\sigma}\delta^{\lambda}_{\nu}$ is the Kronecker delta and for any "vector" T^{σ}_{λ} we have $\delta^{\mu\lambda}_{\nu\sigma} T^{\sigma}_{\lambda} = T^{\mu}_{\nu}$. From (3.8b) it follows that $P^{\mu}_{\sigma} dJ^{\sigma}_{\nu}$ and $\bar{P}^{\mu}_{\sigma} dJ^{\sigma}_{\nu}$ are (1,0)-type and (0,1)-type one-forms on S. Analogously, the holomorphic and antiholomorphic vector fields on S will be

$$\partial^{(1,0)}{}^{\mu}_{\nu} \equiv P^{\lambda}_{\nu} \frac{\partial}{\partial J^{\lambda}_{\mu}} \Leftrightarrow J^{\nu\lambda}_{\mu\sigma} \partial^{(1,0)}{}^{\mu}_{\nu} = i \partial^{(1,0)}{}^{\lambda}_{\sigma}, \tag{3.9a}$$

$$\partial^{(0,1)}{}^{\mu}_{\nu} \equiv \bar{P}^{\lambda}_{\nu} \frac{\partial}{\partial J^{\lambda}_{\mu}} \Leftrightarrow J^{\nu\lambda}_{\mu\sigma} \partial^{(0,1)}{}^{\mu}_{\nu} = -i \partial^{(0,1)}{}^{\lambda}_{\sigma}. \tag{3.9b}$$

Now we can define the complex structure $\mathcal{J}$ on $L^{+}(R^{2n})$ using the decomposition (3.7) by setting $\mathcal{J} = J \otimes 1 \oplus 1 \otimes J_S$ and it is completely defined by the holomorphic (antiholomorphic) vector fields (3.5) on R^{2n} and (3.9) on S.

Symplectic structure $\tilde{\Omega}$ on $L^{+}(R^{2n})$ may also be defined by using the decomposition (3.7):

$$\tilde{\Omega} = \omega \otimes 1 + 1 \otimes \Omega_S,$$

where ω is defined in (3.1) and Ω_S has been described, e.g., in Refs.5,14. In terms of J^{μ}_{ν} and dJ^{μ}_{ν}, the two-form Ω_S has the form:

$$\Omega_S = -\frac{i}{8} P^{\sigma}_{\mu} \bar{P}^{\nu}_{\lambda} dJ^{\mu}_{\nu} \wedge dJ^{\lambda}_{\sigma}. \tag{3.10}$$

It is easy to see that Ω_S is real and compatible with J_S. It is well-known (see, e.g., Ref.5), that $\Omega_S = R_S$, where R_S is a curvature of the determinant bundle over S.

IV. QUANTIZATION OF THE SPACE (R^{2n}, ω, J) AND FOCK BUNDLE

We introduce a (trivial) prequantization bundle $L \simeq R^{2n} \times C$ over R^{2n} with connection

$$\nabla = dx^{\mu}\nabla_{\mu} = dx^{\mu}(\partial_{\mu} + \frac{i}{2}\omega_{\mu\lambda}x^{\lambda}), \tag{4.1}$$

where $\partial_{\mu} \equiv \frac{\partial}{\partial x^{\mu}}$. It is easy to see that the curvature 2-form $F_{\nabla} = \frac{i}{2}[\nabla_{\mu}, \nabla_{\nu}] dx^{\mu} \wedge dx^{\nu}$ of the connection (4.1) coincides with the symplectic 2-form ω.

Theorem 4.1 : *The bundle L is a holomorphic bundle.*

Proof: The two-form ω is of type (1,1) with respect to any complex structure $J \in S$ (see Sec.III). Because ω is of type (1,1) w.r.t. to J, then (0,2) part of the curvature vanishes, so that the connection (4.1) in L endows it with a holomorphic structure.

Take for the Hilbert space of prequatization the space $\mathcal{L}^2(R^{2n}, L)$ defined as the completion of the space of square-integrable smooth section of L over R^{2n} with respect to the inner product

$$(\psi_1, \psi_2) = \int_{R^{2n}} \bar{\psi}_1 \psi_2 \, \omega^n,$$

where $\bar{\psi}_1$ is complex conjugate to ψ_1.

Let us consider the space $\mathcal{O}(R^{2n},\omega,J)$ of holomorphic sections of the bundle L over $R^{2n}_J \equiv (R^{2n},\omega,J)$:

$$\mathcal{O}(R^{2n},\omega,J) = \left\{\psi \in \Gamma(R^{2n},L) : \frac{1}{2}(1+iJ)\nabla_X\psi = 0, \forall \text{ vector field } X \text{ on } R^{2n}\right\} =$$

$$= \left\{\psi \in \Gamma(R^{2n},L) : \frac{1}{2}(\delta^\mu_\nu + iJ^\mu_\nu)\nabla_\mu\psi = 0,\ \mu,\nu = 1,...,2n\right\}.$$

Now we define the Hilbert space H_J of square-integrable holomorphic sections:

$$H_J = \{\psi \in \mathcal{O}(R^{2n},\omega,J) : \int_{R^{2n}} \bar{\psi}\psi\,\omega^n < \infty\}. \tag{4.2}$$

The space H_J is the Hilbert space of quantization. In particular, we shall consider $H_0 \equiv H_{J^0}$, where J^0 is a fixed canonical (see (3.3)) complex structure on R^{2n}.

In Sec.III we introduced the space $L^+(R^{2n}) = R^{2n} \times S$ of all positive Kähler structures on R^{2n} and holomorphic and antiholomorphic vector fields on $L^+(R^{2n})$ defining the complex structure $\mathcal{J}$ on $L^+(R^{2n})$. Denote by $\tilde{L} \to L^+(R^{2n})$ the pull-back of L to $L^+(R^{2n})$.

Theorem 4.2 : *The bundle $\tilde{L}$ is a holomorphic bundle.*

Proof: To prove the assertion denote by $\tilde{\nabla}$ the pull-back of the connection ∇ to $\tilde{L}$. By definition, the pulled back connection $\tilde{\nabla}$ has zero components along fibres:

$$\tilde{\nabla} = \nabla + dJ^\mu_\nu \frac{\partial}{\partial J^\mu_\nu}, \tag{4.3}$$

where we have used J^μ_ν as coordinates, parametrizing $S = Sp(2n,R)/U(n)$. Remind that not all of the components J^μ_ν are independent because they satisfy the equations (3.2) and $J^\mu_\lambda J^\lambda_\nu = -\delta^\mu_\nu$. Now define a $\tilde{\nabla}^{(0,1)}$-operator on sections $\tilde{\psi}$ of $\tilde{L} \to L^+(R^{2n})$ by setting

$$\tilde{\nabla}^{(0,1)}\tilde{\psi} \equiv \left(\nabla^{(0,1)} + dJ^\lambda_\nu \bar{P}^\sigma_\lambda \bar{P}^\mu_\sigma \frac{\partial}{\partial J^\mu_\nu}\right)\tilde{\psi}. \tag{4.4}$$

So $\tilde{\nabla}^{(0,1)}$ is the (0,1)-component of $\tilde{\nabla}$ w.r. to the complex structure $\mathcal{J}$ on $L^+(R^{2n})$ introduced above. The symplectic structure ω on R^{2n} being compatible with all Kähler structures on R^{2n} has the type (1,1) w.r. to any such structure, hence the curvature F_∇ also has the type (1,1) w.r. to any Kähler structure. According to the definition of the complex structure on $L^+(R^{2n})$ it means that the curvature $F_{\tilde{\nabla}}$ of the pulled-back connection $\tilde{\nabla}$ on $\tilde{L}$ has the type (1,1) w.r. to the complex structure on $L^+(R^{2n})$. It follows that

$$(\tilde{\nabla}^{(0,1)})^2\tilde{\psi} = F^{(0,2)}_{\tilde{\nabla}}\tilde{\psi} = 0,$$

i.e. $\tilde{L}$ is holomorphic. It can be also shown by direct calculations, using (4.1), (3.4) and (4.4).

Remark: It is essentially Ward's construction from the twistor theory (cf. Ref.16).

Denote by $\tilde{H}$ the space of square-integrable holomorphic with respect to $\mathcal{J}$ sections of the bundle $\tilde{L} \to L^+(R^{2n})$:

$$\tilde{H} = \{\Psi \in \mathcal{L}^2(L^+(R^{2n}),\tilde{L}) : \frac{1}{2}(1+i\mathcal{J})\tilde{\nabla}_X\Psi = 0,\ \forall \text{ vector field } X \text{ on } L^+(R^{2n})\} =$$

$$= \{\Psi \in \mathcal{L}^2(L^+(R^{2n}), \dot{L}) : \; P_\mu^\lambda \nabla_\lambda \Psi = 0, \; P_\mu^\lambda \frac{\partial}{\partial J_\nu^\lambda} \Psi = 0, \; \mu, \nu, \ldots = 1, \ldots, 2n\} \tag{4.5}$$

Each space (R^{2n}, ω, J) is section of the bundle $L^+(R^{2n}) \to R^{2n}$ and we obtain the Hilbert space of quantization H_J as subspace in $\tilde{H}$ if fix an argument J of holomorphic with respect to $\mathcal{J}$ functions $\Psi(x, J) \in \tilde{H}$.

Now we have the Kähler space $L^+(R^{2n})$, the prequantization bundle $\dot{L} \to L^+(R^{2n})$ over $L^+(R^{2n})$ with connection $\dot{\nabla}$, the space $\mathcal{L}^2(L^+(R^{2n}), \dot{L})$ of prequantization and the space $\tilde{H}$ of quantization associated to $L^+(R^{2n})$. Connection (4.3) is flat along the fibre S in the bundle $L^+(R^{2n}) \to R^{2n}$. Let us add to the connection $dJ_\nu^\mu \frac{\partial}{\partial J_\nu^\mu}$ along S a term $B\, dJ_\nu^\mu \omega^{\nu\lambda} P_\mu^\sigma P_\lambda^\gamma \nabla_\sigma \nabla_\gamma$ which is a one-form on S with values in the algebra of differential operators on R^{2n}, i.e. we introduce the differential operator

$$\mathcal{D}_\mu^\nu = \frac{\partial}{\partial J_\nu^\mu} + B\, \omega^{\nu\lambda} P_\mu^\sigma P_\lambda^\gamma \nabla_\sigma \nabla_\gamma, \tag{4.6}$$

where B is some constant. Then the operator

$$\dot{\nabla}(B) = dx^\mu \nabla_\mu + dJ_\nu^\mu \mathcal{D}_\mu^\nu$$

if $B \neq 0$ can not be interpreted as a connection in the bundle $\dot{L} \to L^+(R^{2n})$ because it is quadratic in derivatives. But it is correctly defined differential operator acting in the space $\mathcal{L}^2(L^+(R^{2n}), \dot{L})$.

By virtue of properties of dJ_ν^μ, which have been discussed in Sec.III, the operator $\mathcal{D}_\mu^\nu$ has only the following nonzero components (holomorphic and antiholomorphic):

$$\mathcal{D}^{(1,0)}{}_\rho^\eta = P_\rho^\mu \mathcal{D}_\mu^\eta = P_\rho^\mu \left(\frac{\partial}{\partial J_\eta^\mu} + B\, \omega^{\eta\lambda} P_\mu^\sigma P_\lambda^\gamma \nabla_\sigma \nabla_\gamma \right) = P_\rho^\mu \frac{\partial}{\partial J_\eta^\mu} + B\, \omega^{\eta\lambda} P_\rho^\sigma P_\lambda^\gamma \nabla_\sigma \nabla_\gamma,$$

$$\mathcal{D}^{(0,1)}{}_\rho^\eta = \bar{P}_\rho^\mu \mathcal{D}_\mu^\eta = \bar{P}_\rho^\mu \frac{\partial}{\partial J_\eta^\mu} = \partial^{(0,1)}{}_\rho^\eta$$

Let us calculate the commutators of the operators $\mathcal{D}^{(1,0)}{}_\rho^\eta$ and $\mathcal{D}^{(0,1)}{}_\rho^\eta$ with the operators $\nabla_\alpha^{(0,1)}$ and $\partial^{(0,1)}{}_\alpha^\beta$ defining the $(0,1)$-components of connection $\dot{\nabla}$ in $\dot{L}$. Because of $\mathcal{D}^{(0,1)}{}_\rho^\eta$ coincides with the $(0,1)$-components of $\dot{\nabla}$ along S then

$$[\mathcal{D}^{(0,1)}{}_\rho^\eta, \mathcal{D}^{(0,1)}{}_\alpha^\beta] = 0. \tag{4.7a}$$

It is not difficult to verify that

$$[\mathcal{D}^{(0,1)}{}_\rho^\eta, \bar{P}_\alpha^\beta \nabla_\beta] = \frac{i}{2} \delta_\alpha^\eta \bar{P}_\rho^\beta \nabla_\beta, \tag{4.7b}$$

i.e. $\mathcal{D}^{(0,1)}{}_\rho^\eta$ preserve the holomorphic structure in the bundle $\dot{L} \to L^+(R^{2n})$ in accordance with the Theorem 4.2.

For the operator $\mathcal{D}^{(1,0)}{}_\rho^\eta$ we have

$$[\mathcal{D}^{(1,0)}{}_\rho^\eta, \bar{P}_\alpha^\beta \nabla_\beta] = \frac{i}{2} \delta_\alpha^\eta \bar{P}_\rho^\sigma \nabla_\sigma - iB P_\alpha^\eta P_\rho^\sigma \nabla_\sigma + iB\, \omega^{\eta\nu} \omega_{\beta\mu} \bar{P}_\alpha^\beta P_\nu^\sigma \nabla_\sigma.$$

Therefore

$$[\mathcal{D}^{(1,0)}, \bar{P}_\alpha^\beta \nabla_\beta] = [dJ_\eta^\rho \mathcal{D}^{(1,0)}{}_\rho^\eta, \bar{P}_\alpha^\beta \nabla_\beta] =$$

$$= \frac{i}{2}\left\{dJ^{\rho}_{\eta}(1-2B)\bar{P}^{\eta}_{\alpha}P^{\sigma}_{\rho} + 2B\, dJ^{\nu}_{\eta}\omega^{\eta\rho}\omega_{\beta\rho}\bar{P}^{\beta}_{\alpha}P^{\sigma}_{\nu}\right\}\nabla_{\sigma} = \frac{i}{2}dJ^{\rho}_{\eta}(1-4B)\bar{P}^{\eta}_{\alpha}P^{\sigma}_{\rho}\nabla_{\sigma}, \quad (4.8)$$

where we have used the property $dJ^{\rho}_{\eta}\omega^{\eta\nu} = dJ^{\nu}_{\eta}\omega^{\eta\rho}$ of dJ^{μ}_{ν}. Thus, the right hand side of (4.8) becomes zero if $B = 1/4$. Later we shall choose this value of the parameter B.

Finally, the commutator $[\mathcal{D}^{(1,0)}, \mathcal{D}^{(0,1)}]$ is equal to

$$[\mathcal{D}^{(1,0)}\,{}^{\eta}_{\rho}, \partial^{(0,1)}\,{}^{\beta}_{\alpha}] = [P^{\mu}_{\rho}\frac{\partial}{\partial J^{\mu}_{\eta}} + \frac{1}{4}\omega^{\eta\nu}P^{\sigma}_{\rho}P^{\gamma}_{\nu}\nabla_{\sigma}\nabla_{\gamma},\ \bar{P}^{\lambda}_{\alpha}\frac{\partial}{\partial J^{\lambda}_{\beta}}] =$$

$$= \frac{i}{8}\omega^{\eta\nu}(\delta^{\beta}_{\rho}P^{\gamma}_{\nu} + \delta^{\beta}_{\nu}P^{\gamma}_{\rho})\bar{P}^{\sigma}_{\alpha}\nabla_{\gamma}\nabla_{\sigma} - \quad (4.9a)$$

$$-\frac{1}{8}\delta^{\beta}_{\rho}\bar{P}^{\eta}_{\alpha}. \quad (4.9b)$$

We see that the commutator is equal to the sum of two terms, where the first term (4.9a) preserves the holomorphic structure in the bundle $\tilde{L} \to L^{+}(R^{2n})$ and the second term (4.9b) does not preserve and does not contain the differential operators and depends only on $J \in S = Sp(2n, R)/U(n)$.

Term (4.9b) may be compensated if one will consider the bundle $\tilde{L}\otimes K^{1/2} \to L^{+}(R^{2n})$ instead of $\tilde{L} \to L^{+}(R^{2n})$, where $K^{1/2} \to L^{+}(R^{2n})$ is the square root of the bundle K, which has the fibre $K_J = \bigwedge^{n}\bar{F}^{*}$ at $(x, J) \in L^{+}(R^{2n})$. Here $\bar{F}^{*} = \{\bar{P}^{\mu}_{\nu}dx^{\nu},\ \mu, ... = 1, ..., 2n\}$. By definition [13,5], a metaplectic structure on symplectic manifold (M, ω) is a line bundle $\delta \to L^{+}M$ such that $\delta^{2} = K$. In our case the restriction of $K^{1/2} \equiv \delta$ to S is simply the square root of the canonical bundle of S. Let $R^{2n}_{J} = (R^{2n}, \omega, J)$ be an arbitrary section of the bundle $L^{+}(R^{2n}) \to R^{2n}$. The pull-back of $K^{1/2}$ to R^{2n}_{J} will be the trivial bundle, because R^{2n}_{J} is the flat Kähler manifold. Transition from the bundle $\tilde{L} \to L^{+}(M)$ to the bundle $\tilde{L}\otimes K^{1/2} \to L^{+}(M)$ corresponds to the metaplectic correction and to the introduction of the half-forms developed in the approach of geometric quantization.[13,5] Notice that Berry's phase (see Ref.17) may be expressed through the curvature of the bundle $K^{1/2} \to S$ after the embedding of the space of external parameters of the quantum system into S.[5]

So, let us calculate $F_{\mathcal{D}} = \frac{i}{2}[\mathcal{D}^{\nu}_{\mu}, \mathcal{D}^{\sigma}_{\lambda}]dJ^{\mu}_{\nu} \wedge dJ^{\lambda}_{\sigma}$. Using formulae (4.7a) and (4.9), we obtain

$$F_{\mathcal{D}} = -\frac{i}{16}P^{\sigma}_{\mu}\bar{P}^{\nu}_{\lambda}dJ^{\mu}_{\nu} \wedge dJ^{\lambda}_{\sigma} = \frac{1}{2}\Omega_{S}, \quad (4.10)$$

where Ω_S is given by (3.10). It is well known (see, e.g., Ref.5), that the curvature of the bundle $K^{1/2} \to S$ is equal to $-\frac{1}{2}\Omega_S$. Therefore, if we add to the operator $\mathcal{D}^{\nu}_{\mu}$ a connection in the bundle $K^{1/2} \to S$ which depends only on J^{ν}_{μ}, then the modified operator $\tilde{\mathcal{D}}^{\nu}_{\mu}$ will preserve the holomorphic structure of the bundle $\tilde{L}\otimes K^{1/2} \to L^{+}(R^{2n})$. The explicit form of this connection in terms of τ_1 and τ_2 from (3.3) is well-known (see, e.g , Refs.18 and 5). But it is unknown in terms of J^{μ}_{ν}, that is why we shall consider below in all formulae the operators $\mathcal{D}^{\nu}_{\mu}$. Our consideration will be true for the "corrected operators" $\tilde{\mathcal{D}}^{\nu}_{\mu}$, too.

Notice, that because (4.9b) depends only on J, the corresponding to (4.9b) global transformations will multiply all functions Ψ from $\tilde{H}$ by $\exp(i\varphi(J))$. Thus, we have the following result.

Theorem 4.3 : *The operator $\mathcal{D} = dJ^{\mu}_{\nu}\mathcal{D}^{\nu}_{\mu}$ preserves the projective space $P(\tilde{H})$ of holomorphic sections of the bundle $\tilde{L} \to L^{+}(R^{2n})$.*

Following Hitchin[19], we may make all further considerations for the space $P(\tilde{H})$.

In virtue of translation invariance of complex structures J on R^{2n}, the bundle $L^+(R^{2n}) \to R^{2n}$ is trivial: $L^+(R^{2n}) = R^{2n} \times S$. That is why we may define a projection ρ: $L^+(R^{2n}) \to S$, $\rho(x, J) = J$. Because of this, in the space $\tilde{H}$ of holomorphic sections of the bundle $\tilde{L} \to L^+(R^{2n})$ and in the projective space $P(\tilde{H})$ there exists the structure of a complex vector bundle over S:

$$\tilde{H} = \bigcup_{J\in S} H_J \to S \qquad (4.11a)$$

$$P(\tilde{H}) = \bigcup_{J\in S} P(H_J) \to S \qquad (4.11b)$$

with the fibres H_J in points $J \in S$ for $\tilde{H}$ and the fibres $P(H_J)$ in points $J \in S$ for $P(\tilde{H})$. Operator $\mathcal{D} = dJ^\mu_\nu \mathcal{D}^\nu_\mu$ was interpreted as the projectively flat connection in the bundle (4.11a) or as the flat connection in the bundle (4.11b).[19–21] It is beautiful and correct interpretation, because $\mathcal{D}$ contains the derivatives of the first order on coordinates of the base S of the bundles (4.11), and the terms $P^\sigma_\mu P^\gamma_\lambda \nabla_\sigma \nabla_\gamma$, which are the part of the generators of the symplectic group $Sp(2n, R)$ acting in the space of sections $\Gamma(S, \tilde{H})$ and $\Gamma(S, P(\tilde{H}))$ of the bundles (4.11). Therefore $F_\mathcal{D}$ in (4.10) is the curvature of this connection.

Having the flat connection $\tilde{\mathcal{D}}$, we can introduce a space $\mathcal{F}$ of covariantly constant sections of the bundle (4.11a):

$$\mathcal{F} = \{\Psi \in \Gamma(S, \tilde{H}) : \ \tilde{\mathcal{D}}\Psi = 0\}. \qquad (4.12)$$

This space is isomorphic to the space H_0 of quantization and may be used as definition of the space of quantization in the case when there is not a single natural choice of the complex structure, but a preferred family S. Analogously, with the help of connection $\mathcal{D}$ (without the metaplectic correction) we can introduce a space $P(\mathcal{F})$ of covariantly constant sections of the bundle (4.11b):

$$P(\mathcal{F}) = \{\Psi \in \Gamma(S, P(\tilde{H})) : \ \mathcal{D}\Psi = 0\}. \qquad (4.13)$$

This space is isomorphic to the projective space $P(H_0)$ of quantization (rays in the Hilbert space H_0).

V. MARSDEN-WEINSTEIN REDUCTION AND G-INVARIANT POLARIZATIONS

On a symplectic manifold M with a 2-form ω for two arbitrary functions f and h one can define a Poisson bracket $\{f, h\} = \omega(X_f, X_h)$. Here a vector field X_f is defined by the formula $X_f \rfloor \omega = -df$, where $X \rfloor \omega$ denotes the contraction of X with ω. A correspondence $f \to X_f$ maps a Lie algebra $C^\infty(M)$ of functions on M (with the Poisson bracket) to the Lie algebra of the Hamiltonian vector fields on M (with the ordinary commutator).

Let G *be a connected Lie group embedded into the group of the symplectomorphisms* of M. Let $\mathcal{G}$ be the Lie algebra of G. Then to each element $\xi \in \mathcal{G}$ one may correspond a symplectic vector field X_ξ on M. The action of G on M is called a Hamiltonian action if to each vector field X_ξ ($\xi \in \mathcal{G}$) there corresponds a function $\varphi_\xi \in C^\infty(M)$, such that

$$X_\xi \rfloor \omega = -d\varphi_\xi \tag{5.1}$$

Let $\mathcal{G}^*$ be a space dual to $\mathcal{G}$. Using functions φ_ξ on M one may define an AdG-equivariant momentum map φ: $M \to \mathcal{G}^*$ by the formula

$$< \varphi(x), \xi > = \varphi_\xi(x),$$

where $\xi \in \mathcal{G}$, $\varphi(x) \in \mathcal{G}^*$.

Let us consider a constraint set $M_0 = \varphi^{-1}(0) = \{x \in M : \varphi_\xi = 0, \ \forall \xi \in \mathcal{G}\}$. We suppose that G acts freely on M_0. In such a situation the reduced phase space M_G is obtained as the quotient[22]

$$M_G = M_0/G. \tag{5.2}$$

For the description of quantization of M_G and conditions under which M_G will be a manifold, see Refs.6,23,24. It may be shown[22,23] that there is a natural symplectic structure ω_G on the space M_G.

There is a canonical representation of the Lie algebra $\mathcal{G}$ on smooth sections of L given by the operators:

$$s(\varphi_\xi) = \nabla_{X_\xi} + i\varphi_\xi, \tag{5.3}$$

where $\varphi_\xi \in C^\infty(M)$ and X_ξ correspond to $\xi \in \mathcal{G}$ (see (5.1)). We suppose that there exists a global action of G on L such that the induced action of $\mathcal{G}$ is given by (5.3).

Consider the submanifolds $M_0 = \varphi^{-1}(0)$ and $M_G = M_0/G$ in M. Let χ: $M_0 \to M_G$ be the projection map and η: $M_0 \to M$ be the inclusion map.

Theorem 5.1 : *There is unique line bundle (L_G, ∇_G) with connection ∇_G on M_G such that*

$$\chi^* L_G = \eta^* L \quad \text{and} \quad \chi^* \nabla_G = \eta^* \nabla.$$

The curvature of the connection ∇_G is the symplectic form ω_G.

For proof see Ref.6.

Since the Hermitian inner product $<,>$ is G-invariant, there is a unique Hermitian inner product $< , >_G$ on L_G such that $\chi^* < , >_G = \eta^* < , >$. Thus L_G, ∇_G and $< , >_G$ are prequantum data on M_G.

Let F be a polarization of M. It is clear that we may associate with F a polarization F_G of the reduced space M_G if and only if the polarization F is invariant with respect to the action of the group G.[6–12] In particular, in Ref.6 the following theorem has been proved:

Theorem 5.2 : *Let G be a connected compact Lie group, M a Hamiltonian G-space and F a G-invariant, positive definite Kähler polarization of M. Then there is canonically associated with F a positive definite polarization F_G of the reduced space M_G.*

Having polarizations F and F_G, one may introduce the following spaces

$$H_F^{\mathcal{G}} = \{\psi \in H_F : \ s(\varphi_\xi)\psi = 0, \ \forall \xi \in \mathcal{G}\}$$

$$H_{F_G} = \left\{\phi \in \mathcal{L}^2(M_G, L_G) : \ X \rfloor \nabla_G \phi = 0, \ \forall X \in \Gamma(M_G, F_G)\right\}$$

In Refs.6–12, it has been proved that these spaces are isomorphic as vector spaces: $H_F^{\mathcal{G}} \cong H_{F_G}$. We shall not discuss here the more difficult questions connected with the introduction of an inner product on $H_F^{\mathcal{G}}$ and H_{F_G} (see Refs. 5–12).

Note that the interesting example of the combination of ideas from Sec.IV and the Marsden-Weinstein reduction is given by Chern-Simons theory in (1+2) dimensions.[19-21] In this theory the initial space M is the space of all gauge fields in principal bundle (with structure group K) over a Riemanian surface of genus g. Complex structure on the infinite dimensional space of connections is induced from the complex structure on the Riemannian surface parametrized by $(6g-6)$ dimensional Teichmüller space T_g when $g > 1$. Symmetry group G of the theory is the infinite dimensional group of gauge transformations. This group preserves the complex structure on the space of all connections, and after the Marsden-Weinstein reduction the theory is described by the finite dimensional Kähler manifold M_G, on which the family of complex structures, parametrized by the space T_g, are defined. There is no preferable point in T_g, therefore we should introduce the bundle of the Fock spaces over T_g with the (projectively) flat connection, permitting one to identify all Fock spaces corresponding to different choices of complex structure.

Generally speaking, it is difficult to properly correlate the quantization of the phase space M and the reduced phase space M_G. In Refs.6–12, it was accomplished by requiring that the auxiliary structures on (M,ω) necessary for quantization (in particular – polarization F) be G-invariant. Then they can be projected to compatible quantization structures on (M_G,ω_G). But the condition of G-invariance of polarization F does not always take place and we shall consider the possibility of its weakening.

VI. QUANTIZATION OF THE SPACE (R^{2n},ω) WITH QUADRATIC FIRST-CLASS CONSTRAINTS

As an example of the Marsden-Weinstein reduction, we shall consider the reduction of the space (R^{2n},ω). This example is very important because it is known[25] that every symplectic manifold (M,Ω) with Ω of finite integral rank can be realized as a reduction of some (R^{2n},ω). Thus, (R^{2n},ω) is universal for symplectic geometry insofar as reduction is concerned.

Let G be an arbitrary connected subgroup of Lie group $Sp(2n,R)$, which acts on (R^{2n},ω). Denote by $T_a = (T_a{}^\mu_\nu)$ the constant matrices of the generators of representation of the Lie group G in the space R^{2n}:

$$[T_a,T_b] = f^c_{ab}T_c \iff T_a{}^\mu_\lambda T_b{}^\lambda_\nu - T_b{}^\mu_\lambda T_a{}^\lambda_\nu = f^c_{ab}\, T_c{}^\mu_\nu ,$$

where f^c_{ab} are the structure constants of G, $a,b,... = 1,...,m = dimG$. Let us also define the realization of the Lie algebra $\mathcal{G}$ of the Lie group G as a subalgebra with the generators X_a in the Lie algebra of Hamiltonian vector fields on (R^{2n},ω):

$$X_a = -T_a{}^\mu_\nu x^\nu\partial_\mu \implies [X_a,X_b] = f^c_{ab}X_c. \tag{6.1}$$

It is easy to see that

$$\mathcal{L}_{X_a}\omega_{\mu\nu} \equiv X_a\omega_{\mu\nu} + \omega_{\lambda\nu}X^\lambda_{a,\mu} + \omega_{\mu\lambda}X^\lambda_{a,\nu} = 0$$

because $T_a \subset sp(2n,R)$, and therefore $T_{a\mu\nu} \equiv T^\lambda_{a\mu}\omega_{\lambda\nu} = T^\lambda_{a\nu}\omega_{\lambda\mu} \equiv T_{a\nu\mu}$.

Let us correspond functions φ_a to the Hamiltonian vector fields X_a by formula (5.1). We have

$$\varphi_a = \frac{1}{2}T_{a\mu\nu}x^\mu x^\nu. \tag{6.2}$$

$$\{\varphi_a, \varphi_b\} = X_a\varphi_b = \omega^{\mu\nu}\partial_\mu\varphi_a\partial_\nu\varphi_b = f^c_{ab}\varphi_c \tag{6.3}$$

and $m = dimG < n$ constraints $\varphi_a \doteq 0$. $a = 1, \dots, m$, define the submanifold $\varphi^{-1}(0)$ in R^{2n} (constraint set). In virtue of (6.3), the Hamiltonian vector fields X_a are tangential to $\varphi^{-1}(0)$. Assuming that for all $x \in \varphi^{-1}(0)$ the stabilizer group of x (a discrete subgroup of G) is trivial, from the description in Sec.V we obtain the reduced phase manifold of G-invariant states of the system as the quotient

$$R^{2n}_G = \varphi^{-1}(0)/G.$$

Now, on R^{2n} let us choose some complex structure J from $S = Sp(2n, R)/U(n)$. Calculate the Lie derivative $\mathcal{L}_{X_a}$ of J^μ_ν:

$$\mathcal{L}_{X_a}J^\mu_\nu \equiv X^\sigma_a J^\mu_{\nu,\sigma} + J^\mu_\sigma X^\sigma_{a,\nu} - J^\sigma_\nu X^\mu_{a,\sigma} = T^\mu_{a\sigma}J^\sigma_\nu - J^\mu_\sigma T^\sigma_{a\nu} = [T_a, J]^\mu_\nu. \tag{6.4}$$

It is important that although the Lie group G does not preserve the fixed complex structure J, but it preserve the family S of the complex structures on (R^{2n}, ω) because $G \subset Sp(2n, R)$.

It turns out that for φ_a from (6.2) the operators $s(\varphi_a) = \nabla_{X_a} + i\varphi_a$ (acting in the space $\mathcal{L}^2(R^{2n}, L)$) coincide with the vector fields X_a: $s(\varphi_a) = X_a$. Therefore we should define the lift $X_a \to \tilde{X}_a$ of the vector fields X_a on the space (R^{2n}, ω) to the vector fields $\tilde{X}_a$ on the space $L^+(R^{2n})$ described in Sec.III. This lift will be defined from the condition of preserving by the lifted vector fields $\tilde{X}_a$ of the antiholomorphic part $\tilde{\nabla}^{(0,1)}$ of the connection $\tilde{\nabla}$ in the bundle $\tilde{L} \to L^+(R^{2n})$ which have been introduced in Sec.IV. In this case the lifted group G will transform a holomorphic section Ψ of the bundle $\tilde{L} \to L^+(R^{2n})$ into the holomorphic sections. Thus, the explicit form of the generators $\tilde{X}_a$ is defined from the conditions

$$[\tilde{X}_a, \bar{P}^\nu_\mu\nabla_\nu]\Psi = [\tilde{X}_a, \bar{P}^\sigma_\mu\frac{\partial}{\partial J^\sigma_\nu}]\Psi = 0,$$

where Ψ is any holomorphic section of the bundle $\tilde{L} \to L^+(R^{2n})$ (for definition see (4.5)).

Proposition 6.1: Vector fields $\tilde{X}_a = X_a + [J, T_a]^\alpha_\beta\frac{\partial}{\partial J^\alpha_\beta}$ on $L^+(R^{2n})$ are infinitesimal automorphisms of the complex structure $\mathcal{J}$ on $L^+(R^{2n})$. Fields $\tilde{X}_a$ preserve the holomorphic structure in the bundle $\tilde{L} \to L^+(R^{2n})$.

Proof: It is relatively easy to check that

$$\mathcal{L}_{\tilde{X}_a}J^\mu_\nu \equiv \tilde{X}_a J^\mu_\nu + J^\mu_\sigma\tilde{X}^\sigma_{a,\nu} - J^\sigma_\nu\tilde{X}^\mu_{a,\sigma} = 0, \tag{6.5a}$$

$$\mathcal{L}_{\tilde{X}_a}J^{\mu\lambda}_{\nu\sigma} \equiv \tilde{X}_a J^{\mu\lambda}_{\nu\sigma} + J^{\mu\alpha}_{\nu\beta}\frac{\partial}{\partial J^\sigma_\lambda}\tilde{X}_a{}^\beta_\alpha - J^{\alpha\lambda}_{\beta\sigma}\frac{\partial}{\partial J^\alpha_\beta}\tilde{X}_a{}^\mu_\nu = 0, \tag{6.5b}$$

$$[\tilde{X}_a, \bar{P}^\nu_\sigma\nabla_\nu] = T_a{}^\nu_\sigma\bar{P}^\lambda_\nu\nabla_\lambda = T_a{}^\nu_\alpha\nabla^{(0,1)}_\nu; \tag{6.6a}$$

$$[\tilde{X}_a, \bar{P}^\sigma_\mu\frac{\partial}{\partial J^\sigma_\nu}] = T_a{}^\lambda_\mu\bar{P}^\sigma_\lambda\frac{\partial}{\partial J^\sigma_\nu} - T_a{}^\nu_\lambda\bar{P}^\sigma_\mu\frac{\partial}{\partial J^\sigma_\lambda} = T_a{}^\lambda_\mu\partial^{(0,1)}{}^\nu_\lambda - T_a{}^\nu_\lambda\partial^{(0,1)}{}^\lambda_\mu. \tag{6.6b}$$

Formulae (6.5) mean that $\mathcal{L}_{\tilde{X}_a}\mathcal{J} = 0$. From (6.6) it is obvious that the lifted vector fields $\tilde{X}_a$ preserve the space $\tilde{H}$ of holomorphic sections of the bundle $\tilde{L} \to L^+(R^{2n})$.

Remark: Notice that if $[J, T_a] = 0$ for some fixed complex structure J, then $T_a \in u_J(n)$. But from this it does not follow that $[J', T_a] = 0$, where J' is another complex structure from $S = Sp(2n, R)/U(n)$.

Now we know the action of the group G in the space $\tilde{H}$ of holomorphic sections of the bundle $\tilde{L} \to L^+(R^{2n})$, therefore we can define a G-invariant subspace $\tilde{H}^G$ in the space $\tilde{H}$. By definition

$$\tilde{H}^G = \{\Psi \in \tilde{H} : \quad \tilde{X}_a \Psi = 0,\ a = 1, ..., m = dimG\}. \tag{6.7}$$

This space is not the space of quantization which may be corresponded to the reduced phase space $R^{2n}_G = \varphi^{-1}(0)/G$, because functions $\Psi \in \tilde{H}^G$ depend on extra variable $J \in S$. But because of the translational invariance of the complex structures J on (R^{2n}, ω) the space $L^+(R^{2n})$ for (R^{2n}, ω) has a structure of double fibration:

$$\begin{array}{ccc} & L^+(R^{2n}) & \xrightarrow{\ p\ } & S \\ \pi & \downarrow & & \\ & R^{2n} & & \end{array} \tag{6.8}$$

That is why in the space $\tilde{H}$ there exists the structure of the bundle $\tilde{H} \to S$, which has been described in Sec.IV. In this bundle the projectively flat connection $\mathcal{D}$ has been introduced. If the action of the group G in $\tilde{H}$ preserves this connection (or, in other words, $[\tilde{X}_a, \mathcal{D}^\nu_\mu] \sim \mathcal{D}^\nu_\mu$) then this action can be pushed down to the action in the space $P(\mathcal{F})$ of covariantly constant with respect to $\mathcal{D}$ sections of the bundle $P(\tilde{H}) \to S$.

Proposition 6.2: The action of the group G preserves the connection $\mathcal{D}$ in the bundle $P(\tilde{H}) \to S$.

Proof: Let us calculate the commutator $[\tilde{X}_a, \mathcal{D}^\nu_\mu]$. It is not hard to verify that

$$[\tilde{X}_a, \mathcal{D}^\nu_\mu] = T_a{}^\lambda_\mu \mathcal{D}^\nu_\lambda - T_a{}^\nu_\lambda \mathcal{D}^\lambda_\mu. \tag{6.9}$$

From (6.9) it is easy to see that $[\tilde{X}_a, \mathcal{D}^\nu_\mu]\Psi = 0$ on the functions Ψ from the space $P(\mathcal{F})$ (see (4.5) and (4.13)).

Thus, the action of the group G with the generators $\tilde{X}_a$ transforms the covariantly constant with respect to $\mathcal{D}$ sections of the bundle $P(\tilde{H}) \to S$ to the covariantly constant ones. Therefore we can introduce the G-invariant subspace $P(\mathcal{F})^G$ in the space $P(\mathcal{F})$:

$$P(\mathcal{F})^G = P(\mathcal{F}) \cap \tilde{H}^G = \{\Psi \in P(\tilde{H}) : \ \mathcal{D}\Psi = 0,\ \tilde{X}_a\Psi = 0,\ a = 1, ..., dimG\}. \tag{6.10}$$

The space $P(\mathcal{F})^G$ will be the projective space of quantization associated with the reduced phase space $R^{2n}_G = \varphi^{-1}(0)/G$. We shall not discuss here the introduction of a Hermitian inner product in $P(\mathcal{F})^G$, which may be induced from the Hermitian inner product in $\tilde{H}$ in a natural way.

Note that the vector fields $\tilde{X}_a$, corresponding to the constraints φ_a, contain the derivatives with respect to J^μ_ν. Components of J^μ_ν may be interpreted as additional "times" and the described above approach is connected with the "multitemporal" approach, developed in Ref.26. Our approach also generalizes the approach to quantization of systems with quadratic first-class constraints, developed in Ref.27, on the case when the symmetry group G does not preserve a polarization.

VII. ON QUANTIZATION OF GENERAL MANIFOLDS (M,ω)

Now we shall discuss the generalization of the quantization scheme of the space (R^{2n},ω_0) described in Sec.VI on arbitrary symplectic manifolds (M,ω). As before, we shall consider the positive Kähler polarizations of M and the bundle $\pi : L^+(M) \to M$ of the (almost) complex structures over M (see Sec.II). Let $L \to M$ be the prequantization bundle over M with connection ∇ having the curvature equal to ω. We have to define the pull-back bundle $\tilde{L} = \pi^* L \to L^+(M)$ and provide it with the complex structure $\mathcal{J}$. Using it, we are able to introduce the space $\tilde{H}$ of holomorphic with respect to $\mathcal{J}$ sections of the bundle $\tilde{L} \to L^+(M)$.

As in the Riemannian case, taking a symplectic connection D on M, we can always provide $L^+(M)$ with a natural *almost* complex structure $\mathcal{J}$.[28–32] Unfortunately, this almost complex structure is almost never integrable (it is integrable $\iff$ M is conformally symplectic flat[28,29,32]). It is therefore appropriate to seek subbundles of $L^+(M)$ picked out by the geometry of M in the hope that some of these are complex manifolds. One way to do this is to restrict the holonomy of M and consider those elements of $L^+(M)$ that are compatible with the holonomy of M.[28,29,31]

So let our $2n$-dimensional symplectic manifold (M,ω) admit a connection D with holonomy group $K \subset Sp(2n,R)$. Let $\mathcal{P}(M,K) \to M$ denote the holonomy bundle, i.e. the reduction of symplectic frame bundle $\mathcal{R}(M,Sp(2n,R))$. The typical fibre $S = Sp(2n,R)/U(n)$ of $L^+(M)$ decomposes into a disjoint union of K-orbits and $L^+(M)$ decomposes into a disjoint union of subbundles, each one associated to $\mathcal{P}$ with such an orbit as typical fibre. We choose a K-invariant symmetric submanifold Q of S, with a K-invariant complex structure J_Q. We denote by $Z = \mathcal{P} \times_K Q$ the associated bundle with fibre Q. Thus we define the symplectic twistor bundle of M with the holonomy group K as the subbundle

$$\pi : Z \to M$$

in $L^+(M)$ with complex fibres Q.

Conditions of the integrability of the almost complex structure on Z are more weak than on $L^+(M)$, and in Refs. 28–31 one may find a number of examples of the manifolds M which are not conformally flat and to which the twistor spaces Z with the integrable complex structure $\mathcal{J}$ correspond. Namely, in Ref.28 it has been shown that the almost complex structure $\mathcal{J}$ on Z is integrable if the curvature R^D and the torsion T^D of the connection D on M satisfy the equations

$$P_x T^D_x(\bar{P}_x X, \bar{P}_x Y) = 0, \qquad (7.1a)$$

$$P_x R^D_x(\bar{P}_x X, \bar{P}_x Y)\bar{P}_x = 0, \qquad (7.1b)$$

for all $J_x \in Q_x = \pi^{-1}(x)$ and $X, Y \in T_x M$. Here P and $\bar{P}$ are the associated projectors onto $\pm i$ eigenspaces of J. For examples of the manifolds, connection D on which satisfies condition (7.1), see Refs. 29–31.

Suppose that a manifold (M,ω) is such that the almost complex structure $\mathcal{J}$ on the associated with it twistor bundle Z is integrable. Thus we can define the space

$$\tilde{H} = \{\Psi \in \Gamma(Z,\tilde{L}) : \frac{1}{2}(1 + i\mathcal{J})\tilde{\nabla}\Psi = 0\}, \qquad (7.2)$$

of the holomorphic with respect to $\mathcal{J}$ sections of the bundle $\tilde{L} \to Z$. In (7.2) we denote by $\tilde{\nabla} = \pi^*\nabla$ the pull-back of the connection ∇ to $\tilde{L}$. Now we have to define the bundle $\tilde{H} \to Q$ with the (projectively) flat connection. Globally it is possible only if on (M, ω) there exists a family of covariantly constant with respect to symplectic connection D complex structures J parametrized by the space Q. This imposes strong restrictions on the geometry of the manifold (M, ω). But such manifolds exist and we shall call them the self-dual manifolds. Now we shall describe an important class of such manifolds considered in Refs. 33, 34, 31.

VIII. MANIFOLDS WITH THE GRASSMANN-SPINOR STRUCTURE

Let our $2n$-dimensional symplectic manifold M admits a connection with a holonomy group $G_1 \times G_2 \subset Sp(2n, R)$, i.e. the holonomy group splits into a product of two normal subgroups G_1 and G_2. Let $\mathcal{P}(M, G_1 \times G_2) \to M$ denote the holonomy bundle, i.e. the reduction of the symplectic frame bundle $\mathcal{R}(M, Sp(2n, R))$. Thus $G_1 \times G_2$ is a connected linear group of transformations of the space $V = R^{2n}$ and $\mathcal{P} \to M$ is a $G_1 \times G_2$-structure.[35]

Let the vector space $V^C = C^{2n}$ may be represented as a tensor product $V^C = E_0 \otimes N_0$, where E_0 and N_0 are some complex representation space of G_1 and G_2 respectively. In addition, group G_1 acts trivially on the vector space N_0 while group G_2 acts trivially on the vector space E_0. The $G_1 \times G_2$-module V is identified with the set of fixed points of some antilinear involution σ : $\sigma^2 = 1$ (*real structure*[36]).

Now one may introduce the following associated with $\mathcal{P}$ vector bundles:

$$E = \mathcal{P} \times_{G_1 \times G_2} E_0, \quad N = \mathcal{P} \times_{G_1 \times G_2} N_0 \tag{8.1}$$

with fibres $E_x \simeq E_0$ and $N_x \simeq N_0$ at each point $x \in M$. From condition $T^C_x M \simeq E_x \otimes N_x$ it follows that we have the isomorphism of the complexified tangent bundle $T^C M$ over M and of the tensor product $E \otimes N$ of the bundles E and N:

$$T^C M \simeq E \otimes N. \tag{8.2}$$

Such manifolds have been considered in Refs.33,34,31 and called the manifolds with grassmann-spinor structure (GS-manifolds in short). Quaternionic Kähler manifolds M with $G_1 = Sp(1)$ and $G_2 = Sp(m)$ ($\dim M = 2n = 4m$) give the simplest example of such manifolds.[37,38]

Definition:[31] *A connected linear Lie group $G_1 \subset GL(E_0)$ is called a group of twistor type if its Lie algebra $\mathcal{G}_1$ has an element J_1 with $J_1^2 = -1$.*

Remark: In other words, J_1 is a complex structure in the vector space E_0 that is considered as a real manifold with $dim_R E_0 = 2dim_C E_0$.

We fix such an element J_1 and denote by $\mathcal{K}_1$ (by $\mathcal{Q}_1$) the subspace of elements $X \in \mathcal{G}_1$ that commute (anticommute) with J_1. Then $\mathcal{Q}_1 = [J_1, \mathcal{G}_1]$ and

$$\mathcal{G}_1 = \mathcal{K}_1 \oplus \mathcal{Q}_1 \tag{8.3}$$

is a symmetric decomposition of the Lie algebra $\mathcal{G}_1$. The left multiplication by J_1 defines an ad$\mathcal{K}_1$-invariant complex structure $J_{\mathcal{Q}_1}$ in $\mathcal{Q}_1$:

$$J_{\mathcal{Q}_1} : X \to J_1 X, \quad X \in \mathcal{Q}_1$$

The symmetric decomposition (8.3) corresponds to the affine symmetric space

$$Q = (AdG_1)J_1 = G_1/K_1,$$

where $K_1 = \mathcal{Z}_{G_1}(J_1)$ is the centralizer of J_1 in the group G_1. The operator J_{Q_1} defines an invariant complex structure J_Q on Q. We say that $Q = G_1/K_1$ is the complex symmetric space associated with a group G_1 of twistor type and a complex structure $J_1 \subset \mathcal{G}_1$. Many examples of such groups one may find in Refs.30,31, but the classification of all semisimple linear groups of twistor type is an open problem.

The Lie algebra of $G_1 \times G_2$ can be represented in the form $\mathcal{G}_1 \otimes 1 \oplus 1 \otimes \mathcal{G}_2 = \mathcal{G}_1 \oplus \mathcal{G}_2$ and the complex structure J_1 in E_0 defines the complex structure operator

$$J = J_1 \otimes 1 \tag{8.4}$$

in the space $V = \frac{1}{2}(1+\sigma)(E_0 \otimes N_0)$.

The group G_1 acts on $Q = G_1/K_1 = G_1 \times G_2/K_1 \times G_2$ by the left translations and G_2 has trivial action on Q. We introduce the *twistor bundle* of M as the bundle $Z \to M$ associated with the principle bundle $\mathcal{P}(M, G_1 \times G_2)$:

$$\pi : Z = \mathcal{P} \times_{G_1 \times G_2} Q \to M.$$

Sections of the bundle $Z \to M$ are identified with almost complex structures on M.

Choose a connection form θ: $T\mathcal{P} \to (\mathcal{G}_1 \oplus \mathcal{G}_2)$ in the $G_1 \times G_2$-structure $\mathcal{P} \to M$. We denote by D the connection (associated with θ) on TM, T^D and R^D its torsion and curvature. Because $T^C M = E \otimes N$, the connection D may be represented as a tensor product $D = D_1 \otimes 1 \oplus 1 \otimes D_2$ of the connection D_1 in the bundle $E \to M$ and of the connection D_2 in the bundle $N \to M$. Moreover, we have[34] $\Lambda^2 T^{*C} M = S^2E^* \otimes \Lambda^2 N^* \oplus \Lambda^2 E^* \otimes S^2 N^* = \Lambda^2_+ T^{*C} M \oplus \Lambda^2_- T^{*C} M$. In particular, for the curvature tensor we have $R^D = R^D_+ \oplus R^D_-$, where $R^D_+ \in \Lambda^2_+ T^{*C} M$ and $R^D_- \in \Lambda^2_- T^{*C} M$.

The $G_1 \times G_2$-connection D generates the splitting of the tangent bundle TZ into the direct sum

$$TZ = \mathcal{H} \oplus \mathcal{V} \tag{8.5}$$

of horizontal and vertical subbundles of TZ. The space $\mathcal{V}_z$ (the vertical subspace) in $z \in Z$ is tangent to the fibre $\pi^{-1}(\pi(z))$ of $Z \to M$ and $\mathcal{H}_z$ (the horizontal subspace) is some supplementary subspace (characterizing by θ). Recall that the fibres of $Z \to M$ are identified with $Q = G_1/K_1 \simeq Q_x$ $(x = \pi(z))$ and π_* induces an isomorphism from $\mathcal{H}_z$ to $T_{\pi(z)}M$.

By definition, points $z \in Z$ are pairs $z = (x, J_x)$, where $x = \pi(z) \in M$ and J_x is an almost complex structure on T_xM. For each $x \in M$ the operator J_x belongs to Lie algebra $(\mathcal{G}_1 \oplus \mathcal{G}_2)_x$ and has the form $J_x = J_1(x) \otimes 1 \in \mathcal{G}_1(x) \otimes 1$. The isomorphism π_* lifts the almost complex structure J_x to an almost complex structure $\mathcal{J}^h_z$ on $\mathcal{H}_z$. So we obtain an almost complex structure $\mathcal{J}^h$ on horizontal subbundle $\mathcal{H}$. There also exists a natural complex structure $\mathcal{J}^v$ on the vertical subbundle $\mathcal{V}$ which equals to the complex structure J_Q on fibre $Q \simeq Q_{\pi(z)}$. Hence we can define an almost complex structure $\mathcal{J}$ on Z using the decomposition (8.5) by setting

$$\mathcal{J} = \mathcal{J}^h \oplus \mathcal{J}^v \tag{8.6}$$

We have the bundle $Z \to M$ of almost Kähler structures on M and suppose that they are positive. The space Z has a natural almost complex structure $\mathcal{J}$ described above. We consider now the conditions (on $G_1 \times G_2$, $\mathcal{P}$ and θ) under which this almost complex structure $\mathcal{J}$ is integrable. For the GS-manifolds under consideration the following theorem has been proved in Ref.31:

Theorem 8.1: *Let $G_1 \times G_2 \subset GL(E_0 \otimes N_0)$ be a group of twistor type and let $J = J_1 \otimes 1$ be a complex structure on V that belong to the ideal $\mathcal{G}_1 \otimes 1$ of the Lie algebra $\mathcal{G}_1 \oplus \mathcal{G}_2$. Let Z be the twistor space of $G_1 \times G_2$-structure with torsionless connection D, associated with J. Assume that $dim N_0 > 2$ or $dim N_0 = 2$ and the second prolongation of the algebra $\mathcal{G}_1 \subset gl(E_0)$ is equal to zero. Then the almost complex structure $\mathcal{J}$ on the twistor space Z is integrable.*

Recall that the kth prolongation $r^{(k)}$ of a linear subspace $r \subset gl(W)$ is defined as the intersection

$$r^{(k)} = (r \otimes S^k W^*) \cap (W \otimes S^{k+1} W^*).$$

The almost complex structure $\mathcal{J}$ on Z depends on the choice of the connection form θ.[28,29,31] Given another $G_1 \times G_2$-connection with covariant derivative D'_X, we shall introduce the tensor

$$A(X,Y) = D'_X Y - D_X Y$$

Theorem 8.2: *The $G_1 \times G_2$-connections D and D' give the same almost complex structure $\mathcal{J}$ on Z if and only if A satisfies*

$$P_x A(\bar{P}_x X, \bar{P}_x Y) = 0 \tag{8.7}$$

for each x in M, vectors X, Y in $T_x M$ and J_x in Q_x.

For proof see Refs.28, 29, 31.

As a corollary, the integrability conditions (7.1) depend only on the class of $G_1 \times G_2$-connections according to the equivalence relations that the tensor A satisfies condition (8.7). We point out the important result that has been proved by Alekseevsky.[39] It has been shown that if the first prolongation $(\mathcal{G}_1 \otimes 1 + 1 \otimes \mathcal{G}_2)^{(1)}$ of the Lie algebra of the holonomy group $G_1 \times G_2$ is equal to zero then in the bundle $\mathcal{P}(M, G_1 \times G_2) \to M$ over M the unique canonical connection D exists, finding of coefficients for which is reduced to the solving of linear equations. For many semisimple Lie groups G_1 and G_2 (in particular, if $G_1 \subset Sp(E_0)$, $G_2 \in Sp(N_0)$) the condition $(\mathcal{G}_1 \otimes 1 + 1 \otimes \mathcal{G}_2)^{(1)} = 0$ is satisfied and on such manifolds a unique canonical connection D exists. Later we shall consider only such symplectic GS-manifolds M which are provided with the unique canonical connection. For more detailed description of the connection between $\mathcal{J}$ and the choice of the connection form θ see Refs. 28, 29, 31.

We described the GS-manifolds, groups of twistor type and twistor spaces Z. We suppose that conditions of the Theorem 8.1 are satisfied and hence the almost complex structure $\mathcal{J}$ on Z is integrable. Grassmann-spinor manifold is called *self-dual* if the bundle $E \to M$ is flat.[34] Remind that the complexified tangent bundle of GS-manifold M has the form $T^C M = E \otimes N$ and the curvature tensor splits: $R^D = R_+^{D_1} \oplus R_-^{D_2}$. Self-duality is equivalent to the condition $R_+^{D_1} = 0$, i.e. connection along the subspaces $E_x \subset T_x^C M$ is flat. The hyper-Kähler manifolds M^{4m} $(n = 2m)$ give the simplest example of self-dual GS-manifolds.[37,38,31]

Let us consider the subbundle $\mathcal{P}_1(M, G_1) \to M$ in the bundle $\mathcal{P}(M, G_1 \times G_2)$ and the bundle

$$\mathcal{P}_1 \times_{G_1} G_1 \longrightarrow M, \tag{8.8}$$

associated with the principal bundle $\mathcal{P}_1$, where G_1 acts on the left on $\mathcal{P}_1$. Condition of self-duality also means that connection in the bundle (8.8) is flat. In the twistor fibre bundle Z, which may be considered as associated with the principal G_1-bundle $\mathcal{P}_1$:

$$Z = \mathcal{P}_1 \times_{G_1} (G_1/K_1) \to M, \tag{8.9}$$

the flat connection is induced. So the bundle (8.9) has the global parallel (w.r. to D_1 and $D = D_1 \otimes 1 \oplus 1 \otimes D_2$) sections (complex structures $J = J_1(x) \otimes 1$). Therefore the twistor space Z of self-dual GS-manifold M is the product manifold $M \times Q$, where $Q = G_1/K_1$.

IX. QUANTIZATION OF REDUCED SELF-DUAL GRASSMANN-SPINOR MANIFOLDS

The D-parallel complex structures J on M under consideration are parametrized by the space Q. That is why we may define a projection

$$\rho : Z \to Q$$

by corresponding a point $(0, J_{x=0})$ of the manifold $Q = G_1/K_1$ to each point (x, J_x) of the manifold Z, transfering J_x D-parallel to the origin $x = 0$, where all non-equivalent complex structures are parametrized by the manifold Q. We shall denote a point $J \in Q$ and the corresponding complex structure on M by the same letter J. The fibre $\rho^{-1}(J)$ in a point $J \in Q$ can be identified with the complex manifold $M_J = (M, J)$, i.e. M provided with the complex structure corresponding to $J \in Q$.

Thus, we have a double fibration

$$\begin{array}{ccc} Z & \xrightarrow{\rho} & Q \\ \pi \downarrow & & \\ M & & \end{array} \tag{9.1}$$

which is crucial for our considerations. Double fibration similar to (9.1) arises naturally in different problems of the twistor theory. In particular, Hitchin has defined[40] a double fibration of the type (9.1) where M is substituted by a $4m$-dimensional hyper-Kähler manifold and $G_1/K_1 = SU(2)/U(1) = CP^1$. In this case Z appears to be a $(2m+1)$-dimensional complex manifold and points of M are identified with real holomorphic sections of $Z \to CP^1$. The main difference between these constructions and our diagram (9.1) is that the twistor spaces, considered in Ref.40, consist of complex structures compatible with a Riemannian metric while ours consist of complex structures compatible with a symplectic form.

Let us describe the definition of the complex structure on Z by the use of the structure of the double fibration (9.1). Consider the bundles $\pi^{-1}(TM)$ and $\rho^{-1}(TQ)$ over Z

which are the pull-backs of the tangent bundles of M and $Q = G_1/K_1$ respectively. The projections π and ρ generate the natural bundle homomorphisms

$$\pi_* : TZ \to \pi^{-1}(TM), \quad \rho_* : TZ \to \rho^{-1}(TQ).$$

We call the kernel of π_* the vertical subbundle $\mathcal{V}$ of TZ and the kernel of ρ_* the horizontal subbundle $\mathcal{H}$ of TZ. Note that the fibre $\mathcal{V}_z$ in a point $z \in Z$ is identified by ρ_* with the tangent space T_JQ in the point $J = \rho(z) \in Q$ and so has the complex structure J_Q defined on Q. Let $\mathcal{J}^h$ be a complex structure equal at a point $z \in Z$ to the complex structure $\mathcal{J}^h_z$ on $\mathcal{H}_z \approx T_{\pi(z)}M$ given by the point $z = (x, J_x)$. Now we can define a complex structure $\mathcal{J}$ on Z by formula (8.6). Note that this complex structure $\mathcal{J}$ is constructed with the help of canonical symplectic connection D on M and that is why it is unique. The projection ρ: $Z \to Q$ will become a holomorphic map w.r. to this complex structure $\mathcal{J}$.

Self-dual GS-manifold M is the Kähler manifold with the positive complex structure J (parametrized by the space Q) and the unique canonical connection D. For complexified cotangent bundle we have $T^{*C}M = T_J^{*(1,0)} \oplus T_J^{*(0,1)}$. In each point $x \in M$ we consider a one-dimensional space $K_{J(x)} = \bigwedge^n T^{*(1,0)}_{J(x)}$. Now let us introduce the canonical bundle $K_J = \bigwedge^n T_J^{*(1,0)}$, sections of which are $K_{J(x)}$. The existence of the metaplectic structure on (M, ω) is equivalent[3–5] to the existence of a line bundle δ_J over M such that $(\delta_J)^2 = K_J$ and we denote $K_J^{1/2} := \delta_J$.

We have the bundle $Z \to M$ of positive Kähler structures on M. Let us introduce the complex line bundle (cf. Ref.13,5)

$$K^{1/2} \to Z \tag{9.2}$$

over Z, which has fibre $K^{1/2}_{J(x)}$ at $(x, J(x)) \in Z$. The bundle (9.2) defines a metaplectic structure on M. The restriction of $K^{1/2}$ to the fibre of Z over each $x \in M$ is the half-form bundle $K_x^{1/2}$ over the space $Q_x \simeq G_1/K_1$. Moreover, this bundle is the restriction of standard half-form bundle over the space $Sp_x(2n, R)/U_x(n)$ discussed in Sec.IV (see Refs. 13, 5). The pull-back of $K^{1/2}$ to a section $M_J = (M, J)$ of the bundle Z is the half-form bundle $K_J^{1/2}$. The bundle $K^{1/2}$ may be called, following the physical tradition, the ghost bundle of Z (or the restricted half-forms bundle).

Let $L \to M$ be the prequantization line bundle over M with the connection ∇. Denote by $\tilde{L} \to Z$ the pull-back of L to Z. Then $\tilde{L}$ is a holomorphic bundle. Proof repeats word by word the proof of Theorem 4.2 from Sec.IV.

We denote by $\tilde{\nabla}$ the pull-back of the connection ∇ to $\tilde{L}$ and define a $\tilde{\nabla}^{(0,1)}$-operator on section $\tilde{\psi}$ of $\tilde{L} \to Z$ by setting

$$\tilde{\nabla}^{(0,1)}\tilde{\psi} = \frac{1}{2}(I + i\mathcal{J})\tilde{\nabla}\tilde{\psi}.$$

The symplectic structure ω on M being compatible with all Kähler structures on M has the type (1,1) w.r. to any such structure, hence the curvature F_∇ also has the type (1,1) w.r. to any Kähler structure. According to the definition of the complex structure on Z it means that the curvature $F_{\tilde{\nabla}}$ of the pulled-back connection $\tilde{\nabla}$ on $\tilde{L}$ has the type (1,1) w.r. to the complex structure of Z. It follows that

$$(\tilde{\nabla}^{(0,1)})^2\tilde{\psi} = F^{(0,2)}_{\tilde{\nabla}}\tilde{\psi} = 0,$$

i.e. $\tilde{L}$ is holomorphic. Finally, we introduce the product of the bundles $\tilde{L}$ and $K^{1/2}$:

$$\tilde{L} \otimes K^{1/2} \longrightarrow Z$$

Bundles $L \otimes K_J^{1/2}$ and $\tilde{L} \otimes K^{1/2}$ are holomorphic.[4,5]

Denote by ∇' a connection in the bundle $K^{1/2} \to Z$. Then $\tilde{\nabla}' = \tilde{\nabla} \otimes 1 + 1 \otimes \nabla'$ will be a connection in the bundle $\tilde{L} \otimes K^{1/2}$. We introduce the space $\tilde{H}$ of holomorphic sections of the bundle $\tilde{L} \otimes K^{1/2}$:

$$\tilde{H} = \{\Psi \in \Gamma(Z, \tilde{L} \otimes K^{1/2}) : \ \frac{1}{2}(1 + i\mathcal{J})\tilde{\nabla}'_X \Psi = 0, \ \forall \ X \in \Gamma(Z, TZ)\}. \qquad (9.3)$$

Since there exists the projection $p : Z \to Q$ (see (9.1)), in the space $\tilde{H}$ there is the structure of a complex vector bundle over Q:

$$\tilde{H} \to Q \qquad (9.4)$$

with the fibres H_J in $J \in Q$. Here H_J is the space of holomorphic with respect to J sections of the bundle $L \otimes K_J^{1/2} \to M$.

Now, let we have a Hamiltonian action of a connected Lie group G on the self-dual GS-manifold M and this group may be embedded as a subgroup into the group G_1. Denote by $\mathcal{G}$ the Lie algebra of the Lie group G and by X_ξ the Hamiltonian vector fields corresponding to $\xi \in \mathcal{G}$. Generally speaking, these vector fields do not preserve the fixed complex structure J on M, but they preserve the family of complex structures on M parametrized by the space Q. Since M is a self-dual GS-manifold, vector fields X_ξ preserve not only the symplectic structure ω ($\mathcal{L}_{X_\xi}\omega = 0$), but also the canonical connection D on M. It is well-known[36] that in this case the unique lift $X_\xi \to \tilde{X}_\xi$ of the vector fields X_ξ on M to the vector fields $\tilde{X}_\xi$ on the twistor space Z exists, and the lifted vector fields $\tilde{X}_\xi$ preserve the complex structure $\mathcal{J}$ on Z. On $Q = G_1/K_1$ the canonical G_1-invariant symplectic structure Ω_Q exists[36,31] and therefore we have the symplectic structure $\Omega_Z = \omega \otimes 1 + 1 \otimes \Omega_Q$ on $Z = M \times Q$ as on the direct product of manifolds. It is obvious that $\mathcal{L}_{\tilde{X}_\xi}\Omega_Z = 0$. Now, to the vector fields $\tilde{X}_\xi$ one may correspond functions $\tilde{\varphi}_\xi \in C^\infty(Z)$ (see Sec.V) and operators $s(\tilde{\varphi}_\xi) = \tilde{\nabla}'_{\tilde{X}_\xi} + i\tilde{\varphi}_\xi$, acting in the space $\tilde{H}$ of sections of the bundle $\tilde{L} \otimes K^{1/2} \to Z$.

Because $\tilde{X}_\xi$ preserve the complex structure $\mathcal{J}$ on Z, the operators $s(\tilde{\varphi}_\xi)$ will preserve the holomorphic structure of the bundle $\tilde{L} \otimes K^{1/2} \to Z$. Hence, $s(\tilde{\varphi}_\xi)$ preserve the space $\tilde{H}$ of holomorphic sections of this bundle, and we can define a G-invariant subspace $\tilde{H}^G$ in the space $\tilde{H}$:

$$\tilde{H}^G = \{\Psi \in \tilde{H} : \ s(\tilde{\varphi}_\xi)\Psi = 0, \ \forall \xi \in \mathcal{G}\} \qquad (9.5)$$

In other words, the operators $s(\tilde{\varphi}_\xi)$ act in the space of sections of the bundle $\tilde{H} \to Q$ and pick out in it the G-invariant subspace $\tilde{H}^G$.

Now we should define a flat connection $\tilde{\mathcal{D}}$ in the bundle $\tilde{H} \to Q$ and introduce the space of covariantly constant sections of the bundle $\tilde{H} \to Q$:

$$\mathcal{F} = \{\Psi \in \Gamma(S, \tilde{H}) : \tilde{\mathcal{D}}\Psi = 0\}. \qquad (9.6)$$

Detailed description of this connection for arbitrary symplectic manifolds (M, ω), on which a family of complex structures locally exists, are given in the paper of Hitchin.[19] The connection exists and because we have introduced the metaplectic correction (we used $\tilde{L} \otimes K^{1/2}$ instead of $\tilde{L}$), it will be flat. For self-dual GS-manifolds the explicit form of this connection is simplified, but nevertheless its description is rather complicated and we shall not give it here. We hope to simplify the description of $\tilde{\mathcal{D}}$ and to give it in a separate paper. It will be also shown that the operators $s(\tilde{\varphi}_\xi)$ preserve the connection $\tilde{\mathcal{D}}$ and therefore for self-dual GS-manifolds M we can introduce the G-invariant subspace $\mathcal{F}^G$ in the space $\mathcal{F}$:

$$\mathcal{F}^G = \{\Psi \in \Gamma(S, \tilde{H}) : \tilde{\mathcal{D}}\Psi = 0,\ s(\tilde{\varphi}_\xi)\Psi = 0,\ \forall \xi \in \mathcal{G}\}. \tag{9.7}$$

The space $\mathcal{F}^G$ will be the physical Fock space of quantization associated with the reduced phase space M_G for the case when M is the self-dual GS-manifold.

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