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I.V.Polubarinov

ON COHERENT STATE REPRESENTATION

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I. INTRODUCTION

I. Many authors (Schrödinger, Wigner, Glauber, Schwinger, Klauder, Sudarshan and others) have elaborated new fruitful approaches exploiting coherent states in many branches of quantum physics. (See refs./1-21/ and further refs.there, see also pioneering articles and some other related refs./22-39/ .) New forms of quantum theory have arisen /1,11,13,19,23,24,30-32/.

Quantum theory may be represented entirely in terms of coherent state expectation values. We call such a formulation the coherent state representation (CSR). Below the relativistic field theory is presented in CSR (we follow refs./21/). Equations of motion for a field operator and for density matrices are written in this representation, and problems on causality and determinism are discussed.

2. Now we define the coherent state representation, the CSR-1, CSR-2, CSR-1' and CSR-2', labelling modifications. One possibility (CSR-1) is to define CSR as such a form of quantum theory, in which every operator is represented only by a set of its coherent state expectation values, i.e., only by its diagonal matrix elements. Non-diagonal elements are superfluous and can be calculated through the diagonal ones.

Other possibilities arise when operators Λ^{ν} (ν being an exponent) are applied to these expectation values (for the definition of Λ see Appendix A). This corresponds to using of decompositions into alternative ordered products. We use only two of the continuum of possible decompositions[13]: the N-product decomposition (ν=0), CSR-1) and the symmetrized product one (ν=-1,CSR-2). CSR-2 distributions coincide, in fact,with the Wigner distributions in a phase space[23] (see Sec.4).

CSR makes the quantum theory language close to the classical theory one: the commuting coordinate and momentum both (in fact, the coherent state expectation values of the coordinate and momentum operators) are used simultaneously. Equations in CSR-1' and CSR-2' , besides these variables, contain the derivatives with respect to them (in quantum field theory those are naturally functional derivatives). The Planck constant $\hbar$ enters into the theory as a factor for these derivatives (like a "coupling constant"). For example, the quantum-mechanical Hamiltonian in CSR-2' is

$$\mathcal{H} = \frac{\left(p - i\frac{\hbar}{2}\frac{\partial}{\partial x}\right)^2}{2m} + V\left(x + i\frac{\hbar}{2}\frac{\partial}{\partial p}\right) , \tag{1}$$

where in a many-dimensional case $x=(\vec{x}_1 \ldots \vec{x}_n), p=(\vec{p}_1 \ldots \vec{p}_n)$ and the summation is implied in the first term. For another examples see point 7 below, in particular, eqs. (45) and (46), and ref.[21d].

3. The coherent states we are interested in are written in quantum mechanics and in the scalar field quantum theory as follows

$$|xp\rangle = e^{i\hbar^{-1}\left(p(t)\hat{x}(t) - x(t)\hat{p}(t)\right)}|0\rangle = e^{i\hbar^{-1}\int dt\, f(t)\hat{x}(t)}|0\rangle , \tag{2}$$

$$|\varphi\rangle = e^{-\hbar^{-1}(\varphi,\hat{\varphi})}|0\rangle , \quad (\varphi,\hat{\varphi}) = \int d^3x\, \varphi(x)\overleftrightarrow{\partial_4}\hat{\varphi}(x) = -i\int d^4x\, J(x)\hat{\varphi}(x), \tag{3}$$
$$\left(\varphi\overleftrightarrow{\partial_4}\hat{\varphi} = \partial_4\varphi\cdot\hat{\varphi} - \varphi\,\partial_4\hat{\varphi}\right),$$

where $\hat{x}(t)$ and $\hat{\varphi}(x)$ are the quantum mechanical coordinate and scalar field free Heisenberg operators, $\hat{p}(t) = m\dot{\hat{x}}(t)$ and $\dot{\hat{\varphi}}(x)$ are the corresponding momentum operators, and

$$x(t) = \langle xp|\hat{x}(t)|xp\rangle , \quad p(t) = \langle xp|\hat{p}(t)|xp\rangle , \quad \varphi(x) = \langle\varphi|\hat{\varphi}(x)|\varphi\rangle , \quad \dot{\varphi}(x) = \langle\varphi|\dot{\hat{\varphi}}(x)|\varphi\rangle$$

are the corresponding "classical" counterparts, the expectation values of the above operators in states $|xp\rangle$ and $|\varphi\rangle$ x). These coordinates and momenta are independent quantities xx), so that the initial values $x(t')$, $p(t')$ or $\varphi(\vec{x},t')$, $\dot{\varphi}(\vec{x},t')$ must be given simultaneously and may be chosen arbitrary at any initial time t' , like in classics.

These initial values xxx) define the coherent state and therefore the coherent state expectation values of all the operators.

In quantum mechanics these quantities are functions of two variables $x(t')$ and $p(t')$ or, equivalently, functionals of one function $f(t)$.

In the scalar field theory they are functionals of two functions $\varphi(\vec{x},t')$ and $\dot{\varphi}(\vec{x},t')$ of 3-argument $\vec{x}$ or, equivalently, functionals of one scalar function $J(x)$ of 4-argument x_μ (the latter is more convenient due to covariance).

With all the conceivable initial values we obtain a complete (in fact, overcomplete) set of the states and all diagonal matrix elements of all the operators, i.e., all the operators in CSR-1. The completeness relations are written as follows

$$\frac{1}{(2\pi\hbar)^3}\int d^3x\, d^3p\, |xp\rangle\langle xp| = \frac{1}{(2\pi\hbar)^3}\int d^3x\, d^3p\, \Lambda^{-1}|xp\rangle\langle xp| = 1 \tag{4}$$

$$\int \delta^2\varphi\, |\varphi\rangle\langle\varphi| = \int \delta^2\varphi\, \Lambda^{-1}|\varphi\rangle\langle\varphi| = 1 \quad \left(\delta^2\varphi = \delta\frac{\varphi(\vec{x},t')}{\sqrt{2\pi\hbar}}\,\delta\frac{\dot{\varphi}(\vec{x},t')}{\sqrt{2\pi\hbar}}\right) \tag{5}$$

The well-known useful property is

$$\langle xp| : \hat{x}(t_1)\ldots\hat{x}(t_n) : |xp\rangle = x(t_1)\ldots x(t_n) , \tag{6}$$

$$\langle\varphi| : \hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n) : |\varphi\rangle = \varphi(x_1)\cdots\varphi(x_n) \tag{7}$$

x) Due to the free equations of motion the exponents in eqs. (I) and (2) and the states themselves are conserved in time.
xx) Unlike the mean squared coordinate and momentum, which are subjected to the uncertainty relation.
xxx) Or values at any other fixed time.

where :: denote the N-product. For other properties see Appendix A.

4. The evolution of the coherent state expectation values for the coordinate and fields in the free case and in the cases of simplest interactions is given by the following equations.

I) The free case

$$x(t) = mD(t-t')\overleftrightarrow{\frac{\partial}{\partial t'}}x(t') \qquad \left(= -m\int ds\, D(t-s)f(s)\right) \tag{8}$$

$$\varphi(x) = i\int d^3x'\, \Delta(x-x')\overleftrightarrow{\partial_4}\varphi(x') \qquad \left(= -\int d^4y\, \Delta(x-y)J(y)\right) \tag{9}$$

$$\psi(x) = -i\int d^3x'\, S(x-x')\gamma_4\psi(x') \qquad \left(= \int d^4y\, S(x-y)\eta(y)\right) \tag{10}$$

(in parentheses the definitions of functions $f(t)$, $J(x)$ and $\eta(x)$ are given).

2) The interaction of the scalar field with an external current $j(x)$ (for the electromagnetic field analogously)

$$\varphi(x) = \varphi(x) + \int_{t'} d^4y\, \Delta_{ret}(x-y)j(y)\,. \tag{11}$$

Here $\varphi(x) = \langle\varphi|\hat\varphi(x)|\varphi\rangle$, $\hat\varphi(x)$ is the Heisenberg scalar field operator.

3) The interaction of the spinor field with an external electromagnetic field $A_\mu(x)$

$$\psi(x) = \psi(x) - ie\int_{t'} d^4y\, S_{ret}(x-y)\gamma_\mu A_\mu(y)\psi(y)\,. \tag{12}$$

Here $\psi(x) = \langle\psi|\hat\psi(x)|\psi\rangle$, $\hat\psi(x)$ is the Heisenberg spinor field operator.

The above equations differ from the corresponding ones for operators only by absence of the ^ . Of course, it is the case for expectation values in any other states (in the spirit of the Ehrenfest theorem). However, only for the coherent state expectation values the above equations become quite classical ones, solutions of which are defined by arbitrary given initial values (e.g., $\varphi(\vec{x},t')$ and $\dot\varphi(\vec{x},t')$, or equivalently $J(x)$ for equations (9) and (11)), the equations themselves and their solutions being of the manifestly causal character. So, the solution of eq. (12) may be written as follows

$$\psi(x) = i\int d^3x'\, S^A_{ret}(x,x')\gamma_4\psi(x')\,. \tag{13}$$

It is remarkable that according to a coherent state expectation value theorem (see below) only these coherent state expectation values and the above equations are equivalent to the original operators and to the equations for them. One can re-establish the operators, using only their coherent state expectation values. Therefore, causality of the above equations exhausts causality of these theories /21a,b/ (the signal velocity does not exceed the light velocity).

5. The coherent state expectation value theorem: every operator is determined by its coherent state expectation values (i.e., only by diagonal matrix elements!). Let us show it in terms of the relativistic quantum field theory /21b/ . Given some operator $\hat Q$ (e.g., S-matrix) by its decomposition into the N-products

$$\hat Q = \sum_{n=0}^{\infty}\int d^4x_1\cdots d^4x_n\, K(x_1\cdots x_n):\hat\varphi(x_1)\cdots\hat\varphi(x_n): \tag{14}$$

Then using eq. (7), we obtain

$$\langle\varphi|\hat Q|\varphi\rangle = \sum_{n=0}^{\infty}\int d^4x_1\cdots d^4x_n\, K(x_1\cdots x_n)\varphi(x_1)\cdots\varphi(x_n)\,. \tag{15}$$

After taking functional derivatives[x] of arbitrary order with respect to $J(x)$ and equating $J=0$, we find

$$\frac{\delta}{\delta J(y_1)}\cdots\frac{\delta}{\delta J(y_n)}\langle\varphi|\hat Q|\varphi\rangle\Big|_{J=0} = (-1)^n n!\int d^4x_1\cdots d^4x_n\, K(x_1\cdots x_n)\Delta(x_1-y_1)\cdots\Delta(x_n-y_n) \tag{16}$$

This is the most general matrix element of $\hat Q$ with n external lines. The theorem is shown, since each operator is exhausted by the set of all its matrix elements with all possible numbers of the external lines (i.e., of the initial and final quanta). For the proof we have made a transition from CSR-1 (the set of all possible coherent state expectation values of $\hat Q$) into the Fock representation (the set of the matrix elements of $\hat Q$

x) For the operator interpretation of these derivatives see eq.(52).

between all possible states with definite numbersof the quanta).

Any operator $\hat{Q}$ and its coherent state non-diagonal matrix elements are expliocitly given in terms of its coherent state expectation values as follows

$$\hat{Q} = :\exp\left(\hat{x}(t')\frac{\partial}{\partial x(t')} + \dot{\hat{x}}(t')\frac{\partial}{\partial \dot{x}(t')}\right)\langle xp|\hat{Q}|xp\rangle_{|x(t')=p(t')=0} =$$

$$= :\exp\left(\hat{x}(\tau)\overleftrightarrow{\frac{\partial}{\partial\tau}}\frac{\delta}{\delta f(\tau)}\right):\langle xp|\hat{Q}|xp\rangle_{|f=0} \tag{I7}$$

$$\langle x_2p_2|\hat{Q}|x_1p_1\rangle =$$

$$=\langle x_2p_2|x_1p_1\rangle\sum_{n=0}^{\infty}\frac{1}{n!}x_{21}(\tau_1)\dots x_{21}(\tau_n)\overleftrightarrow{\frac{\partial}{\partial\tau_1}}\dots\overleftrightarrow{\frac{\partial}{\partial\tau_n}}\frac{\delta}{\delta f(\tau_1)}\dots\frac{\delta}{\delta f(\tau_n)}\langle xp|\hat{Q}|xp\rangle_{|f=0} \tag{I8}$$

$$\hat{Q} = :\exp\int d^3x'\left(\hat{\varphi}(x')\frac{\delta}{\delta\varphi(x')} + \dot{\hat{\varphi}}(x')\frac{\delta}{\delta\dot{\varphi}(x')}\right):\langle\varphi|\hat{Q}|\varphi\rangle_{|\varphi(\vec{y},t')=\dot{\varphi}(\vec{y},t')=0} =$$

$$= :\exp\left(i\int d^3y\,\hat{\varphi}(y)\overleftrightarrow{\partial}_4\frac{\delta}{\delta J(y)}\right):\langle\varphi|\hat{Q}|\varphi\rangle_{|J=0} =$$

$$=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^3y_1\dots d^3y_n:\hat{\varphi}(y_1)\dots\hat{\varphi}(y_n):\overleftrightarrow{\partial}_{14}\dots\overleftrightarrow{\partial}_{n4}\frac{\delta}{\delta J(y_1)}\dots\frac{\delta}{\delta J(y_n)}\langle\varphi|\hat{Q}|\varphi\rangle_{|J=0} \tag{I9}$$

$$\langle\varphi_2|\hat{Q}|\varphi_1\rangle =$$

$$=\langle\varphi_2|\varphi_1\rangle\sum_{n=0}^{\infty}\frac{i^n}{n!}\int d^3y_1\dots d^3y_n\,\varphi_{21}(y_1)\dots\varphi_{21}(y_n)\overleftrightarrow{\partial}_{14}\dots\overleftrightarrow{\partial}_{n4}\frac{\delta}{\delta J(y_1)}\dots\frac{\delta}{\delta J(y_n)}\langle\varphi_2|\hat{Q}|\varphi_1\rangle_{|J=0}. \tag{20}$$

The functions $x_{21}(t)$ and $\varphi_{21}(x)$ and the soalar products $\langle x_2p_2|x_1p_1\rangle$ and $\langle\varphi_2|\varphi_1\rangle$ are given in Appendix A. Equations (I9) and (20) follow from eq. (I6). The last row of eq. (19) gives the exact meaning to the preceeding one, the times being non-equal.

6. The following three classes of theories are of interest:

A) Quantum mechanics of the one-quantum states. The CSR's are constructed by means of the states $|xp\rangle$. The theory may be entirely reformulated in terms of the coherent state expectation values $\langle xp|\hat{Q}|xp\rangle$ of all the operators (CSR-1). They permit one to re-establish $\langle x_2p_2|\hat{Q}|x_1p_1\rangle$, $\langle x''|\hat{Q}|x'\rangle$, $\langle p''|\hat{Q}|p'\rangle$ and so on.

B) Quantum mechanics of the n-quantum states, when the coservation law for the number of quanta are valid. Here the

"CSR's" based on the n-quantum states $|x_1p_1\dots x_np_n\rangle = \hat{a}^+(x_1p_1)\dots\hat{a}^+(x_np_n)|0\rangle$ x) where each of n quanta is in the coherent state of type $|xp\rangle$. Then the theory is exhausted only by diagonal matrix elements $\langle x_1p_1\dots x_np_n|\hat{Q}|x_1p_1\dots x_np_n\rangle$ of all the operators (and, if one likes, one may reestablish $\langle\bar{x}_1\bar{p}_1\dots\bar{x}_n\bar{p}_n|\hat{Q}|x_1p_1\dots x_np_n\rangle$, $\langle x_1''\dots x_n''|\hat{Q}|x_1'\dots x_n'\rangle$ and so on).

To support the statement that it is sufficient to use only diagonal elements in coherent states, let us give the following analogy. In classics a system with n degrees of freedom is characterized by 2n variables such as x_i and p_i ($i = 1 \dots n$). In quantum meohanics eaoh operator $\hat{Q}$ for a similar system is also characterized by 2n variables, for example, $\langle x_1''\dots x_n''|\hat{Q}|x_1'\dots x_n'\rangle$ in the x-representation or $\langle p_1''\dots p_n''|\hat{Q}|x_1'\dots x_n'\rangle$ in the mixed, x-, p-representation xx), and so on. The same is valid for the coherent state expectation values $\langle x_1p_1\dots x_np_n|\hat{Q}|x_1p_1\dots x_np_n\rangle$ too, in contrast to non-diagonal matrix elements, depending on 4n variables. Contrary to other above matrix elements, the expectation values are real quantities like classical ones. These arguments hold in the theories A) and C) too.

C) Relativistic quantum field theory with nonconservation of the number of quanta. The CSR's are constructed out, using the coherent states of field $|\varphi\rangle$ (which are superpositions over the number of quanta n) . The theory is exhausted by the coherent state expectation values of all the operators xxx) .

Thus, in any case, choosing suitable coherent states, we may <u>represent</u> quantum theory only in terms of diagonal matrix elements

x) For definition of $\hat{a}^+(xp)$ see /2Ie/.

xx) Such a representation for the density matrix has been considered by Blokhintsev /40/.

xxx) One oan continue this list by transitions to D) with n-field states $|\varphi_1\dots\varphi_n\rangle$ and to E) with the superpositions of those over n like the above transitions from A) to B) and from B) to C).

of all operators, including the density matrix $\hat{\rho}(x)$. The diagonal matrix elements of $\hat{\rho}(t)$ always are probabilities (or probability densities) unlike the nondiagonal ones. Thus, in CSR all the density matrix elements $\langle x_1 p_1 \cdots x_n p_n | \hat{\rho}(t) | x_1 p_1 \cdots x_n p_n \rangle$ or $\langle \varphi | \hat{\rho}(t) | \varphi \rangle$ are probability densities and have meaning like that of classical phase space densities.

To obtain usual transition probabilities (i.e., the diagonal in n elements of the density matrix) in the case C) also it is sufficient to know only the expectation values $\langle x_1 p_1 \cdots x_n p_n | \hat{\rho}(t) | x_1 p_1 \cdots x_n p_n \rangle$ of the same type as those in the case B). However, knowledge of only such expectation values with all possible n is insufficient to re-establish fully operators noncommuting with n, namely, to find their non-diagonal in n matrix elements. In particular, this concerns $\hat{\rho}(t)$ and $\langle \varphi | \hat{\rho}(t) | \varphi \rangle$.

7. Non-linearity in the field operators (of the current, the 4-momentum tensor, equations of motion, e.g., the equations

$$\hat{x}(t) = \hat{\lambda}(t) + \int_{t'} ds\, D_{ret}(t-s)\, F(\hat{x}(s)) \tag{21}$$

$$\hat{\varphi}(x) = \hat{\varphi}(x) + \int_{t'} d^4y\, \Delta_{ret}(x-y)\, j(\hat{\varphi}(y)) \tag{22}$$

where F and j are implied to be non-linear) leads to the following three difficulties (i),ii) and iii)).

i) Acausality. After iterating eq. (22) infinitely many times and N-ordering (to calculate coherent state expectation values), one obtains that not only the causal functions Δ_{ret} enter into coefficient functions, but also the non-causal ones $\Delta^{(1)}$, unlike theories 1)-3). In general form this has been shown by Bialynicki-Birula/I3/ by means of functional method (see App.B).

The solution. We introduce an operator Λ^{-1}(for an explicit form of Λ see Appendix A and refs./2I d/) which operates on the above initial value arguments of the coherent state expectation

values, and removes the $\Delta^{(1)}$(the acausal free dispersion), and gives a causal result. We call this the CSR-2 /2Id/. For example,

$$\langle xp|\hat{x}(t)|xp\rangle \quad \text{and} \quad \Lambda^{-1}\langle xp|\hat{x}(t)|xp\rangle \tag{23}$$

$$\langle \varphi|\hat{\varphi}(x)|\varphi\rangle \quad \text{and} \quad \Lambda^{-1}\langle \varphi|\hat{\varphi}(x)|\varphi\rangle \tag{24}$$

$$\rho(t) = \langle xp|\hat{\rho}(t)|xp\rangle \quad \text{and} \quad \rho(t) = \Lambda^{-1}\langle xp|\hat{\rho}(t)|xp\rangle \tag{25}$$

$$\rho(t) = \langle \varphi|\hat{\rho}(t)|\varphi\rangle \quad \text{and} \quad \rho(t) = \Lambda^{-1}\langle \varphi|\hat{\rho}(t)|\varphi\rangle \tag{26}$$

are the CSR-1 and CSR-2 for $\hat{x}(t)$, $\hat{\varphi}(x)$ and the density matrices in quantum mechanics and in quantum field theory.

The operation Λ^{-1} corresponds to replacement of symmetrized products by the N-products without changing coefficient functions. Therefore, the coefficient functions in CSR-2 are those for the symmetrized product decompositions. For the latter the theorems, like Dyson and Wick ones for the N-product decompositions, can be shown by induction, and now pairings are causal:x)

$$\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n) = \frac{1}{n!}\{\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\} + \sum_{\text{one pairing}} \frac{i\hbar}{2}\Delta(x_1-x_2)\frac{1}{(n-2)!}\{\hat{\varphi}(x_3)\cdots\hat{\varphi}(x_n)\} + \sum_{\text{two pairings}}\cdots + \cdots \tag{27}$$

$$T\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n) = \frac{1}{n!}\{\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\} + \sum(-i\hbar)\Delta_{sym}(x_1-x_2)\frac{1}{(n-2)!}\{\hat{\varphi}(x_3)\cdots\hat{\varphi}(x_n)\} + \sum\cdots+\cdots \tag{28}$$

where $\Delta_{sym}(x_1-x_2) = \frac{1}{2}\Delta_{ret}(x_1-x_2) + \frac{1}{2}\Delta_{adv}(x_1-x_2)$. The $\Delta^{(1)}$'s arise, when we decompose symmetrized products into N-products:

$$\frac{1}{n!}\{\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\} = :\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n): + \sum\frac{\hbar}{2}\Delta^{(1)}(x_1-x_2):\hat{\varphi}(x_3)\cdots\hat{\varphi}(x_n): + \sum\cdots+\cdots \tag{29}$$

and this leads to the usual pairings $i\hbar\,\Delta^{(-)}(x_1-x_2)$ and $-i\hbar\Delta_+(x_1-x_2)$

The decomposition, inverse to eq.(29), is

$$:\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n): = \frac{1}{n!}\{\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\} + \sum\left(-\frac{\hbar}{2}\right)\Delta^{(1)}(x_1-x_2)\frac{1}{(n-2)!}\{\hat{\varphi}(x_3)\cdots\hat{\varphi}(x_n)\} + \sum\cdots+\cdots \tag{30}$$

It is useful to introduce "primed" operators, for which

$$\Lambda^{-1}\langle\varphi|\hat{Q}|\varphi\rangle = \langle\varphi|\hat{Q}'|\varphi\rangle \tag{31}$$

for example,

x) Slightly changing the Dyson statement, every product is identically equal to the sum of its symmetrized constituents.

$$\left(\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\right)' = :\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n): + \sum \frac{i\hbar}{2}\Delta(x_1-x_2):\hat{\varphi}(x_3)\cdots\hat{\varphi}(x_n): + \sum \cdots + \cdots \tag{32}$$

$$\left(T\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\right)' = :\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n): + \sum (-i\hbar)\Delta_{sym}(x_1-x_2):\hat{\varphi}(x_3)\cdots\hat{\varphi}(x_n): + \sum \cdots + \cdots \tag{33}$$

$$\left(\frac{1}{n!}\{\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\}\right)' = :\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n): \tag{34}$$

For some quantities such a substitution of the N-produot for the symmetrized ones was, in faot, proposed in book[41].

The N-product decomposition of the S'-matrix and the CSR-2 representative of S, $\Lambda^{-1}\langle\varphi|\hat{U}(t,t')|\varphi\rangle = \langle\varphi|\hat{U}'(t,t')|\varphi\rangle$, are given in terms of the functions Δ_{sym} , instead of Δ_+ in the S-matrix, (see Appendix B and ref.[21d]). It is noticeable that the S-matrix in CSR-2 is constructed in terms of the half-retarded-half-advanced functions like the Fokker-Wheeler-Feynman "action"[42]. As to the N-product decomposition of the primed Heisenberg field $\hat{\varphi}'(x)$ and the CSR-2 representative of $\hat{\varphi}(x)$, $\Lambda^{-1}\langle\varphi|\hat{\varphi}(x)|\varphi\rangle = \langle\varphi|\hat{\varphi}'(x)|\varphi\rangle$, these are constructed only of the Δ_{ret}-functions (see Appendix B and ref.[21d]). The $\hbar$-independent part of the latter is a Neumann series of the classical equation of form (22).

The operations Λ and Λ^{-1} have operator equivalents[21d]

$$\hat{Q} = \hat{\Lambda}\hat{Q}'\hat{\Lambda}^{-1} = \hat{Q}' + \frac{i}{4\hbar}\int d^3x'\,[\hat{\varphi}(x')\overleftrightarrow{\partial}_4'\,[\hat{\varphi}^{(1)}(x'),\hat{Q}']] + \tag{35}$$
$$+ \frac{1}{2!}\left(\frac{i}{4\hbar}\right)^2\int d^3x'\,d^3x''\,[\hat{\varphi}(x'')\overleftrightarrow{\partial}_4''\,[\hat{\varphi}^{(1)}(x'')[\hat{\varphi}(x')\overleftrightarrow{\partial}_4'\,[\hat{\varphi}^{(1)}(x'),\hat{Q}']]]] + \cdots$$

where $\hat{Q}$ is an arbitrary operator, and $\hat{\varphi}^{(1)}(x)$ is the Hilbert transform of $\hat{\varphi}(x)$ (see Appendix A). For example,

$$\frac{1}{n!}\{\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n)\} = \hat{\Lambda}:\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n):\hat{\Lambda}^{-1} \,. \tag{36}$$

When operating on a linear functionals, Λ^{-1} and Λ reduce to unity. Therefore, in theories I)-3) with linear equations Λ^{-1} gives nothing, applied to the ooherent state expectation values of the fields and to equations (8)-(12). However, it operates nontrivially on quantities, which are non-linear in fields (the energy-momentum tensor, the S-matrix, etc.).

Thus, all the theories (linear and non-linear) become manifestly causal ones in CSR-2, at least, more causal than usual ones.

The transition to the primed operators and to CSR-2 may be interpreted as a transition from an "analytical signal" $(\Delta^{(-)})$ to a "real one" (Δ) [21d].

ii) Absence of closed equations of a simple structure for $\langle\varphi|\hat{\varphi}(x)|\varphi\rangle$ and $\Lambda^{-1}\langle\varphi|\hat{\varphi}(x)|\varphi\rangle$ [x], expect for the limit case $\hbar = 0$.

iii) The latter is a consequenoe of the general peouliarity: in CSR-1 and CSR-2 it is not easy to indicate a representative for a product of two or more operators[xx], i.e., to write coherent state expectation values of products of operators in terms of expectation values of each of them. If we involve the non-diagonal elements, then a solution is given by the "matrix product". The non-diagonal elements are representable through the diagonal ones according to eqs. (18) and (20). In principle, the problem is solved. However, such a oonstruotion is cumbersome.

It appears that an operator realization in spirit of the Dirac representation theory[43](applied, however to the expeotation values[xxx]) is more concise than the matrix one. We mean that operators are represented as those operating on the initial value arguments of the coherent state expeotation values (such as $\varphi(\vec{x},t')$, $\dot{\varphi}(\vec{x},t')$ or $J(x)$). So, the representatives of the free ooordinate and field operators, $\hat{x}_k(t)$ and $\hat{\varphi}(x)$, are given by

$$\left.\begin{matrix}\langle xp|\hat{x}_k(t)\hat{Q}|xp\rangle \\ \langle xp|\hat{Q}\,\hat{x}_k(t)|xp\rangle\end{matrix}\right\} = \left(x_k(t) \pm i\hbar D_{k\ell}^{(\mp)}(t,s)\frac{\overleftrightarrow{\partial}}{\partial s}\frac{\delta}{\delta f_\ell(s)}\right)\langle xp|\hat{Q}|xp\rangle = \tag{37}$$
$$= \Lambda\left(x_k(t) \pm i\frac{\hbar}{2}\frac{\delta}{\delta f_k(t)}\right)\Lambda^{-1}\langle xp|\hat{Q}|xp\rangle$$

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x) Opposite assertion in ref.[21c] was incorreot (see ref.[21d]).

xx) See, however, ref.[13].

xxx) Dirac considered representations such as x- or p-representations where diagonal matrix elements are insufficient (see Sec.3).

$$\left.\begin{array}{l}\langle\varphi|\hat{\varphi}(x)\hat{Q}|\varphi\rangle\\ \langle\varphi|\hat{Q}\,\hat{\varphi}(x)|\varphi\rangle\end{array}\right\} = \left(\varphi(x) \mp \hbar\int d^3\xi\,\Delta^{(\mp)}(x-\xi)\overleftrightarrow{\partial}_4\frac{\delta}{\delta\mathcal{J}(\xi)}\right)\langle\varphi|\hat{Q}|\varphi\rangle =$$

$$= \Lambda\left(\varphi(x) \pm i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(x)}\right)\Lambda^{-1}\langle\varphi|\hat{Q}|\varphi\rangle \qquad (38)$$

where $\hat{Q}$ is an arbitrary operator. The CSR-1' and CSR-2' (left and right) representatives are in the first and second parantheses. The prime is used to distinguish these operator forms of CSR. In eqs. (37) and (38) there are used the relations

$$\Lambda x_k(t)\Lambda^{-1} = x_k(t) + \frac{\hbar}{2}D^{(1)}_{kl}(t,s)\overleftrightarrow{\frac{\partial}{\partial s}}\frac{\delta}{\delta f(s)} \quad (a) \qquad \frac{\delta}{\delta f_k(t)} = D(t-s)\overleftrightarrow{\frac{\partial}{\partial s}}\frac{\delta}{\delta f_k(s)} \quad (b) \qquad (39)$$

$$\Lambda\varphi(x)\Lambda^{-1} = \varphi(x) + i\frac{\hbar}{2}\int d^3\xi\,\Delta^{(1)}(x-\xi)\overleftrightarrow{\partial}_4\frac{\delta}{\delta\mathcal{J}(\xi)} \quad (a) \qquad \frac{\delta}{\delta\mathcal{J}(x)} = i\int d^3\xi\,\Delta(x-\xi)\overleftrightarrow{\partial}_4\frac{\delta}{\delta\mathcal{J}(\xi)} \quad (b) \qquad (40)$$

Equations (39.b) and (40.b) are valid, taking into account a structure of objects, the operator $\delta/\delta\mathcal{J}(x)$ operates on.

These representatives of $\hat{x}(t)$ and $\hat{\varphi}(x)$ satisfy the commutation relations

$$\left[x_m(t) \pm i\frac{\hbar}{2}\frac{\delta}{\delta f_m(t)},\ x_n(s) \pm i\frac{\hbar}{2}\frac{\delta}{\delta f_n(s)}\right] = \pm i\hbar\,\delta_{mn}D(t-s)$$

$$\left[x_m(t) + i\frac{\hbar}{2}\frac{\delta}{\delta f_m(t)},\ x_n(t) - i\frac{\hbar}{2}\frac{\delta}{\delta f_n(s)}\right] = 0 \qquad (41)$$

$$\left[\varphi(x) \pm i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(x)},\ \varphi(y) \pm i\frac{\hbar}{2}\frac{\delta}{\delta\varphi(y)}\right] = \pm i\hbar\,\Delta(x-y)$$

$$\left[\varphi(x) + i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(x)},\ \varphi(y) - i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(y)}\right] = 0 \qquad (42)$$

Note the equal-time (Schrödinger) CSR-I and CSR-2 representatives of $\hat{x}_k = \hat{x}_k(t)$, $\hat{p}_k = \hat{p}_k(t)$, $\hat{\varphi}(\vec{x}) = \hat{\varphi}(\vec{x},t)$ and $\hat{\dot{\varphi}}(\vec{x}) = \hat{\dot{\varphi}}(\vec{x},t)$

$$x_k + \frac{\hbar}{2}A^{-1}_{kl}\frac{\partial}{\partial x_l} \pm i\frac{\hbar}{2}\frac{\partial}{\partial p_k} = \Lambda\left(x_l \pm i\frac{\hbar}{2}\frac{\partial}{\partial p_k}\right)\Lambda^{-1}$$

$$p_k + \frac{\hbar}{2}A_{kl}\frac{\partial}{\partial p_l} \mp i\frac{\hbar}{2}\frac{\partial}{\partial x_k} = \Lambda\left(p_k \mp i\frac{\hbar}{2}\frac{\partial}{\partial x_k}\right)\Lambda^{-1} \qquad (43)$$

$$\varphi(\vec{x}) + \frac{\hbar}{2}\int d^3\xi\,\Delta^{(1)}(\vec{x}-\vec{\xi},0)\frac{\delta}{\delta\varphi(\vec{\xi})} \pm i\frac{\hbar}{2}\frac{\delta}{\delta\dot{\varphi}(\vec{x})} = \Lambda\left(\varphi(\vec{x}) \pm i\frac{\hbar}{2}\frac{\delta}{\delta\dot{\varphi}(\vec{x})}\right)\Lambda^{-1} \qquad (44)$$

$$\dot{\varphi}(\vec{x}) - \frac{\hbar}{2}\int d^3\xi\,\ddot{\Delta}^{(1)}(\vec{x}-\vec{\xi},0)\frac{\delta}{\delta\dot{\varphi}(\vec{\xi})} \mp i\frac{\hbar}{2}\frac{\delta}{\delta\varphi(\vec{x})} = \Lambda\left(\dot{\varphi}(\vec{x}) \mp i\frac{\hbar}{2}\frac{\delta}{\delta\varphi(\vec{x})}\right)\Lambda^{-1}$$

The knowledge of representatives of the free field operator (or the free coordinate operator in quantum mechanics) permits us to write all the quantities and equations like those in the usual operator theory. So, the equations of motion for the Heisenberg coordinate and scalar field operator representatives $\mathcal{X}_k(t)$ and $\Phi(x)$ are written in CSR-2' like eqs. (21) and (22)

$$\mathcal{X}_k(t) = x_k(t) + i\frac{\hbar}{2}\frac{\delta}{\delta f_k(t)} + \int_{t'} ds\, D_{ret\,kl}(t-s)F_l(\mathcal{X}(s)), \qquad (45)$$

$$\Phi(x) = \varphi(x) + i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(x)} + \int_{t'} d^4y\,\Delta_{ret}(x-y)\,j(\Phi(y)). \qquad (46)$$

These equations differ from the corresponding classical ones only in the quantum terms $i\frac{\hbar}{2}\frac{\delta}{\delta f_k(t)}$ and $i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(x)}$ (besides that the Planck constant enters into eq. (46) only through the Compton wave length). All the results, using solutions of these equations, are defined by initial values $x_k(t')$, $\dot{x}_k(t')$, $\varphi(\vec{x},t')$ and $\dot{\varphi}(\vec{x},t')$ like in classics. Equation (46) leads to causal results in any order of perturbation theory (see /2Id/).

In CSR-1' equations of motion are also of forms (45) and (46), but instead of $x_m(t) + i\frac{\hbar}{2}\frac{\delta}{\delta f_m(t)}$ and $\varphi(x) + i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(x)}$ they contain the above CSR-1' representatives. The origin of the acausality due to the "acausal free dispersion" $\Delta^{(1)}(x_1-x_2)$ is the following

$$\int d^3\xi\,\Delta^{(-)}(x-\xi)\overleftrightarrow{\partial}_4\frac{\delta}{\delta\mathcal{J}(\xi)} = \frac{1}{2}\int d^3\xi\left[\Delta(x-\xi) - i\Delta^{(1)}(x-\xi)\right]\overleftrightarrow{\partial}_4\frac{\delta}{\delta\mathcal{J}(\xi)} =$$

$$= -\frac{i}{2}\left(\frac{\delta}{\delta\mathcal{J}(x)} + \int d^3\xi\,\Delta^{(1)}(x-\xi)\overleftrightarrow{\partial}_4\frac{\delta}{\delta\mathcal{J}(\xi)}\right). \qquad (47)$$

Equations (45) and (46) are equivalent to the following differential equations with the initial values

$$\left.\begin{array}{l} m\ddot{\mathcal{X}}_k(t) = F_k(\mathcal{X}(t)) \\ \mathcal{X}_k(t') = x_k(t') + i\frac{\hbar}{2}\frac{\partial}{\partial p_k(t')}, \quad m\dot{\mathcal{X}}_k(t') = p_k(t') - i\frac{\hbar}{2}\frac{\partial}{\partial x_k(t')} \end{array}\right\} \qquad (48)$$

$$\left.\begin{array}{l} (\Box - m^2)\Phi(x) = -j(\Phi(x)) \\ \Phi(\vec{x},t') = \varphi(\vec{x},t') + i\frac{\hbar}{2}\frac{\delta}{\delta\dot{\varphi}(\vec{x},t')}, \quad \dot{\Phi}(\vec{x},t') = \dot{\varphi}(\vec{x},t') - i\frac{\hbar}{2}\frac{\delta}{\delta\varphi(\vec{x},t')} \end{array}\right\} \qquad (49)$$

For the CSR-1′ and CSR-2′ representatives of ordinary products and T-products of the field operators see eqs. (31)-(33) in ref./2Id/.

The left and right CSR-2′ representatives of the S-matrix are

$$\mathcal{U}(t,t';+)=\mathcal{U}\left(t,t';\varphi(y)+i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(y)}\right)=T\exp\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\mathcal{L}_I\left(\varphi(x)+i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(x)}\right), \quad (50)$$

$$\widetilde{\mathcal{U}}(t,t';-)=\widetilde{\mathcal{U}}\left(t,t';\varphi(y)-i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(y)}\right)=\overline{T}\exp\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\mathcal{L}_I\left(\varphi(x)-i\frac{\hbar}{2}\frac{\delta}{\delta\mathcal{J}(x)}\right). \quad (51)$$

In CSR-1′ the only distinction is in the form of the representative of $\hat{\varphi}(x)$.

The CSR-1′ and CSR-2′ representatives of the multiple commutators $[\hat{\varphi}(x_n)\dots[\hat{\varphi}(x_1),\hat{Q}]\dots]$ and anticommutators $\{\hat{\varphi}(x_n)\dots\{\hat{\varphi}(x_1),\hat{Q}\}\dots\}$ x) are clear from the formulas

$$\langle\varphi|[\varphi(x_n)[\varphi(x_{n-1})\dots[\hat{\varphi}(x_1),\hat{Q}]\dots]]|\varphi\rangle=(i\hbar)^n\frac{\delta}{\delta\mathcal{J}(x_1)}\dots\frac{\delta}{\delta\mathcal{J}(x_n)}\langle\varphi|\hat{Q}|\varphi\rangle, \quad (52)$$

$$\frac{1}{2^n}\langle\varphi|\{\hat{\varphi}(x_n)\{\hat{\varphi}(x_{n-1})\dots\{\hat{\varphi}(x_1),\hat{Q}\}\dots\}\}|\varphi\rangle=\Lambda\varphi(x_1)\dots\varphi(x_n)\Lambda^{-1}\langle\varphi|\hat{Q}|\varphi\rangle. \quad (53)$$

The coherent state expectation values of any operators are obtained simply by applying the representatives to unity /2Id/.

The transition into CSR-2 and CSR-2′ removes the free dispersion and acausality and simplifies all the quantities and equations.

8. Now we make some remarks concerning calculations of effects.

a) Eigenvalue spectra of operators (e.g., the energy spectrum of hydrogen atom) are, of course, independent of a representation and of using density matrix (when we correctly formulate boundary conditions).

b) The Planck formula. Two different ways give the same result for the statistical sum

$$Z=\frac{1}{2\pi\hbar}\int dx\,dp\langle xp|e^{-\beta\hat{H}}|xp\rangle=\frac{1}{2\pi\hbar}\int dx\,dp\,\Lambda^{-1}\langle xp|e^{-\beta\hat{H}}|xp\rangle=\frac{1}{1-\exp(-\beta\hbar\omega)}. \quad (54)$$

The first expression (in CSE-1) is clear. As to the second one, the rule to use the CSR-2 representative as the integrand is confirmed

x) Both are symmetric in x_i due to the Jacobi identity and the identity $\{a\{bc\}\}=\{b\{ac\}\}+[c[ba]]$.

by direct calculation (using the Fourier transform of $\langle xp|e^{-\beta\hat{H}}|xp\rangle$ over x and p) or by means of any of the identities /2Id,e/ x)

$$(2\pi\hbar)^{-3n}\int dx\,\Lambda^{-1}|xp\rangle\langle xp|=|p\rangle\langle p|, \quad (55)$$

$$(2\pi\hbar)^{-3n}\int dp\,\Lambda^{-1}|xp\rangle\langle xp|=|x\rangle\langle x|, \quad (56)$$

where the 3n-dimensional case is implied: $|xp\rangle=|\vec{x}_1\vec{p}_1\dots\vec{x}_n\vec{p}_n\rangle$, $dx=d^3x_1\dots d^3x_n$ and $|p\rangle$ and $|x\rangle$ are the eigenvectors of momentum and coordinate operators with the eigenvalues $p=(\vec{p}_1,\dots\vec{p}_n)$ and $x=(\vec{x}_1,\dots\vec{x}_n)$, respectively. For proof of the identities see Appendix C.

c) It may be shown by means of identity (55) that the distributions in CSR-2 over the momentum expectation values coincide with the distributions over the momentum eigenvalues, and, therefore, reproduce any usual transition probabilities and cross sections for any reactions (e.g., for the Compton cross section)/2Ie/. This way of calculation includes "summation over initial and final coordinates", like one does in classical problems on beams of particles.

Note that in b) and c) we deal only with the states $|x_1p_1\dots x_np_n\rangle$ whatever the case might be A), B) or C).

The last form of the completeness relation (4), which corresponds to CSR-2, follows immediately from eq.(55). The use of the CSR-2 representative does not change the normalization

$$(2\pi\hbar)^{-3n}\int dx\,dp\langle xp|\hat{\rho}|xp\rangle=(2\pi\hbar)^{-3n}\int dx\,dp\,\Lambda^{-1}\langle xp|\hat{\rho}|xp\rangle. \quad (57)$$

$$\int\delta^2\varphi\langle\varphi|\hat{\rho}|\varphi\rangle=\int\delta^2\varphi\,\Lambda^{-1}\langle\varphi|\hat{\rho}|\varphi\rangle. \quad (58)$$

For more detailed exposition of CSR-1′ and CSR-2′ see ref./2Id/ (CSR-1 and CSR-2 there).

x) The third "justification": when integrating by parts, Λ^{-1} reduces to unity.

2. CSR DYNAMICS

In CSR there are applicable all forms of dynamios, elaborated for amplitudes (cf. /35,36/).

Schrődinger pioture. Evolution may be represented by a differential or integral (Markovian) operators[x)]

$$\varrho(xpt)=\left\{\begin{array}{ll}\langle xp|\,e^{-i\hat{H}(t-t')}\,\hat{\varrho}(t')\,e^{i\hat{H}(t-t')}|xp\rangle & \text{(CSR-I)}\\ \Lambda^{-1}\langle xp|\,e^{-i\hat{H}(t-t')}\,\varrho(t')\,e^{i\hat{H}(t-t')}\,|xp\rangle & \text{(CSR-2)}\end{array}\right\}=$$

$$=e^{-i\hbar^{-1}(t-t')(\mathcal{H}-\mathcal{H}^{+})}\varrho(xpt')=$$

$$=\int dx'dp'\,G(xpt,x'p't')\,\varrho(x'p't')=$$

$$=\int dx'dp'dt'dx''dp''dt''\,G_{o}(xpt,x''p''t'')\,K(x''p''t'',x'p't')\,\varrho(x'p't'), \qquad (59)$$

where we do not fix a representation (CSR-I or CSR-2)

$$G(xpt,x'p't')=\theta(t-t')\,e^{-i\hbar^{-1}(t-t')(\mathcal{H}-\mathcal{H}^{+})}\,\delta(x-x')\delta(p-p') \qquad (60)$$

$$G_{o}(xpt,x'p't')=\theta(t-t')\,e^{-i\hbar^{-1}(t-t')(\mathcal{H}_{o}-\mathcal{H}_{o}^{+})}\,\delta(x-x')\delta(p-p') \qquad (61)$$

$$\varrho(xpt)=e^{-i\hbar^{-1}(t-t')(\mathcal{H}_{o}-\mathcal{H}_{o}^{+})}\varrho(xpt')=$$

$$=\int dx'\,dp'\,G_{o}(xpt,x'p't')\,\varrho(x'p't') \qquad (62)$$

($\mathcal{H}_o$ is the free Hamiltonian, $\mathcal{H}=\mathcal{H}_o+\mathcal{H}_1$).

The densities satisfy the generalized Liouville equation with and without an interaction

$$\left[\frac{\partial}{\partial t}+i\hbar^{-1}(\mathcal{H}-\mathcal{H}^{+})\right]\varrho(xpt)=0 \qquad (63)$$

$$\left[\frac{\partial}{\partial t}+i\hbar^{-1}(\mathcal{H}_o-\mathcal{H}_o^{+})\right]\varrho(xpt)=0 \qquad (64)$$

and the kernels G and G_o are the oorresponding Green funotions

---

x) The many-dimensional case is implied: $\varrho(xpt)=\varrho(\vec{x}_1\vec{p}_1\cdots\vec{x}_n\vec{p}_n t)$, $\delta(x-x')\delta(p-p')=\delta(\vec{x}_1-\vec{x}_1')\delta(\vec{p}_1-\vec{p}_1')\ldots\delta(\vec{x}_n-\vec{x}_n')\delta(\vec{p}_n-\vec{p}_n')$.

$$\left[\frac{\partial}{\partial t}+i\hbar^{-1}(\mathcal{H}-\mathcal{H}^{+})\right]G(xpt,x'p't')=\delta(t-t')\delta(x-x')\delta(p-p'), \qquad (65)$$

$$\left[\frac{\partial}{\partial t}+i\hbar^{-1}(\mathcal{H}_o-\mathcal{H}_o^{+})\right]G_o(xpt,x'p't')=\delta(t-t')\delta(x-x')\delta(p-p'). \qquad (66)$$

One oan transform the differential equations into the following integral ones

$$\varrho(xpt)=\varrho(xpt)+\hbar^{-1}\int dx'dp'dt'\,G_o(xpt,x'p't')(\mathcal{H}_1-\mathcal{H}_1^{+})\,\varrho(x'p't'), \qquad (67)$$

$$G(xpt,x'p't')=G_o(xpt,x'p't')+\hbar^{-1}\int dx''dp''dt''\,G_o(xpt,x''p''t'')(\mathcal{H}_1-\mathcal{H}_1^{+})\,G(x''p''t'',x'p't'), \qquad (68)$$

which are oonvenient for obtaining the perturbation theory approximations.

From equations (63) and (64) we conclude that

$$L=\hbar^{-1}(\mathcal{H}-\mathcal{H}^{+})=\hbar^{-1}(\mathcal{H}_o-\mathcal{H}_o^{+}+\mathcal{H}_1-\mathcal{H}_1^{+}),$$
$$L_o=\hbar^{-1}(\mathcal{H}_o-\mathcal{H}_o^{+}),\qquad L_1=\hbar^{-1}(\mathcal{H}_1-\mathcal{H}_1^{+}) \qquad (69)$$

are the generalized total, free and interaotion Liouvillians. One oan oonvert the interaotion Liouvillian into an integral operator

For the free motion

$$L=\hbar^{-1}(\mathcal{H}-\mathcal{H}^{+})=\frac{\left(p+\frac{\hbar}{2}\Lambda\frac{\partial}{\partial p}\right)_n}{m}\frac{\partial}{\partial x_n}=\Lambda\frac{p_n}{m}\frac{\partial}{\partial x_n}\Lambda^{-1}, \qquad \text{(CSR-I')}\ (70)$$

$$L=\hbar^{-1}(\mathcal{H}-\mathcal{H}^{+})=\frac{p_n}{m}\frac{\partial}{\partial x_n}, \qquad \text{(CSR-2')}\ (71)$$

and henoe

$$G_o(xpt,x'p't')=\theta(t-t')\,e^{-(t-t')\frac{\left(p+\frac{\hbar}{2}\Lambda\frac{\partial}{\partial p}\right)_n}{m}\frac{\partial}{\partial x_n}}\,\delta(x-x')\delta(p-p')= \qquad (72.a)$$

$$=\theta(t-t')\,e^{-(t-t')\frac{p}{m}\frac{\partial}{\partial x}}\,e^{-(t-t')\frac{\hbar}{2m}\Lambda\frac{\partial}{\partial p}\frac{\partial}{\partial x}}\,e^{\frac{\hbar(t-t')^2}{4m}\Lambda\frac{\partial}{\partial x}\frac{\partial}{\partial x}}\,\delta(x-x')\delta(p-p') \qquad (72.b)$$

$$=\pi^{-3}\theta(t-t')\,e^{-\hbar\Lambda\left(x-x'-\frac{p'}{m}(t-t')\right)^2-\hbar\Lambda^{-1}(p-p')^2}\,\Lambda^{-1}, \qquad \text{(CSR-I')}\ (72.c)$$

$$G_o(xpt,x'p't')=\theta(t-t')\,e^{-(t-t')\frac{p_n}{m}\frac{\partial}{\partial x_n}}\,\delta(x-x')\delta(p-p')= \qquad (73.a)$$

$$=\theta(t-t')\,\delta\!\left(x-x'-\frac{p}{m}(t-t')\right)\delta(p-p'). \qquad \text{(CSR-2')}\ (73.b)$$

When $\delta(x-x')\delta(p-p')$ being represented by Fourier integrals, expression (72.b) does not exist (in fact, because of involving the "bad" operator Λ^{-1}). For a oorreot treatment see ref. /44/.

In CSR-2' G_0 turns to be $\hbar$-independent and coincides formally with the corresponding classical Green function. The same holds if we take oscillator motions as zero approximation (see Appendix A and ref. /21e/).

In quantum field theory we can write analogous equations:

$$\Lambda^{-1}\langle\varphi|\hat{\rho}(t)|\varphi\rangle = e^{-i\hbar^{-1}(t-t')(\mathcal{H}-\mathcal{H}^{+})}\Lambda^{-1}\langle\varphi|\hat{\rho}(t')|\varphi\rangle = \int\delta^2\varphi' G[\varphi\dot{\varphi}t,\varphi'\dot{\varphi}'t']\Lambda'^{-1}\langle\varphi'|\hat{\rho}(t')|\varphi'\rangle \quad (74)$$

where

$$G[\varphi\dot{\varphi}t,\varphi'\dot{\varphi}'t'] = \theta(t-t')e^{-i\hbar^{-1}(t-t')(\mathcal{H}-\mathcal{H}^{+})}\Lambda^{-1}\Lambda'^{-1}|\langle\varphi|\varphi'\rangle|^2 \quad (75)$$

$$\left(\int\delta^2\varphi\,\Lambda^{-1}\Lambda'^{-1}|\langle\varphi|\varphi'\rangle|^2=1,\ \int\delta^2\varphi\,\Lambda^{-1}\Lambda'^{-1}|\langle\varphi|\varphi'\rangle|^2F[\varphi,\dot{\varphi}]=F[\varphi',\dot{\varphi}']\right). \quad (76)$$

This Green function satisfies the equation

$$G[\varphi\dot{\varphi}t,\varphi'\dot{\varphi}'t'] = G_0[\varphi\dot{\varphi}t,\varphi'\dot{\varphi}'t'] + \int\delta^2\varphi''dt''G_0[\varphi\dot{\varphi}t,\varphi''\dot{\varphi}''t'']L''G[\varphi''\dot{\varphi}''t'',\varphi'\dot{\varphi}'t'] \quad (77)$$

Interaction picture.

$$\rho^{int}(t) = \Lambda^{-1}\langle\varphi|\hat{\rho}^{int}(t)|\varphi\rangle = U(t,t';+)\tilde{U}^{-1}(t,t';-)\,\rho^{int}(t'). \quad (78)$$

According to eqs. (50) and (51)

$$U(t,t';+)\tilde{U}^{-1}(t,t';-) = T\exp\frac{i}{\hbar}\int_{t'}^{t}d^4x\left[\mathcal{L}_I\left(\varphi(x)+i\frac{\hbar}{2}\frac{\delta}{\delta J(x)}\right)-\mathcal{L}_I\left(\varphi(x)-i\frac{\hbar}{2}\frac{\delta}{\delta J(x)}\right)\right]. \quad (79)$$

Equation of motion for evolution operator (79) is the Liouville equation in the interaction picture

$$\left\{\frac{d}{dt}+i\hbar^{-1}\int d^3x\left[\mathcal{L}_I\left(\varphi(x)+i\frac{\hbar}{2}\frac{\delta}{\delta J(x)}\right)-\mathcal{L}_I\left(\varphi(x)-i\frac{\hbar}{2}\frac{\delta}{\delta J(x)}\right)\right]\right\}U(t,t';+)\tilde{U}^{-1}(t,t';-)=0. \quad (80)$$

CSR-I differs in the form of the field representative.

Probabilities of transitions between n-quantum states follow from (in CSR-I)

$$\frac{\delta}{\delta J(y_1)}\cdots\frac{\delta}{\delta J(y_n)}U(t,t';+)\tilde{U}^{-1}(t,t';-)\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J'(x_m)}|\langle\varphi|\varphi'\rangle|^2\Big|_{J=J'=0} =$$

$$=(-1)^m\frac{\delta}{\delta J(y_1)}\cdots\frac{\delta}{\delta J(y_n)}U(t,t';+)\tilde{U}^{-1}(t,t';-)\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_m)}|\langle\varphi|0\rangle|^2\Big|_{J=0}. \quad (81)$$

For details see Appendix C.

3. ON DETERMINISM

1. An attempt is suggested to describe an individual particle by the density matrix (see /21e/)

$$\hat{\rho}_{\vec{x}\vec{p}} = \Lambda^{-1}|\vec{x}\vec{p}\rangle\langle\vec{x}\vec{p}| \quad (82)$$

and a beam ("ensemble") of particles with definite momentum by the partial density matrix

$$\hat{\rho}_{\vec{p}} = \frac{1}{(2\pi\hbar)^3}\int d^3x\,\Lambda^{-1}|\vec{x}\vec{p}\rangle\langle\vec{x}\vec{p}| \quad (83)$$

(the latter according to eq. (55)). Hence the beam may be described by the state vector $|\vec{p}\rangle$, as is usually done in standard quantum mechanics. This is the origin of amplitudes (see also the footnote on the next page).

Given $\hat{\rho}(t') = \Lambda'^{-1}|x'p'\rangle\langle x'p'|$ at an initial time t'. Then for the phase space density in CSR-2 we find /21d,e/ *)

$$\rho(xpt') = \Lambda^{-1}\Lambda'^{-1}|\langle xp|x'p'\rangle|^2 = (2\pi\hbar)^{3n}\delta(x-x')\delta(p-p') \quad (84)$$

$$\rho(xpt) = e^{-i\hbar^{-1}(t-t')(\mathcal{H}-\mathcal{H}^{+})}\rho(xpt') = (2\pi\hbar)^{3n}\delta\left(x-x'-\frac{p}{m}(t-t')\right)\delta(p-p') \quad (85)$$

the latter for the free evolution, which turns to be the same as in classics. The same holds for "free" oscillator motions (see Appendix A and ref. /21e/). In all these cases quantum corrections appear in CSR-2 only due to interaction (they are only a part of all corrections in CSR-I). However, the relativistic free motion of state (82) is modified by quantum corrections (see Appendix D).

*) In many-dimensional case

$$\delta\left(x-x'-\frac{p}{m}(t-t')\right)\delta(p-p') = \prod_i\delta\left(\vec{x}_i-\vec{x}_i'-\frac{\vec{p}_i}{m}(t-t')\right)\delta(\vec{p}_i-\vec{p}_i').$$

The usual predictions *)

$$\rho(x\,t) = \langle x|\,\hat\rho(t)|x\rangle = (2\pi)^{-3n}\int d\,p\,\Lambda^{-1}\langle xp|\,\hat\rho(t)|xp\rangle, \tag{86}$$

$$\rho(p\,t) = \langle p|\,\hat\rho(t)\,|p\rangle = (2\pi)^{-3n}\int d\,x\,\Lambda^{-1}\langle xp|\,\hat\rho(t)|xp\rangle, \tag{87}$$

being partial distributions (i.e., integrals over some canonical variables) are "indeterministic" ones.

If one likes, they are representable in terms of the usual amplitudes. The evolution of such partial distribution is representable not by itself but either through the total phase density evolution (according to eqs. (86) or (87)) or as a results of the evolution of the amplitudes. This is a pay for the loss of information***) , i.e. for the incompleteness of sets of variables,

*) The density matrix, e.g., $\langle p''|\,\hat\rho(t)|p'\rangle$ may be found as follows

$$\langle p''|\hat\rho(t)|p'\rangle = (2\pi)^{-6n}\int d\,x_1 d\,p_1 d\,x_2 d\,p_2 \langle p''|x_2p_2\rangle\langle x_2p_2|\,\hat\rho(t)|x_1p_1\rangle\langle x_1p_1|p'\rangle,$$

where it is implied that the non-diagonal elements $\langle x_2p_2|\hat\rho(t)|x_1p_1\rangle$ have been reestablished according to eq. (18). Usually only the diagonal elements are of interest, and the rule (87) for them seems to be more elegant.

***) Only the density matrix as a whole $\langle x'''|\hat\rho(t)|x''\rangle$, or $\langle p'''|\,\hat\rho(t)\,|p'\rangle$ (i.e., with the non-diagonal matrix elements in these representations) contains all the information: it involves the complete set of canonical variables and undergoes the Markovian evolution

$$\langle x'''|\,\hat\rho\,(t)\,|x'\rangle = \langle x'''|\,e^{-i\hat H(t-t')}\,\hat\rho(t')\,e^{i\hat H(t-t')}|x''\rangle =$$

$$= e^{-iH'''(t-t')+iH''(t-t')}\langle x'''|\,\hat\rho(t')|x''\rangle \qquad \left(H'' = \frac{(i\frac{\partial}{\partial x''})^2}{2m} + V(x'')\right)$$

unlike the evolution of the probability densities $\rho(x''t) = \langle x''|\hat\rho(t)|x''\rangle$ (the direct transition $x''\to x'$ is unrealizable). For the states $\hat\rho(t') = |x'\rangle\langle x'|$ (or $|p'\rangle\langle p'|$) the above expression is factorized and the evolution of the probability densities is given in terms of the pseudo-Markovian evolution of the amplitudes as follows

which are usually used to characterize states (e.g., only x'', or only p').

Both the density matrix (82) and usual state $|p\rangle$ give only idealized descriptions, details (uncertainties of real objects) being neglected. However, it seems that one can relate eq. (82) to an individual particle *) .

Such an approach is a refined formalism of wave packets.

2. In CSR the quantum theory may be formulated entirely in terms of probabilities (the coherent state expectation values of density matrix) without necessity to appeal to the amplitudes. Here we have some sort of determinism, because a complete set of "canonical" variables is used simultaneously, and there is no need in "hidden variables". In CSR-2 quantum theory becomes manifestly causal. Although all this takes place in terms of the expectation values, the theory stays to be complete, because we change only representation. All we say here concerns only the free case.

The above considerations are close to those due to Moyal/24/ who has investigated the quantum-mechanical phase space in terms of the Wigner distribution function/23/. In fact, the CSR-2 coincides with the Wigner representation (see the next Sec.) and, therefore, suffers from the same difficulty: the phase space density is not positive definite for some states (see ref./24/ and Appendix C), unlike the CSR-1 phase space density.

$$\rho(x''t) = \langle x''|\hat\rho(t)|x'\rangle = |\langle x''|e^{-iH(t-t')}|x'\rangle|^2 = |e^{-iH''(t-t')}\langle x''|x'\rangle|^2$$

However, from other point of view/45/ the non-Markovian evolution of usual probability density $\rho(x''t)$ and representability of it through the pseudo-Markovian evolution of the amplitudes is a consequence of the pseudo-Euclidean space-time metric.

*) For instance, let a particle cross three counters. After measuring times of crossing of the first two of them, one can predict time of crossing of the last counter.

4. EQUIVALENCE OF CSR-2 (CSR-2') AND WIGNER REPRESENTATION

The equivalence of both representations becomes clear from the relation

$$\Lambda^{-1}|xp\rangle\langle xp| = \hbar^3 \int d\,a\, e^{-ipa} |x-\tfrac{\hbar}{2}a\rangle\langle x+\tfrac{\hbar}{2}a| = \qquad (88)$$

$$= \hbar^3 \int d\,k\, e^{ixk}\, |p-\tfrac{\hbar}{2}k\rangle\langle p+\tfrac{\hbar}{2}k| \;,$$

which is shown in Appendix C. In turn, using eq.(88) it is easy to verify some of the above relations: the last equality of eq.(4), eqs. (55), (56), (57) and (84).

Consider the evolution of the phase space density matrix in CSR-2

$$\rho(xpt) = \Lambda^{-1}\langle xp|\hat{\rho}(t)|xp\rangle = \hbar^3 \int d\,a\, e^{-ipa}\langle x+\tfrac{\hbar}{2}a|\hat{\rho}(t)|x-\tfrac{\hbar}{2}a\rangle =$$

$$= \hbar^3 \int d\,a\, e^{-ipa}\langle x+\tfrac{\hbar}{2}a| e^{-i\hbar^{-1}(t-t')\hat{H}}\hat{\rho}(t')e^{i\hbar^{-1}(t-t')\hat{H}}|x-\tfrac{\hbar}{2}a\rangle =$$

$$= \hbar^3 \int d\,a\, e^{-ipa}\, e^{-i\hbar^{-1}(t-t')(H_2-H_1)}\langle x+\tfrac{\hbar}{2}a|\hat{\rho}(t')|x-\tfrac{\hbar}{2}a\rangle =$$

$$= e^{-i\hbar^{-1}(t-t')(\mathcal{H}-\mathcal{H}^{+})}\,\hbar^3 \int d\,a\, e^{-ipa}\langle x+\tfrac{\hbar}{2}a|\hat{\rho}(t')|x-\tfrac{\hbar}{2}a\rangle =$$

$$= e^{-i\hbar^{-1}(t-t')(\mathcal{H}-\mathcal{H}^{+})}\rho(xpt')\;, \qquad (89)$$

where $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$, (90)

$$H_1 = \frac{1}{2m}\left(\frac{\partial}{\partial a} - \frac{\hbar}{2}\frac{\partial}{\partial x}\right)^2 + V(x-\tfrac{\hbar}{2}a), \quad H_2 = \frac{1}{2m}\left(\frac{\partial}{\partial a} + \frac{\hbar}{2}\frac{\partial}{\partial x}\right)^2 + V(x+\tfrac{\hbar}{2}a) \qquad (91)$$

and $\mathcal{H}$ is Hamiltonian (1). By the way, we see that in the Wigner representation the representatives of the operators $\hat{x}_n$ and $\hat{p}_n$ are $x_n \pm i\frac{\hbar}{2}\frac{\partial}{\partial p_n}$, and $p_n \mp i\frac{\hbar}{2}\frac{\partial}{\partial x_n}$, respectively, i.e., the same as in CSR-2'.

CSR interpretation of the Wigner representation removes question /24/ on distributions for non-commutative variables and makes clear that the Wigner phase space densities are diagonal matrix elements of density matrix. A possible solution of another problem, that the Wigner (CSR-2) densities are not positive definite for some states (see ref. /24/ and Appendix C), is to turn into CSR-1, where all the phase space densities $\langle xp|\hat{\rho}|xp\rangle$ are certainly positive definite (but, however, the above causal properties become implicit).

Note that every CSR-I distribution is the convolution of corresponding CSR-2 one with the normal distribution (the Weierstrass transform /44/):

$$\rho_1(xpt) = \Lambda\rho_2(xpt) = \pi^{-3n}\int dx'dp'\, e^{-\hbar^{-1}(A(x-x')^2 + A^{-1}(p-p')^2)}\rho_2(x'p't)$$

$$\langle\varphi|\hat{\rho}(t)|\varphi\rangle = C\int\delta^2\varphi'\exp\Big(-\hbar^{-1}\int d^3x\, d^3y\,\big[-\ddot{\Delta}(\vec{x}-\vec{y},0)(\varphi(\vec{x})-\varphi'(\vec{x}))(\varphi(\vec{y})-\varphi'(\vec{y}))+$$

$$+\Delta^{(1)}(\vec{x}-\vec{y},0)(\dot{\varphi}(\vec{x})-\dot{\varphi}'(\vec{x}))(\dot{\varphi}(\vec{y})-\dot{\varphi}'(\vec{y}))\big]\Big)\Lambda'^{-1}\langle\varphi'|\hat{\rho}(t)|\varphi'\rangle \qquad (92)$$

(some sort of averaging over vacuum fluctuations).

Note that according to Appendix C for arbitrary $\nu \geqslant -1$

$$\Lambda^{\nu}|xp\rangle\langle xp| = \hbar^{3n}\int dy\, da\, e^{-ipa}\, e^{-\frac{\nu+1}{4}\hbar A a^2}\cdot$$

$$\cdot\begin{cases}(\pi(\nu+1)\hbar)^{-\frac{3n}{2}}(\det A)^{\frac{n}{2}}\, e^{-\frac{1}{(\nu+1)\hbar}A(x-y)^2} & /\nu>-1/\\ \delta(x-y) & /\nu=-1/\end{cases}|y-\tfrac{\hbar}{2}a\rangle\langle y-\tfrac{\hbar}{2}a| = \qquad (93)$$

$$= \hbar^{3n}\int dq\, dk\, e^{ixk}\, e^{-\frac{\nu+1}{4}\hbar A^{-1}k^2}\cdot$$

$$\cdot\begin{cases}(\pi(\nu+1)\hbar)^{-\frac{3n}{2}}(\det A)^{-\frac{n}{2}}\, e^{-\frac{1}{(\nu+1)\hbar}A^{-1}(p-q)^2} & /\nu>-1/\\ \delta(p-q) & /\nu=-1/\end{cases}|q-\tfrac{\hbar}{2}k\rangle\langle q+\tfrac{\hbar}{2}k| .$$

APPENDIX A

I. Coherent states (3) are defined by the minimal uncertainty condition and the normalization one

$$\hat{\varphi}^{(-)}(x)|\varphi\rangle = \varphi^{(-)}(x)|\varphi\rangle\ , \qquad \text{(A.I)}$$

$$\langle\varphi|\varphi\rangle = 1\ . \qquad \text{(A.2)}$$

Those states may be written also as follows

$$|\varphi\rangle = e^{-\hbar^{-1}(\varphi^{(-)},\hat{\varphi}^{(-)})-\hbar^{-1}(\varphi^{(+)},\hat{\varphi}^{(+)})}|0\rangle = e^{-\frac{\hbar^{-1}}{2}(\varphi^{(-)},\varphi^{(-)})}\, e^{-\hbar^{-1}(\varphi^{(+)},\hat{\varphi}^{(+)})}|0\rangle = \qquad \text{(A.3.a)}$$

$$= e^{\frac{i}{\hbar}\int d^4x\left(J\hat{\varphi}^{(-)}+J\hat{\varphi}^{(+)}\right)}|0\rangle = e^{-\frac{1}{4\hbar}\int d^4y\, d^4z\, J(y)\Delta^{(1)}(y-z)J(z)}\, e^{\frac{i}{\hbar}\int d^4x\, J(x)\hat{\varphi}^{(+)}(x)}|0\rangle\ . \qquad \text{(A.3.b)}$$

The vacuum state $|0\rangle$ is one of the coherent states: $\hat{\varphi}^{(-)}(x)|0\rangle = 0$. The quantity

$$\hbar^{-1}(\varphi^{(-)},\varphi^{(-)}) = \frac{1}{2\hbar}\int d^4x\, d^4y\, J(x)\Delta^{(1)}(x-y)J(y) \qquad \text{(A.4)}$$

is the coherent state expectation value of the operator of the number of quanta.

From (A.3.b) it follows

$$\hat{\varphi}^{(+)}(x)|\varphi\rangle = \left(-i\hbar\frac{\delta}{\delta J(x)} - i\,\frac{1}{2}\int d^4y\,\Delta^{(1)}(x-y)J(y)\right)|\varphi\rangle =$$

$$= \left(-i\hbar\frac{\delta}{\delta J(x)} + \frac{1}{2}\varphi^{(+)}(x) - \frac{1}{2}\varphi^{(-)}(x)\right)|\varphi\rangle =$$

$$= \left(\frac{1}{2}\varphi^{(+)}(x) + \hbar\int d^3\xi\,\Delta^{(+)}(x-\xi)\overleftrightarrow{\partial_4}\frac{\delta}{\delta J(\xi)}\right)|\varphi\rangle \qquad \text{(A.5)}$$

and this together with (A.I) leads to eqs. (38). Note the relations

$$\langle\varphi|\{\Delta\varphi(x),\Delta\varphi(y)\}|\varphi\rangle = \hbar\Delta^{(1)}(x-y), \qquad \Delta\varphi(x) = \hat{\varphi}(x)-\varphi(x) \qquad \text{(A.6)}$$

$$\langle\varphi_2|\hat{\varphi}(x)|\varphi_1\rangle = \varphi_{21}(x)\langle\varphi_2|\varphi_1\rangle\ , \qquad \text{(A.7.a)}$$

$$\langle\varphi_2|:\hat{\varphi}(x_1)\cdots\hat{\varphi}(x_n):|\varphi_1\rangle = \varphi_{21}(x_1)\cdots\varphi_{21}(x_n)\langle\varphi_2|\varphi_1\rangle\ , \qquad \text{(A.7.b)}$$

where

$$\varphi_{21}(x) = \varphi_1^{(-)}(x) + \varphi_2^{(+)}(x) \qquad \text{(A.8)}$$

$$\langle\varphi_2|\varphi_1\rangle = e^{-\frac{\hbar^{-1}}{2}(\varphi_1^{(-)},\varphi_1^{(-)}) - \frac{\hbar^{-1}}{2}(\varphi_2^{(-)},\varphi_2^{(-)}) + \hbar^{-1}(\varphi_2^{(-)},\varphi_1^{(-)})} =$$

$$= e^{-\frac{1}{4\hbar}\int d^4x\, d^4y\,(J_1(x)-J_2(x))\Delta^{(1)}(x-y)(J_1(y)-J_2(y)) + \frac{i}{2\hbar}\int d^4x\, d^4y\, J_2(x)\Delta(x-y)J_1(y)}. \qquad \text{(A.9)}$$

Here the positive- and negative-frequency parts $\varphi^{(\mp)}(x)$ are defined according to the Schrödinger equations

$$\partial_4\varphi^{(\mp)}(x) = \mp\sqrt{-\Delta+m^2}\,\varphi^{(\mp)}(x) \qquad \text{(A.I0)}$$

and the splitting into those and the Hilbert transform are given by eqs.

$$\varphi^{(\mp)}(x) = \pm\frac{1}{2\pi i}\int\frac{ds}{t-s\mp i\varepsilon}\varphi(\vec{x},s) = \qquad (\varepsilon\to+0)$$

$$= i\int d^3x'\,\Delta^{(\mp)}(x-x')\overleftrightarrow{\partial_4'}\varphi(x') = \left(= -\int d^4y\,\Delta^{(\mp)}(x-y)J(y)\right)$$

$$= \frac{1}{2}\left(1\mp\frac{\partial_4}{H}\right)\varphi(x) = \frac{1}{2}\varphi(x) \mp \frac{i}{2}\varphi^{(1)}(x) \qquad \left(H=\sqrt{-\Delta+m^2}\right) \qquad \text{(A.II)}$$

$$\varphi^{(1)}(x) = \frac{1}{\pi}P\int\frac{ds}{t-s}\varphi(\vec{x},s) =$$

$$= i\int d^3x'\,\Delta^{(1)}(x-x')\overleftrightarrow{\partial_4'}\varphi(x') = \left(= -\int d^4y\,\Delta^{(1)}(x-y)J(y)\right)$$

$$= \frac{\partial_4}{H}\varphi(x) = i\left(\varphi^{(-)}(x)-\varphi^{(+)}(x)\right) \qquad \text{(A.I2)}$$

(The Hilbert transform is, in fact, the operator of sign of energy.) In particular,

$$\Delta^{(\mp)}(x) = \pm\frac{1}{2\pi i}\int\frac{ds}{t-s\mp i\varepsilon}\Delta(\vec{x},s), \qquad \Delta^{(1)}(x) = \frac{1}{\pi}P\int\frac{ds}{t-s}\Delta(\vec{x},s)\ . \qquad \text{(A.I3)}$$

The Δ -functions are defined as follows

$$[\hat{\varphi}^{(\mp)}(x),\hat{\varphi}^{(\pm)}(y)] = \pm\frac{\hbar}{(2\pi)^3}\int\frac{d^3k}{2\omega}e^{\pm ik(x-y)} = i\hbar\Delta^{(\mp)}(x-y) \qquad \left(\omega=\sqrt{\vec{k}^2+m^2}\right) \qquad \text{(A.I4)}$$

$$[\hat{\varphi}(x),\hat{\varphi}(y)] = i\hbar\Delta(x-y),\ [\hat{\varphi}^{(1)}(x),\hat{\varphi}(y)] = i\hbar\Delta^{(1)}(x-y),\ [\hat{\varphi}^{(1)}(x),\hat{\varphi}^{(1)}(y)] = i\hbar\Delta(x-y) \qquad \text{(A.I5)}$$

$$\Delta_{ret}(x) = -\theta(t)\Delta(x) = \Delta_{sym}(x) - \frac{1}{2}\Delta(x) \qquad = \frac{1}{(2\pi)^4}\int d^4k\,\frac{\exp(ikx)}{k^2+m^2-i\varepsilon k_0}$$

$$\Delta_{adv}(x) = \theta(-t)\Delta(x) = \Delta_{sym}(x) + \frac{1}{2}\Delta(x) \qquad = \frac{1}{(2\pi)^4}\int d^4k\,\frac{\exp(ikx)}{k^2+m^2+i\varepsilon k_0}$$

$$\Delta_{sym}(x) = -\frac{1}{2}\varepsilon(t)\Delta(x) = \frac{1}{2}\left(\Delta_{ret}(x)+\Delta_{adv}(x)\right) \qquad = \frac{1}{(2\pi)^4}P\int d^4k\,\frac{\exp(ikx)}{k^2+m^2}$$

$$\Delta_+(x) = -\theta(t)\Delta^{(-)}(x)+\theta(-t)\Delta^{(+)}(x) = \Delta_{sym}(x)+\frac{i}{2}\Delta^{(1)}(x) = \frac{1}{(2\pi)^4}\int d^4k\,\frac{\exp(ikx)}{k^2+m^2-i\varepsilon}$$

$$\Delta_-(x) = -\theta(t)\Delta^{(+)}(x)+\theta(-t)\Delta^{(-)}(x) = \Delta_{sym}(x)-\frac{i}{2}\Delta^{(1)}(x) = \frac{1}{(2\pi)^4}\int d^4k\,\frac{\exp(ikx)}{k^2+m^2+i\varepsilon}$$

$$S_{(\ )}(x) = (\gamma\partial-m)\Delta_{(\ )}(x). \qquad \text{(A.I6)}$$

2. The quantum-mechanical coherent states of form (2) are discussed in Appendices of refs. /21d,e/.

They correspond to the linear equations: to the free equations

$$m\ddot{\hat{x}}_k(t)=0,\quad \hat{x}_k(t)=\hat{x}_k(t')+\frac{\hat{p}_k}{m}(t-t'),\quad x_k(t)=x_k(t')+\frac{p_k}{m}(t-t') \qquad (A.17)$$

and to the oscillator equations (the linear oscillator, rotation, other Lissajous curves, etc.)[x)]

$$m\ddot{\hat{x}}_k(t)+m\omega^2\hat{x}_k(t)=0 \qquad (A.18)$$

$$\hat{x}_k(t)=\hat{x}_k(t')\cos\omega(t-t')+\frac{\hat{p}_k(t')}{m\omega}\sin\omega(t-t'),\quad \hat{p}_k(t)=\hat{p}_k(t')\cos\omega(t-t')-m\omega\hat{x}_k(t')\sin\omega(t-t')$$

$$x_k(t)=x_k(t')\cos\omega(t-t')+\frac{p_k(t')}{m\omega}\sin\omega(t-t'),\quad p_k(t)=p_k(t')\cos\omega(t-t')-m\omega x_k(t')\sin\omega(t-t')$$

In both the cases the exponent of eq. (2), $p_k(t)\hat{x}_k(t)-x_k(t)\hat{p}_k(t)$ is conserved in time.

The definitionsof D-functions in quantum mechanics are

$$[\hat{x}_k^{(\mp)}(t),\hat{x}_\ell^{(\pm)}(s)]=i\hbar D_{k\ell}^{(\mp)}(t,s),\quad D_{k\ell}^{(\mp)}(t,s)=\frac{1}{2}D_{k\ell}(t-s)\mp\frac{i}{2}D_{k\ell}^{(1)}(t,s)$$

$$[\hat{x}_k(t),\hat{x}_\ell(s)]=i\hbar D_{k\ell}(t-s),\quad [\hat{x}_k^{(1)}(t),\hat{x}_\ell(s)]=i\hbar D_{k\ell}^{(1)}(t,s) \qquad \begin{matrix}(A.19)\\(A.20)\end{matrix}$$

$$D_{ret}(t-s)=-\theta(t-s)D(t-s) \qquad (A.21)$$

For the free case:

$$D_{k\ell}(t-s)=-\delta_{k\ell}\frac{t-s}{m}, \qquad (A.22)$$

$$D_{k\ell}^{(1)}(t,s)=\hbar^{-1}\langle xp|\{\Delta x_k(t),\Delta x_\ell(s)\}|xp\rangle=A_{k\ell}^{-1}+A_{k\ell}\frac{(t-t')(s-t')}{m^2}. \qquad (A.23)$$

and for the oscillator case:

$$D_{k\ell}(t-s)=-\delta_{k\ell}\frac{\sin\omega(t-s)}{m\omega}, \qquad (A.24)$$

$$D_{k\ell}^{(1)}(t,s)=A_{k\ell}^{-1}\cos\omega(t-t')\cos\omega(s-t')+A_{k\ell}\frac{\sin\omega(t-t')\sin\omega(s-t')}{m^2\omega^2}. \qquad (A.25)$$

In both the cases $\hat{x}_k^{(\mp)}(t)=mD_{k\ell}^{(\mp)}(t,t')\overleftrightarrow{\frac{\partial}{\partial t'}}\hat{x}_\ell(t'),\ \Delta x_k(t)=\hat{x}_k(t)-x_k(t)$ (A.26)

$$A=2\hbar^{-1}\|\langle xp|\Delta p_k(t')\Delta p_\ell(t')|xp\rangle\|,\quad A^{-1}=C=2\hbar^{-1}\|\langle xp|\Delta x_k(t')\Delta x_\ell(t')|xp\rangle\| \qquad (A.27)$$

For other details see the above refs.

In connection with eq. (18) note the formulas

$$\langle x_2p_2|\hat{x}(t)|x_1p_1\rangle=x_{21}(t)\langle x_2p_2|x_1p_1\rangle,\quad x_{21}(t)=x_1^{(-)}(t)+x_2^{(+)}(t), \qquad (A.28)$$

$$\langle x_2p_2|:\hat{x}(t_1)\ldots\hat{x}(t_n):|x_1p_1\rangle=x_{21}(t_1)\ldots x_{21}(t_n)\langle x_2p_2|x_1p_1\rangle, \qquad (A.29)$$

$$\langle x_2p_2|x_1p_1\rangle=\exp\left(-\frac{1}{4}\left(A(\vec{x}_1-\vec{x}_2)^2+A^{-1}(\vec{p}_1-\vec{p}_2)^2\right)+\frac{i}{2}\hbar^{-1}(\vec{p}_1\vec{x}_2-\vec{p}_2\vec{x}_1)\right). \qquad (A.30)$$

3. The operators Λ may be written as follows

$$\Lambda=\exp\left(\frac{\hbar}{4}\left(A^{-1}\frac{\partial}{\partial x}\frac{\partial}{\partial x}+A\frac{\partial}{\partial p}\frac{\partial}{\partial p}\right)\right)=\exp\left(\frac{\hbar}{2}\frac{\partial}{\partial a}\frac{\partial}{\partial a^*}\right)=$$

$$=\exp\left(\frac{\hbar}{4}D^{(1)}(\tau,\tau')\left(\frac{\partial}{\partial x(t')}+\overleftarrow{\frac{\partial}{\partial\tau}}\frac{\partial}{\partial\dot{x}(t')}\right)\left(\frac{\partial}{\partial x(t')}+\overleftarrow{\frac{\partial}{\partial\tau'}}\frac{\partial}{\partial\dot{x}(t')}\right)\right)=\quad(\tau=\tau'=t')$$

$$=\exp\left(\frac{\hbar}{4}\frac{\delta}{\delta f(\tau)}\overleftrightarrow{\frac{\partial}{\partial\tau}}D^{(1)}(\tau,\tau')\overleftrightarrow{\frac{\partial}{\partial\tau'}}\frac{\delta}{\delta f(\tau')}\right)\ (\tau,\tau'\ \text{are arbitrary}), \qquad (A.31)$$

$$\Lambda=\exp\left(\frac{\hbar}{4}\int d^3\xi'd^3\zeta'\Delta^{(1)}(\xi'-\zeta')\left(\frac{\delta}{\delta\varphi(\xi')}+\overleftarrow{\partial}_4^{\xi'}\frac{\delta}{\delta\partial_4\varphi(\xi')}\right)\left(\frac{\delta}{\delta\varphi(\zeta')}+\overleftarrow{\partial}_4^{\zeta'}\frac{\delta}{\delta\partial_4\varphi(\zeta')}\right)\right)=$$

$$=\exp\left(\frac{\hbar}{4}\int d^3\xi'd^3\zeta'\left(\Delta^{(1)}(\xi'-\zeta')\frac{\delta}{\delta\varphi(\xi')}\frac{\delta}{\delta\varphi(\zeta')}+\partial_4^{\xi'}\partial_4^{\zeta'}\Delta^{(1)}(\xi'-\zeta')\frac{\delta}{\delta\partial_4\varphi(\xi')}\frac{\delta}{\delta\partial_4\varphi(\zeta')}\right)\right)=(\xi_0'=\zeta_0'=t')$$

$$=\exp\left(\frac{\hbar}{4}\int d^3\xi d^3\zeta\frac{\delta}{\delta\chi(\xi)}\overleftrightarrow{\partial}_4^{\xi}\Delta^{(1)}(\xi-\zeta)\overleftrightarrow{\partial}_4^{\zeta}\frac{\delta}{\delta\chi(\zeta)}\right)\ (\xi_0,\zeta_0\ \text{are arbitrary}). \qquad (A.32)$$

Note the minimal uncertainty condition

$$-\int d^3\xi\,\Delta^{(1)}(\vec{x}-\vec{\xi},0)\ddot{\Delta}^{(1)}(\vec{\xi}-\vec{y},0)=\delta(\vec{x}-\vec{y}), \qquad (A.33)$$

similar to the relation $A_{k\ell}A^{-1}_{\ell m}=\delta_{km}$.

APPENDIX B

Using the Hori functional approach the N-ordered form of S-matrices $\hat{U}(t,t')$ and $\hat{U}'(t,t')$ may be written as follows

$$\hat{U}(t,t')=:\exp\left(\int\hat{\varphi}\frac{\delta}{\delta\Phi}\right):\exp\left(\frac{\hbar}{2}\int\frac{\delta}{\delta\Phi}(-i)\Delta_+\frac{\delta}{\delta\Phi}\right)\exp\left(\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\mathcal{L}_I(\Phi(x))\right)\Big|_{\Phi=0}=$$

$$=:\exp\left(\int\hat{\varphi}\frac{\delta}{\delta\Phi}\right):\exp\left(\frac{\hbar}{4}\int\frac{\delta}{\delta\Phi}\Delta^{(1)}\frac{\delta}{\delta\Phi}\right)\exp\left(\frac{\hbar}{2}\int\frac{\delta}{\delta\Phi}(-i)\Delta_{sym}\frac{\delta}{\delta\Phi}\right)\exp\left(\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\mathcal{L}_I(\Phi(x))\right)\Big|_{\Phi=0} \qquad (B.1)$$

$$\hat{U}'(t,t')=:\exp\left(\int\hat{\varphi}\frac{\delta}{\delta\Phi}\right):\exp\left(\frac{\hbar}{2}\int\frac{\delta}{\delta\Phi}(-i)\Delta_{sym}\frac{\delta}{\delta\Phi}\right)\exp\left(\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\mathcal{L}_I(\Phi(x))\right)\Big|_{\Phi=0} \qquad (B.2)$$

The S-matrix in CSR-I and in CSR-2 is given by

x) The generalization $\omega\to\omega_k$ is evident.

$$\langle\varphi|\hat{U}(t,t')|\varphi\rangle=\exp\left(\int\varphi\frac{\delta}{\delta\phi}\right)\exp\left(\frac{\hbar}{2}\int\frac{\delta}{\delta\phi}(-i)\Delta_{+}\frac{\delta}{\delta\phi}\right)\exp\left(\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\mathcal{L}_I(\phi(x))\right)\Big|_{\phi=0}=$$

$$=\exp\left(\frac{\hbar}{2}\int\frac{\delta}{\delta\phi}(-i)\Delta_{+}\frac{\delta}{\delta\phi}\right)\exp\left(\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\mathcal{L}_I(\phi(x))\right)\Big|_{\phi=\varphi} \qquad (B.3)$$

$$\Lambda^{-1}\langle\varphi|\hat{U}(t,t')|\varphi\rangle=\langle\varphi|\hat{U}'(t,t')|\varphi\rangle=$$

$$=\exp\left(\int\varphi\frac{\delta}{\delta\phi}\right)\exp\left(\frac{\hbar}{2}\int\frac{\delta}{\delta\phi}(-i)\Delta_{sym}\frac{\delta}{\delta\phi}\right)\exp\left(\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\mathcal{L}_I(\phi(x))\right)\Big|_{\phi=0}=$$

$$=\exp\left(\frac{\hbar}{2}\int\frac{\delta}{\delta\phi}(-i)\Delta_{sym}\frac{\delta}{\delta\phi}\right)\exp\left(\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\mathcal{L}_I(\phi(x))\right)\Big|_{\phi=\varphi} \qquad (B.4)$$

The N-ordered form of the Heisenberg field operators $\hat{\varphi}(x)$ and $\hat{\varphi}'(x)$, and $\hat{\varphi}(x)$ in CSR-I and in CSR-2 are given by

$$\hat{\varphi}(x)=:\exp\left(\int\hat{\varphi}\frac{\delta}{\delta\phi}\right):\exp\left(\frac{\hbar}{4}\int\frac{\delta}{\delta\phi}\Delta^{(1)}\frac{\delta}{\delta\phi}\right)\exp\left(i\int\frac{\delta}{\delta\phi}\Delta_{ret}\frac{\delta}{\delta\tilde{\Phi}}\right)$$

$$\exp\left(-\frac{i}{\hbar}\int_{t'}^{t}d^4x\left[\mathcal{L}_I\left(\phi(x)+\frac{\hbar}{2}\tilde{\Phi}(x)\right)-\mathcal{L}_I\left(\phi(x)-\frac{\hbar}{2}\tilde{\Phi}(x)\right)\right]\right)\phi(x)\Big|_{\phi=\tilde{\Phi}=0} \qquad (B.5)$$

$$\hat{\varphi}'(x)=:\exp\left(\int\hat{\varphi}\frac{\delta}{\delta\phi}\right):\exp\left(i\int\frac{\delta}{\delta\phi}\Delta_{ret}\frac{\delta}{\delta\tilde{\Phi}}\right)$$

$$\exp\left(-\frac{i}{\hbar}\int_{t'}^{t}d^4x\left[\mathcal{L}_I\left(\phi(x)+\frac{\hbar}{2}\tilde{\Phi}(x)\right)-\mathcal{L}_I\left(\phi(x)-\frac{\hbar}{2}\tilde{\Phi}(x)\right)\right]\right)\phi(x)\Big|_{\phi=\tilde{\Phi}=0} \qquad (B.6)$$

$$\langle\varphi|\hat{\varphi}(x)|\varphi\rangle=\exp\left(\int\varphi\frac{\delta}{\delta\phi}\right)\exp\left(\frac{\hbar}{4}\int\frac{\delta}{\delta\phi}\Delta^{(1)}\frac{\delta}{\delta\phi}\right)\exp\left(i\int\frac{\delta}{\delta\phi}\Delta_{ret}\frac{\delta}{\delta\tilde{\Phi}}\right)$$

$$\exp\left(-\frac{i}{\hbar}\int_{t'}^{t}d^4x\left[\mathcal{L}_I\left(\phi(x)+\frac{\hbar}{2}\tilde{\Phi}(x)\right)-\mathcal{L}_I\left(\phi(x)-\frac{\hbar}{2}\tilde{\Phi}(x)\right)\right]\right)\phi(x)\Big|_{\phi=\tilde{\Phi}=0}=$$

$$=\exp\left(\frac{\hbar}{4}\int\frac{\delta}{\delta\phi}\Delta^{(1)}\frac{\delta}{\delta\phi}\right)\exp\left(i\int\frac{\delta}{\delta\phi}\Delta_{ret}\frac{\delta}{\delta\tilde{\Phi}}\right)$$

$$\exp\left(-\frac{i}{\hbar}\int_{t'}^{t}d^4x\left[\mathcal{L}_I\left(\phi(x)+\frac{\hbar}{2}\tilde{\Phi}(x)\right)-\mathcal{L}_I\left(\phi(x)-\frac{\hbar}{2}\tilde{\Phi}(x)\right)\right]\right)\phi(x)\Big|_{\substack{\tilde{\Phi}=0\\ \phi=\varphi}} \qquad (B.7)$$

$$\Lambda^{-1}\langle\varphi|\hat{\varphi}(x)|\varphi\rangle=\langle\varphi|\hat{\varphi}'(x)|\varphi\rangle=\exp\left(\int\varphi\frac{\delta}{\delta\phi}\right)\left(i\int\frac{\delta}{\delta\phi}\Delta_{ret}\frac{\delta}{\delta\tilde{\Phi}}\right)$$

$$\exp\left(-\frac{i}{\hbar}\int_{t'}^{t}d^4x\left[\mathcal{L}_I\left(\phi(x)+\frac{\hbar}{2}\tilde{\Phi}(x)\right)-\mathcal{L}_I\left(\phi(x)-\frac{\hbar}{2}\tilde{\Phi}(x)\right)\right]\right)\phi(x)\Big|_{\phi=\tilde{\Phi}=0}=$$

$$=\exp\left(i\int\frac{\delta}{\delta\phi}\Delta_{ret}\frac{\delta}{\delta\tilde{\Phi}}\right)\exp\left(-\frac{i}{\hbar}\int d^4x\left[\mathcal{L}_I\left(\phi(x)+\frac{\hbar}{2}\tilde{\Phi}(x)\right)-\mathcal{L}_I\left(\phi(x)-\frac{\hbar}{2}\tilde{\Phi}(x)\right)\right]\right)\phi(x)\Big|_{\substack{\tilde{\Phi}=0\\ \phi=\varphi}} \qquad (B.8)$$

Equations (B.5) and (B.7) are due to Bialynicki-Birula [15].

The fields $\phi(x)$ and $\tilde{\Phi}(x)$ are not realistic ones, but arbitrary c-number functions of x_μ, unlike $\varphi(x)$. Except for this Appendix, we avoid mixing of these two sorts of the functional dependences, and consider only the functionals of the field $\varphi(x)$. The operator

$$\exp\left(\frac{\hbar}{4}\int\frac{\delta}{\delta\phi}\Delta^{(1)}\frac{\delta}{\delta\phi}\right) \qquad (B.9)$$

represents Λ, but we use its representation only in terms of $\varphi(x)$.

Electrodynamics in CSR was considered in ref. [21c,d] and the situation is analogous to that in the scalar field theory. If the QED S-matrix is represented as follows

$$\hat{U}(t,t')=T\exp\frac{i}{\hbar}\int_{t'}^{t}d^4x\,\hat{j}_\mu(x)\hat{A}_\mu(x)=\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{i}{\hbar}\right)^m\int d^4x_1\ldots d^4x_m:\hat{A}_{\mu_1}(x_1)\ldots\hat{A}_{\mu_m}(x_m):\cdot$$

$$\cdot T\,\hat{j}_{\mu_1}(x_1)\ldots\hat{j}_{\mu_m}(x_m)\exp\left(-\frac{1}{2\hbar}\int d^4y\,d^4z\,\hat{j}_\nu(y)\overline{\hat{A}_\nu(y)\hat{A}_\lambda(z)}\hat{j}_\lambda(z)\right), \qquad (B.10)$$

where in the last line the T-ordering concerns only the spinor field, one easily obtains the expectation values over vector field coherent states

$$\langle A|\hat{U}|A\rangle=T\exp\left(-\frac{1}{2\hbar}\int d^4y\,d^4z\,\hat{j}_\nu(y)(-i\hbar)\Delta_{+}(y-z)\hat{j}_\nu(z)+\frac{i}{\hbar}\int d^4x\,\hat{j}_\mu(x)A_\mu(x)\right), \qquad (B.11)$$

$$\Lambda_A^{-1}\langle A|\hat{U}|A\rangle=T\exp\left(-\frac{1}{2\hbar}\int d^4y\,d^4z\,\hat{j}_\nu(y)(-i\hbar)\Delta_{sym}(y-z)\hat{j}_\nu(z)+\frac{i}{\hbar}\int d^4x\,\hat{j}_\mu(x)A_\mu(x)\right). \qquad (B.12)$$

Further we can produce the N-product decomposition for the spinor field and take the expectation values over spinor field coherent states, and the propagators $S_{+}(x-y)$ and $S_{sym}(x-y)$ enter in CSR-1 and CSR-2, respectively [21d].

Equations (B.11) and (B.12) are, however, final for the case of an external current $(\hat{j}_\mu(x)\to j^e_\mu(x))$.

For the S-matrix and the Heisenberg fields in QED we can also use the above functional method.

APPENDIX C

Derive eqs. (55),(56),(88) and (93). Using eq. (Б.I9) of ref./2Id/, for any dimension[x)] we obtain

$$\langle x''|(\Lambda^{\nu}|xp\rangle\langle xp|)|x'\rangle = \Lambda^{\nu}\langle x''|xp\rangle\langle xp|x'\rangle =$$

$$= N_1^2\Lambda^{\nu}\, e^{-(2\hbar)^{-1}A(x''-x)^2-(2\hbar)^{-1}A(x'-x)^2+i\hbar^{-1}p(x''-x')} \tag{C.I}$$

Further, we have

$$e^{\nu\frac{\hbar}{4}A\frac{\partial}{\partial p}\frac{\partial}{\partial p}}\, e^{i\hbar^{-1}p(x''-x')} = e^{-\frac{\nu}{4\hbar}A(x''-x')^2+i\hbar^{-1}p(x''-x')} \tag{C.2}$$

$$e^{\nu\frac{\hbar}{4}A^{-1}\frac{\partial}{\partial x}\frac{\partial}{\partial x}}\, e^{-(2\hbar)^{-1}A(x''-x)^2-(2\hbar)^{-1}A(x'-x)^2} =$$

$$= e^{\nu\frac{\hbar}{4}A^{-1}\frac{\partial}{\partial x}\frac{\partial}{\partial x}}\, N^2\int du\,dv\, e^{i(x''-x)u+i(x'-x)v-\frac{\hbar}{2}A^{-1}(u^2+v^2)} =$$

$$= N^2\int du\,dv\, e^{i(x''-x)u+i(x'-x)v-\frac{\hbar}{4}A^{-1}\left[(\nu+1)(u+v)^2+(u-v)^2\right]}. \tag{C.3}$$

Therefore,

$$\Lambda^{\nu}\langle x''|xp\rangle\langle xp|x'\rangle = e^{i\hbar^{-1}p(x''-x')}e^{-\frac{\nu+1}{4\hbar}A(x''-x')^2}\begin{cases}(\pi(\nu+1)\hbar)^{-\frac{3n}{2}}(\det A)^{\frac{n}{2}}e^{-\frac{1}{(\nu+1)\hbar}A\left(x-\frac{x'+x''}{2}\right)^2}, & \nu>-1,\\ \delta\left(x-\frac{x'+x''}{2}\right), & \nu=-1,\end{cases} \tag{C.4}$$

Analogously one obtains

$$\Lambda^{\nu}\langle p''|xp\rangle\langle xp|p'\rangle = e^{-i\hbar^{-1}x(p''-p')}e^{-\frac{\nu+1}{4\hbar}A^{-1}(p''-p')^2}\begin{cases}(\pi(\nu+1)\hbar)^{-\frac{3n}{2}}(\det A)^{-\frac{n}{2}}e^{-\frac{1}{(\nu+1)\hbar}A^{-1}\left(p-\frac{p'+p''}{2}\right)^2}, & \nu>-1,\\ \delta\left(p-\frac{p'+p''}{2}\right), & \nu=-1.\end{cases} \tag{C.5}$$

Hence,

$$\frac{1}{(2\pi\hbar)^{3n}}\int dp\,\Lambda^{\nu}\langle x''|xp\rangle\langle xp|x'\rangle = \delta(x''-x')\begin{cases}(\pi(\nu+1)\hbar)^{-\frac{3n}{2}}(\det A)^{\frac{n}{2}}e^{-\frac{1}{(\nu+1)\hbar}A(x-x')^2}, & \nu>-1\\ \delta(x-x'), & \nu=-1,\end{cases} \tag{C.6}$$

$$\frac{1}{(2\pi\hbar)^{3n}}\int dx\,\Lambda^{\nu}\langle p''|xp\rangle\langle xp|p'\rangle = \delta(p''-p')\begin{cases}(\pi(\nu+1)\hbar)^{-\frac{3n}{2}}(\det A)^{-\frac{n}{2}}e^{-\frac{1}{(\nu+1)\hbar}A^{-1}(p-p')^2}, & \nu>-1\\ \delta(p-p'), & \nu=-1\end{cases} \tag{C.7}$$

Using the formulas found, it is easy to obtain eqs. (55),(56),(88) and (93), for example, for

$$(2\pi\hbar)^{-3n}\int dp\,\Lambda^{-1}|xp\rangle\langle xp| = (2\pi\hbar)^{-3n}\int dp\,dx''\,dx'\,|x''\rangle\Lambda^{-1}\langle x''|xp\rangle\langle xp|x'\rangle\langle x'| = |x\rangle\langle x|. \tag{C.8}$$

---

x) $px = \vec{p}_1\vec{x}_1+\ldots+\vec{p}_n\vec{x}_n,\ A(x'-x)^2 \equiv A_{k\ell}\sum_{i=1}^{n}(x'_i-x_i)_k(x'_i-x_i)_\ell \quad (k,\ell=1,2,3)$
$A^{-1}(p'-p)^2 = (A^{-1})_{k\ell}\sum_{i=1}^{n}(p'_i-p_i)_k(p'_i-p_i)_\ell.$

$$\Lambda^{-1}|xp\rangle\langle xp| = \int dx''\,dx'\,|x''\rangle\Lambda^{-1}\langle x''|xp\rangle\langle xp|x'\rangle\langle x'| =$$

$$= \hbar^3\int dy\,da\, e^{-ipa}\,\delta(x-y)\,|y-\tfrac{\hbar}{2}a\rangle\langle y+\tfrac{\hbar}{2}a| =$$

$$= \hbar^3\int da\, e^{-ipa}\,|x-\tfrac{\hbar}{2}a\rangle\langle x+\tfrac{\hbar}{2}a| \tag{C.9}$$

Expectation values of $\hat{\rho} = |x'\rangle\langle x'|,\ |p'\rangle\langle p'|,\ |x_1p_1\rangle\langle x_1p_1|$ and $\Lambda^{-1}|x_1p_1\rangle\langle x_1p_1|$ are positive definite in x- and p-representations and in CSR-I and CSR-2 . However, the eigenstates $|n\rangle\langle n|$ of oscillator Hamiltonian, or $\hat{N}=\hat{a}^+\hat{a}$, are positive definite in CSR-I, but, in general, not positive definite in CSR-2:

$$|\langle xp|n\rangle|^2 = \frac{(aa^*)^n}{n!}|\langle xp|0\rangle|^2 = \frac{(aa^*)^n}{n!}e^{-a^*a} = \frac{(Ax^2+A^{-1}p^2)^n}{2^n n!\hbar^n}e^{-\frac{\hbar^{-1}}{2}(Ax^2+A^{-1}p^2)} \geq 0 \tag{C.10}$$

$$\Lambda^{-1}|\langle xp|0\rangle|^2 = 2\, e^{-\hbar^{-1}(Ax^2+A^{-1}p^2)} \qquad (n=0) \tag{C.11}$$

$$\Lambda^{-1}|\langle xp|1\rangle|^2 = \left[-2+4\hbar^{-1}(Ax^2+A^{-1}p^2)\right]e^{-\hbar^{-1}(Ax^2+A^{-1}p^2)} \qquad (n=1) \tag{C.12}$$

The latter density is not positive definite.

Two ways to obtain CSR-I and CSR-2 for $\hat{\rho}=|n\rangle\langle n|$ are

1) $$|\langle xp|n\rangle|^2 = \frac{(-1)^n}{n!}\left(\frac{d}{d\alpha}\right)^n g(xp\alpha)\Big|_{\alpha=1},\quad g(xp\alpha) = e^{-\alpha\frac{\hbar^{-1}}{2}(Ax^2+A^{-1}p^2)} \tag{C.13}$$

$$\Lambda^{-1}|\langle xp|n\rangle|^2 = \frac{(-1)^n}{n!}\left(\frac{d}{d\alpha}\right)^n\Lambda^{-1}g(xp\alpha)\Big|_{\alpha=1},\ \Lambda^{-1}g(xp\alpha) = \frac{2}{2-\alpha}e^{-\frac{\alpha\hbar^{-1}}{2-\alpha}(Ax^2+A^{-1}p^2)} \tag{C.14}$$

2) $$\langle n|xp\rangle\langle xp|m\rangle = \frac{1}{\sqrt{m!n!}}a^{*m}a^n|\langle xp|0\rangle|^2 \tag{C.15}$$

$$\Lambda^{-1}\langle n|xp\rangle\langle xp|m\rangle = \frac{1}{\sqrt{m!n!}}\left(a^*-\frac{\hbar}{2}\frac{\partial}{\partial a}\right)^m\left(a-\frac{\hbar}{2}\frac{\partial}{\partial a^*}\right)^n\Lambda^{-1}|\langle xp|0\rangle|^2$$

$$= \frac{1}{\sqrt{m!n!}}a^{*m}a^n\left[1-\frac{\hbar}{4r}\left(\frac{\partial}{\partial r}+\frac{2n}{r}\right)\right]^m\left[1-\frac{\hbar}{4r}\frac{\partial}{\partial r}\right]^n\Lambda^{-1}|\langle xp|0\rangle|^2 \tag{C.16}$$

where in the last equation the action-angle variables

$$a = r e^{i\varphi},\quad \frac{\partial}{\partial a} = \frac{a^*}{2r}\left(\frac{\partial}{\partial r}-\frac{i}{r}\frac{\partial}{\partial\varphi}\right) \tag{C.17}$$

are used. (Note that in eq. (C.16) it is easy to average over φ, and this gives $\delta_{mn}\Lambda^{-1}|\langle xp|n\rangle|^2$.)

In the many-dimensional case

$$|\langle xp|p_1\dots p_n\rangle|^2 = |a_{p_1}|^2\dots|a_{p_n}|^2\, e^{-a_p^* a_p}\,, \tag{C.18}$$

$$g(xp\alpha) = \exp\left(-a_\mu^* \alpha_{\mu\nu} a_\nu\right) = \exp\left(-(2\hbar)^{-1}\left(\sqrt{A}\,\alpha\sqrt{A}\,x^2 + \sqrt{A^{-1}}\,\alpha\sqrt{A^{-1}}\,p^2\right)\right), \tag{C.19}$$

$$\Lambda^{-1} g(xp\alpha) = \left[\det\left(1-\tfrac{1}{2}\alpha\right)\right]^{-1} \exp\left(-(2\hbar)^{-1}\left(\sqrt{A}\left(\alpha^{-1}-\tfrac{1}{2}\right)^{-1}\sqrt{A}\,x^2 + \sqrt{A^{-1}}\left(\alpha^{-1}-\tfrac{1}{2}\right)^{-1}\sqrt{A^{-1}}\,p^2\right)\right), \tag{C.20}$$

where α is a diagonal matrix.

In connection with eq. (81) note (cf. /26/)

$$\frac{\delta}{\delta J(x)}|\langle\varphi|0\rangle|^2 = -\varphi^{(1)}(x)\,|\langle\varphi|0\rangle|^2\,,$$

$$\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}|\langle\varphi|0\rangle|^2 = \left(\varphi^{(1)}(x_1)\varphi^{(1)}(x_2) - \Delta^{(1)}(x_1-x_2)\right)|\langle\varphi|0\rangle|^2,$$

$$\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}|\langle\varphi|0\rangle|^2 = -\left(\varphi^{(1)}(x_1)\Delta^{(1)}(x_2-x_3) + \varphi^{(1)}(x_2)\Delta^{(1)}\dots\right)|\langle\varphi|0\rangle|^2,$$

$$\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}\frac{\delta}{\delta J(x_3)}\frac{\delta}{\delta J(x_4)}|\langle\varphi|0\rangle|^2 = |\langle\varphi|0\rangle|^2\Big(\varphi^{(1)}(x_1)\varphi^{(1)}(x_2)\varphi^{(1)}(x_3)\varphi^{(1)}(x_4) -$$
$$- \varphi^{(1)}(x_1)\varphi^{(1)}(x_2)\Delta^{(1)}(x_3-x_4) - \varphi^{(1)}(x_3)\varphi^{(1)}(x_4)\Delta^{(1)}(x_1-x_2) - \dots$$
$$+\Delta^{(1)}(x_1-x_2)\Delta^{(1)}(x_3-x_4) + \Delta^{(1)}(x_1-x_3)\Delta^{(1)}(x_2-x_4) + \Delta^{(1)}(x_1-x_4)\Delta^{(1)}(x_2-x_3)\Big), \tag{C.21}$$

$$\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}|\langle\varphi|0\rangle|^2\Big|_{J=0} = \begin{cases} 0 & n \text{ odd} \\ (-1)^{\frac{n}{2}} \displaystyle\sum_{\substack{\text{over distinct} \\ \text{permutations} \\ \text{of } 1\dots n}} \Delta^{(1)}(x_1-x_2)\cdots\Delta^{(1)}(x_{n-1}-x_n) & n \text{ even} \end{cases} \quad \text{(permanent)} \tag{C.22}$$

Suitable positive and negative frequency projections of eq.(C.22) give free evolution transition probabilities of n-quantum states. Using eq. (C.22) one can represent $|0\rangle\langle 0|$ by the N-product decomposition (19):

$$|0\rangle\langle 0| = :e^{-\frac{i}{2\hbar}\int d^3x\, \hat\varphi^{(1)}(x)\overleftrightarrow{\partial_4}\hat\varphi(x)}: \tag{C.22}$$

This presentation is clear from

$$|\langle\varphi|0\rangle|^2 = \exp\left(-\frac{1}{2\hbar}\int d^4x\,d^4y\, J(x)\Delta^{(1)}(x-y)J(y)\right) = \exp\left(-\frac{i}{2\hbar}\int d^3x\,\varphi^{(1)}(x)\overleftrightarrow{\partial_4}\varphi(x)\right). \tag{C.23}$$

Note that

$$\hat N = \frac{i}{2\hbar}\int d^3x :\hat\varphi^{(1)}(x)\overleftrightarrow{\partial_4}\hat\varphi(x): \tag{C.24}$$

is one of the possible presentations of the operator of number of quanta. For others see ref./21b/ (Appendix Б) and ref./21d/ (eqs.(A.11)).

From eq. (C.22) it follows

$$|\varphi\rangle\langle\varphi| = e^{-\frac{1}{2\hbar}\int d^4x\,d^4y\, J(x)\Delta^{(1)}(x-y)J(y)} :e^{-\frac{i}{\hbar}\int d^3x\,\hat\varphi^{(1)}(x)\overleftrightarrow{\partial_4}\hat\varphi(x) - \frac{1}{\hbar}\int d^4x\, J(x)\hat\varphi^{(1)}(x)}: =$$
$$= :e^{-\frac{i}{2\hbar}\int d^3x\,(\hat\varphi^{(1)}(x)-\varphi^{(1)}(x))\overleftrightarrow{\partial_4}(\hat\varphi(x)-\varphi(x))}: \tag{C.25}$$

The derivatives $\dfrac{\delta}{\delta J(x_1)}\cdots\dfrac{\delta}{\delta J(x_{2n})}\dfrac{|\varphi\rangle\langle\varphi|}{|\langle\varphi|0\rangle|^2}$ (C.26)

and

$$\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_{2n})}\frac{\delta}{\delta J'(y_1)}\cdots\frac{\delta}{\delta J'(y_{2m})}\frac{|\langle\varphi|\hat U(t,t')|\varphi'\rangle|^2}{|\langle\varphi|0\rangle|^2\,|\langle\varphi'|0\rangle|^2} \tag{C.27}$$

and suitable projections give the n-quantum states $:\hat\varphi(x_1)\dots\hat\varphi(x_n):|0\rangle\langle 0|:\hat\varphi(x_{n+1})\dots\hat\varphi(x_{2n}):$ and probabilities for transitions between such states.

APPENDIX D

In the relativistic case($\hat H = \sqrt{\hat p^2+m^2}$) in CSR-2' $\left(\mathcal{H} = \sqrt{\left(p - i\frac{\hbar}{2}\frac{\partial}{\partial x}\right)_n^2 + m^2}\right)$ the free evolution can be represented by

$$G_0(\vec x\vec p t, \vec x'\vec p' t') = e^{-i\hbar^{-1}(t-t')(\mathcal{H}-\mathcal{H}^+)}\delta(\vec x-\vec x')\delta(\vec p-\vec p') = \tag{D.1.a}$$
$$= \delta(\vec p-\vec p')\,\hbar^3\int d\vec a\, e^{-i\vec p\vec a}\langle x+\tfrac{\hbar}{2}a|x'\rangle\langle x'|x-\tfrac{\hbar}{2}a\rangle = \tag{D.1.b}$$
$$= \delta(\vec p-\vec p')\,\hbar^3\int d^3a\, e^{-i\vec p\vec a}\left(-2i\partial_4^x\Delta^{(-)}(x-x'+\tfrac{\hbar}{2}a)\right)\left(-2i\partial_4^x\Delta^{(+)}(x-x'-\tfrac{\hbar}{2}a)\right) \tag{D.1.c}$$

where $x=(\vec x,t)$, $x'=(\vec x',t')$, $a=(\vec a,0)$. Equation (D.1.b) holds in the non-relativistic free case too, and also leads to eq. (85)[x]. Expression (D.1.c) may be presented via the covariant function:

$$e^{-i\hbar^{-1}(t-t')(\mathcal{H}-\mathcal{H}^+)}\delta(\vec x-\vec x') = \frac{1}{2\pi\hbar}\int dp_0\,\hbar^4\int d^4a\, e^{-ipa}\left(2i\partial_4^x\Delta^{(-)}(x-x'+\tfrac{\hbar}{2}a)\right)\left(2i\partial_4^x\Delta^{(+)}(x-x'-\tfrac{\hbar}{2}a)\right) =$$
$$= -\frac{1}{2\pi\hbar}\int dp_0\, 4\left(p_4 - i\frac{\hbar}{2}\frac{\partial}{\partial x_4}\right)\left(p_4 + i\frac{\hbar}{2}\frac{\partial}{\partial x_4}\right)J(x-x',a) \tag{D.2}$$

$$J(x,a) = \hbar^4\int d^4a\, e^{-ipa}\Delta^{(-)}(x+\tfrac{\hbar}{2}a)\Delta^{(+)}(x-\tfrac{\hbar}{2}a) =$$
$$= \frac{1}{\pi i\sqrt{-p^2}}\theta(p_0)\int_{-\infty}^{\infty} q\,dq\, e^{i2q\tau}\,\delta(p^2+m^2+q^2) \quad \left(\tau^2 = x^2 - \frac{(xp)^2}{p^2}\right) \tag{D.3}$$

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[x] The non-relativistic transformation function $\langle x''|x'\rangle$ is well-known. For the relativistic one see ref./21a/, eq. (Б.10).

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