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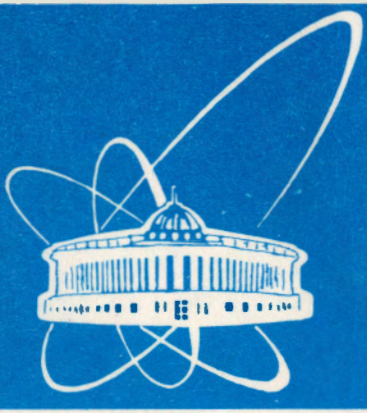

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ON EXACT INTEGRABILITY OF 2D POINCARE GRAVITY

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It is well known problem in two-dimensions how to determine the dynamical description of gravity. One possible way is to use the gauge approach previously developed for four-dimensional case [1]. This leads to description of gravity in terms of zweibeins $e^a = e^a_\mu dx^\mu$ and the Lorentz connection one-form $\omega^a_b = \omega^a{}_{b,\mu} dx^\mu$ as independent variables. The theory described by quadratic in curvature and torsion action was suggested [2] and integrability of equations of motion was investigated in the conformal gauge. The light cone gauge was studied in [3]. The different aspects of quantization of the model was considered in [4]. In [5] was shown that choosing appropriate coordinate system on 2D manifold M^2 one finds the exact solution of the gravitational equations. In this note we give more transparent and strict proof of the exact integrability and describe some issues which were shadowed in previous consideration [5].

In two dimensions the gauge gravity is described in terms of zweibeins $e^a = e^a_\mu dz^\mu, a = 0, 1$. The 2D metric on the surface M^2 has the form $g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab}$ $\eta_{ab} = diag(+1, (-1)^s)$ and Lorentz connection one-form $\omega^a{}_b = \omega \varepsilon^a{}_b$, $\omega = \omega_\mu dz^\mu$ ($\varepsilon_{ab} = -\varepsilon_{ba}$, $\varepsilon_{01} = 1$). For $s = 0$ we have a Euclidean and for $s = 1$ a Minkowskian signature. The curvature and torsion two-forms are: $R = d\omega$, $T^a = de^a + \varepsilon^a{}_b \omega \wedge e^b$.

The dynamics of gravitational variables (e^a, ω) is determined by the action [2,5]:

$$S_{gr} = \int\limits_{M^2} \frac{\alpha}{2} * T^a \wedge T^a + \frac{\beta}{2} * R \wedge R + (-1)^s \frac{\lambda}{4} \varepsilon_{ab} e^a \wedge e^b \qquad (1)$$

where $*$ is the Hodge dualization and α, β, λ are arbitrary constants. The theory (1) with $\beta = \lambda = 0$ was analyzed in [6] and was shown to describe some kind of topological gravity. Here we consider the case when these constants are not zero and fix $\beta = 1$. Generally one may add to (1) the term proportional to R. But this term is boundary one and it does not affect the equations of motion.

It is convenient to consider variables $\rho = *R$ and $q^a = *T^a$. Variation of action (1) with respect to zweibeins e^a and Lorentz connection ω leads

to the following equations of motion [5]:

$$d\rho = -\alpha q^a \varepsilon_{ab} e^b \tag{2}$$

$$\nabla q^a = \frac{(-1)^s}{2\alpha} \Phi(q^2, \rho) \varepsilon^a{}_b e^b, \tag{3}$$

where $\nabla q^a \equiv dq^a + \omega \varepsilon^a{}_b q^b$; here $q^2 = q^a q^b \eta_{ab}$. In (2), (3) the following notation was introduced: $\Phi(q^2, \rho) = \rho^2 + \alpha q^2 - \alpha^2 - \Lambda\alpha$, where $\Lambda = \frac{\lambda}{\alpha} - \alpha$.

It follows from eq.(3) that

$$dq^2 = \frac{(-1)^{s+1}}{\alpha^2} \Phi(q^2, \rho) d\rho \tag{4}$$

It is differential equation for $q^2(\rho)$, the solution is function [5]

$$q^2(\rho) = -\frac{1}{\alpha}(\rho + (-1)^{s+1}\alpha)^2 + \Lambda + \epsilon e^{(-1)^{s+1}\frac{\rho}{\alpha}}$$

where ϵ is the integrating constant.

It should be noted that below analysis is not dependent on the concrete form of functions $\Phi(q^2, \rho)$ and $q^2(\rho)$. In Euclidean case, of course, our analysis is valid not for any ρ but only in the regions where $q^2(\rho) \geq 0$.

The general solution of eqs.(2-3) is one of two types. The first one is the space-time with zero torsion squared $q^2 \equiv 0$. From eqs.(2)-(3) we obtain then that torsion is zero $q^a \equiv 0$ and curvature is constant: $\rho = \pm\sqrt{\lambda}$. The second type of solutions is characterized by that the torsion q^a is not zero identically on M^2. To analyze it let us now consider the following one-form:

$$\xi = \frac{1}{q^2} e^{(-1)^{s+1}\frac{\rho}{\alpha}} q_a e^a \tag{5}$$

It is direct consequence of gravitational equations (2)-(3) that this one-form is closed $d\xi = 0$. Indeed, straightforward calculation gives:

$$d\xi = \frac{e^{(-1)^{s+1}\frac{\rho}{\alpha}}}{q^2}(\nabla q_a e^a + q_a \nabla e^a + (-1)^{s+1}(\frac{1}{\alpha} - \frac{\Phi}{\alpha^2 q^2}) d\rho \wedge q_a e^a) \tag{6}$$

From eq.(2) we obtain:

$$d\rho \wedge (q_a e^a) = \alpha q^2 V \tag{7}$$

where $V = \frac{1}{2} e^a \varepsilon_{ab} e^b$ is volume 2-form and we used the identity $e^b \wedge e^c = (-1)^s \varepsilon^{bc} \frac{1}{2} \varepsilon_{\alpha\beta} e^\alpha \wedge e^\beta$. We also have that $\nabla e^a = (-1)^s q^a V$.

Thus substituting eqs.(3) and (7) into eq.(6) we obtain that

$$d\xi = 0.$$

Now assuming that the 2D space-time M^2 has trivial topology (namely the first cohomology group is trivial: $H^1(M^2) = 0$) there exists globally defined one-valued scalar field ϕ such that

$$\xi = d\phi \tag{8}$$

If $H^1(M^2) \neq 0$, say M^2 is cylinder, such a field ϕ still exists and is just angle variable(which changes in the interval $0 \leq \phi \leq 2\pi$) corresponding to the relevant nontrivial circle. Below we assume M^2 to be topologically trivial or direct product $M^2 = S^1 \oplus R^1$.

From eqs. (3), (5), (8) we obtain that

$$\begin{aligned} q^a \varepsilon_{ab} e^b &= -\frac{1}{\alpha} d\rho \\ q_a e^a &= q^2 e^{(-)^s \frac{\rho}{\alpha}} d\phi \end{aligned} \tag{9}$$

One can see that fields (ρ, ϕ) determine the natural coordinate system on M^2.

Equations (9) are easily solved with respect to the orthonormal basis e^a:

$$e^a = q^a e^{(-1)^s \frac{\rho}{\alpha}} d\phi + \frac{(-1)^s}{\alpha q^2} \varepsilon^a{}_b q^b d\rho \tag{10}$$

and metric $ds^2 = e^a_\mu e^b_\nu \eta_{ab} dx^\mu dx^\nu$ takes the form

$$ds^2 = q^2 e^{(-1)^s \frac{2\rho}{\alpha}} (d\phi)^2 + \frac{(-1)^s}{\alpha^2 q^2} (d\rho)^2, \tag{11}$$

where function $q^2(\rho)$ is found from eq.(4).

The initial system (2-3) is differential equations for zweibeins e^a and the Lorentz connection ω as unknown functions. Let us now find the Lorentz

connection ω. One can do this in two different (but equivalent) ways. The first one is to solve the equation

$$de^a + \varepsilon^a{}_b \omega \wedge e^b = T^a$$

with respect to ω assuming that zweibeins e^a are already known and in coordinates (ρ, ϕ) expressed accordingly to (9).

The other way is to consider the equation

$$q^a \varepsilon_{ab} \nabla q^b = (-1)^{s+1} \frac{\Phi}{2\alpha} q_a e^a \tag{12}$$

obtained by multiplication eq.(3) on $q^b \varepsilon_{ba}$.

Let for definiteness $q^2 > 0$ (for $s = 1$), then one can introduce the new field θ in following way

$$\begin{aligned} q^0 &= q\cos\theta,\ q^1 = q\sin\theta \quad for \quad s = 0 \\ q^0 &= q\cosh\theta,\ q^1 = q\sinh\theta \quad for \quad s = 1 \end{aligned} \tag{13}$$

where $q \equiv \sqrt{q^2}$. Then one has

$$q^a \varepsilon_{ab} dq^b = q^2 d\theta \tag{14}$$

Solving eq.(12) with respect to ω one finally obtains

$$\omega + (-1)^{s+1} d\theta = -\frac{\alpha}{2} (q^2)'_\rho e^{(-1)^s \frac{\rho}{\alpha}} d\phi \tag{15}$$

This completes the proof of exact integrability of eqs.(2-3).

It should be noted that to this moment everything was Lorentz invariant. Under local Lorentz rotations on angle η the Lorentz connection one-form ω and θ transform as follows

$$\omega \to \omega + (-1)^s d\eta,\ \theta \to \theta + \eta \tag{16}$$

So the eq.(15) is indeed Lorentz invariant.

Thus we obtain that in coordinates (ρ, ϕ) solution of equations (2-3) is given by (10) and (15) and depends on an arbitrary field θ. This arbitrariness is just reflection of the underlying local Lorentz symmetry.

Now we may use this freedom and choose the gauge fixing. One can suggest the following choices.

A. $\phi = F(\theta)$.

F is arbitrary analytic and monotonic function. Then $d\phi = B_0(\theta) d\theta$ and from eqs.(10)(15) takes the form

$$\begin{aligned} ds^2 &= q^2 e^{(-1)^s \frac{2\rho}{\alpha}} B_0^2(\theta) d\theta^2 + \frac{(-1)^s}{\alpha^2 q^2} d\rho^2 \\ \omega &= -((-1)^{s+1} + \frac{\alpha}{2} (q^2)'_\rho e^{(-1)^s \frac{\rho}{\alpha}} B_0(\theta)) d\theta \end{aligned} \tag{17}$$

It is the gauge which was used in the [5] and expressions (17) are exactly that of obtained in [5].

B. $\theta = 0$.

This gauge is equivalent to the condition $q^1 = 0$. Then $q = q^0$ and expression for ω is given by

$$\omega = -\frac{\alpha}{2} (q^2)'_\rho e^{(-1)^s \frac{\rho}{\alpha}} d\phi \tag{18}$$

It is worth noting that in this gauge the equations (2-3) are essentially simplified and solution is obtained at once. This gauge seems to be very useful in solving eqs.(2-3) when the coupling to matter is taken into account.

In paper [7] was suggested the general 2D Poincaré gravity

$$S = \int_{M^2} U(\rho, q^2) \frac{1}{2} e^a \varepsilon_{ab} \wedge e^b \tag{19}$$

with the Lagrangian U being arbitrary function of curvature ρ and torsion q^2. Variation of (19) with respect to e^a and ω gives the equations (both for $s = 0$ and $s = 1$):

$$\begin{aligned} d(U'_\rho) &= -2U'_{q^2} q^a \varepsilon_{ab} e^b \\ \nabla(U'_{q^2} q^a) &= \frac{1}{2}(\rho U'_\rho + 2q^2 U'_{q^2} - U) \varepsilon^a{}_b e^b \end{aligned} \tag{20}$$

One can prove the exact integrability of these equations along the same line as before. We will do this in quite different way than in the [7].

Indeed, let us introduce new variables $\bar{q}^a, \bar{\rho}$:

$$\bar{q}^a = V'_{q^2} q^a, \ \bar{\rho} = \frac{1}{2} U'_\rho \tag{21}$$

We assume that transformation $(q^2, \rho) \to (\bar{q}^2, \bar{\rho})$ is correct and relevant Hessian is not zero. This means, in particular, that there exists the inverse transformation such that one gets $q^2 = q^2(\bar{q}^2, \bar{\rho}), \rho = \rho(\bar{q}^2, \bar{\rho})$. Then the eqs.(20) are rewritten in terms of the variables $\bar{q}, \bar{\rho}$ as follows

$$\begin{aligned} d\bar{\rho} &= -\alpha \bar{q}^a \varepsilon_{ab} e^b \\ \nabla \bar{q}^a &= -\frac{1}{2} \Phi(\bar{q}^2, \bar{\rho}) \varepsilon^a{}_b e^b, \end{aligned} \tag{22}$$

Function $\Phi(\bar{q}^2, \bar{\rho})$ is $(U - \rho U'_\rho - 2q^2 U'_{q^2})$ under condition that q^2, ρ are expressed in terms of $\bar{q}^2, \bar{\rho}$. One can see that eqs.(22) take the form (2-3). The above proof of integrability of system (2-3) was made for general function Φ. Really the form of the solution (10), (15) is not dependent on the concrete form of the function Φ. The later influences only on the dependence $\bar{q}^2(\bar{\rho})$ as solution of differential equation $d\bar{q}^2/d\bar{\rho} = \Phi(\bar{q}^2, \bar{\rho})$. So we have some type of universality [8] and the exact solution of eqs.(22) again takes the form (10), (15)

Thus, the 2D Poincaré gravity gives us an unique example when the equations are integrated and the exact general solution takes analytically simple form.

The previous analysis concerned the non-compact manifolds. Let us consider now the Euclidean compact closed two-dimensional differential manifold M^2_g of genus g. It is worth noting that one-form, dual to ξ:

$$\begin{aligned} *\xi &= \frac{1}{q^2} e^{\frac{-\rho}{\alpha}} q^a \varepsilon_{ab} e^b \\ &= -\frac{d\rho}{\alpha q^2(\rho)} e^{-\frac{\rho}{\alpha}} \end{aligned} \tag{23}$$

is obviously closed:

$$d(*\xi) = 0$$

Thus, we obtain that ξ and $*\xi$ are both closed one-forms. For closed compact differential manifold it means that one-form ξ (or/and $*\xi$) is the harmonic one and hence it represents of the first cohomology group of M^2_g: $\xi \in H^1(M^2_g)$. If M^2 is topologically sphere ($g = 0$), i.e. $dim H^1(M^2) = 0$, then $\xi = 0$. This means that torsion is identically zero on M^2. Hence, only solution of the first type ($q^a \stackrel{\circ}{\equiv} 0$ and $\rho = -\sqrt{\lambda}$) is realized.

Generally, one should note that really the form $*\xi$ is exact:

$$*\xi = d\psi$$

where

$$\psi = -\int e^{-\frac{\rho}{\alpha}} \frac{d\rho}{\alpha q^2(\rho)}$$

Since curvature ρ is globally defined one-valued on M^2 function, ψ is one-valued function too. Though, ψ can take infinite values at points where $q^2(\rho) = 0$. Hence $\xi = 0$ for any genus g.

Thus, in general case the following Theorem is valid.

Theorem: Let M^2_g is compact closed two-dimensional differential manifold of genus g. The equations (2), (3) being considered on M^2_g can have only solutions of the first type, i.e. the torsion is identically zero $q^a = 0$ and curvature is constant: $\rho = -\sqrt{\lambda}$ for $g = 0$, $\rho = 0$ for $g = 1$, $\rho = \sqrt{\lambda}$ for $g > 1$.

The value of constant curvature is determined in agree with the Euler number of M^2_g, $\chi = 2(g - 1)$.

However, this doesn't exhaust all possible solutions in the class of closed, not necessary differential, manifolds: the metric (11) can describe compact closed manifolds with conic singularities at points where $q^2(\rho) = 0$. The complete analysis of the Euclidian solutions will be given elsewhere.

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