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E2-93-71

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ON THE PROBLEM OF UNBOUNDEDNESS FROM BELOW OF THE SPINOR QED HAMILTONIAN

Submitted to «Physica Scripta»

1993

INTRODUCTION

F.Palumbo [1] showed that the spinor QED Hamiltonian H_{QED} is unbounded from below. He got this interesting result considering, in fact, the operator

$$\overline{H} = \frac{\int dS(0)\Omega^*_{pr} H_{QED} \Omega_{pr}}{\int dS(0)\Omega^*_{pr}\Omega_{pr}}, \tag{0.1}$$

here $dS(0) \equiv \prod_{\mathbf{k}\neq 0,\lambda=1,2} dq(\mathbf{k},\lambda)$, and Ω_{pr} is a trial function depending on the transversal photon variables $q(\mathbf{k},\lambda), \mathbf{k}\neq 0, |\mathbf{k}| < l, \lambda = 1,2$, on electron and positron degrees of freedom and on the zero momentum mode vector potential variable $\mathbf{q}(0)$ [1],[2]. Thus, $\overline{H}$ depends on $\mathbf{q}(0)$ and $\partial/\partial\mathbf{q}(0)$. The simplest choice of the probe function enabled F.Palumbo to get the operator $\overline{H}$ of the form

$$H_{QED} \rightarrow \overline{H}_1 = -\frac{1}{2}(\frac{\partial}{\partial \mathbf{q}(0)})^2 + \mathbf{q}(0)\cdot \mathbf{a}. \tag{0.2}$$

Here, $\mathbf{a}$ does not depend on $\mathbf{q}(0)$. The operator $\overline{H}_1$ is obviously unbounded from below. This result is obliged entirely to the new term H_p, see eq. (1.8). F.Palumbo has shown that *this term has to be introduced into the QED Hamiltonian that is given in text books* – see, e.g., [3]. In this work I substitute

$$dS(\mathbf{r}) \equiv d\mathbf{q}(0) \prod_{\mathbf{k}\neq 0, \mathbf{k}\neq\pm\mathbf{r}, \lambda=1,2} dq(\mathbf{k},\lambda) \tag{0.3a}$$

for $dS(0)$ in eq.(0.1) (here $\mathbf{r}$ is a fixed value of the photon momentum) and use a more sophisticated choice of the trial function (see eqs. (2.1), (2.2), (2.7), (2.8), (2.19) and (2.23)). The result is the formula

$$H_{QED} \rightarrow \overline{H}_2 = -\sum_{\lambda=1,2}[\frac{\partial}{\partial q(\mathbf{r},\lambda)}\frac{\partial}{\partial q(-\mathbf{r},\lambda)} + q(\mathbf{r},\lambda)q(-\mathbf{r},\lambda)\gamma^2]$$

$$+const + O(1/V), \tag{0.3}$$

$$\gamma^2 \equiv [e^2 c(m/l, r/l) + e^4 d(m/l, r/l)] l^2. \tag{0.4}$$

Here, e^2 is the nonrenormalized coupling constant, m is the electron mass parameter in the Lagrangian of the spinor QED, l is the momentum cut-off parameter, V is a large periodicity volume, $l \to \infty$, $V \to \infty$, $c(x,y)$ and $d(x,y)$ are some functions, $c(x,0)$ being positive , $c(0,0) > 0$, $d(x,y)$ is bounded if $x \geq 0, y \geq 0$. Equations (0.3) and (0.4), if $e^2 \ll 1$, indicate the existence of the negative photon squared mass of the order of magnitude $\sim e^2 l^2$ and unboundedness from below of the operators $\overline{H}_2$ and H_{QED}, as well as the operator h of the abstract.

The article is organized as follows. Section **I** contains rather voluminous preliminary explanations concerning: a) the spinor QED Hamiltonian and gauge-transformational properties of the variables, this Hamiltonian depends on, b) my method of the cut-off, c) the problem of **competibility** of the realistic cut-off with the Lorentz invariance and d) the idea of my proof. Section **II** contains the proof of the statement of the abstract, i.e. the derivation of eqs. (0.3) and (0.4). Appendix A contains derivation of some formulas which are necessary for the proof of Sec.**II**.

SEC.I. SOME PRELIMINARY EXPLANATIONS

Here, I shall consider the Hamiltonian h,

$$h = H_{QED} + H_2 \tag{1.1}$$

which is the sum of the QED cut-off Hamiltonian H_{QED} [1],[4],

$$H_{QED} = H_{0ph} + H_{0f} + H_1 + H_c + H_p, \tag{1.2}$$

and the positive term H_2:

$$H_{0ph} = \int [(\dot{\mathbf{B}}^{tr})^2 + (rot\mathbf{B}^{tr})^2] d\mathbf{x}/2$$

$$= \sum_{\mathbf{k}\neq 0,\lambda=1,2} [-\frac{\partial}{\partial q(\mathbf{k},\lambda)}\frac{\partial}{\partial q(-\mathbf{k},\lambda)} + k^2 q(\mathbf{k},\lambda)q(-\mathbf{k},\lambda)]/2, \tag{1.3}$$

$$H_{0f} = \sum E(\mathbf{p})[a^*(\mathbf{p},\sigma)a(\mathbf{p},\sigma) + b^*(\mathbf{p},\sigma)b(\mathbf{p},\sigma)]$$
$$= \int \psi_1^*(\mathbf{x})[-i\hat{\alpha}\nabla + \beta m]\psi_1(\mathbf{x})d\mathbf{x}, \tag{1.4}$$

here $E(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$,

$$H_1 = e \int \psi_1^*(\mathbf{x})\hat{\alpha}\psi_1(\mathbf{x})\mathbf{B}^{tr}(\mathbf{x})d\mathbf{x}, \tag{1.5}$$

$$\mathbf{B}^{tr}(\mathbf{x}) = \sum_{\mathbf{k}\neq 0,\lambda=1,2} \mathbf{e}(\mathbf{k},\lambda)q(\mathbf{k},\lambda)e^{i\mathbf{kx}}/\sqrt{V}, \tag{1.6}$$

$$H_c = e^2 \sum_{\mathbf{k}\neq 0} \frac{\rho(\mathbf{k})\rho(-\mathbf{k})}{2V\mathbf{k}^2}, \tag{1.7}$$

here $\rho(\mathbf{k}) = \int \psi_1^*(\mathbf{x})\psi_1(\mathbf{x})e^{i\mathbf{kx}}d\mathbf{x}$,

$$H_p = -\frac{1}{2}(\frac{\partial}{\partial \mathbf{q}(0)})^2 + e\mathbf{q}(0)\int \psi_1^*(\mathbf{x})\hat{\alpha}\psi_1(\mathbf{x})d\mathbf{x}/\sqrt{V}, \tag{1.8}$$

$$H_2 = M^2/2 \int [\mathbf{q}(0)/\sqrt{V} + \mathbf{B}^{tr}(\mathbf{x})]^2 d\mathbf{x}, \tag{1.9}$$

here $M^2 > 0$. In equations (1.2)-(1.9) H_{0ph} is the Hamiltonian of free transversal photons, $\mathbf{e}(\mathbf{k},\lambda)$ is the polarization vector of a photon with the momentum $\mathbf{k}$ and polarization index λ, $(\mathbf{k}\cdot\mathbf{e}(\mathbf{k},\lambda)) = 0$, $(\mathbf{e}^*(\mathbf{k},\lambda 1)\cdot\mathbf{e}(\mathbf{k},\lambda 2)) = \delta_{\lambda 1,\lambda 2}$, $\mathbf{e}^*(\mathbf{k},\lambda) = \mathbf{e}(-\mathbf{k},\lambda)$, $\mathbf{q}(0)$ is the spatially independent zero momentum mode of the vector potential [1], [2], [4], $\mathbf{q}(0) = \int \mathbf{B}(\mathbf{x})d^3x/\sqrt{V}$, H_{0f} is the Hamiltonian of free electrons and positrons, m is the fermion mass parameter, $a(\mathbf{p},\sigma)$ and $b(\mathbf{p},\sigma)$ are the annihilation operators of the electron and positron with the momentum $\mathbf{p}$ and spin projection σ, $\psi_1(\mathbf{x}) =$

$\sum[u(\mathbf{p},\sigma)a(\mathbf{p},\sigma)+v(\mathbf{p},\sigma)b^*(-\mathbf{p},\sigma)]e^{i\mathbf{px}}/\sqrt{V}$, $u(\mathbf{p},\sigma)$ and $v(\mathbf{p},\sigma)$ are the solutions of the Dirac equation with the energy $\pm E(\mathbf{p})$. The Hamiltonian H_1 describes the interaction of photons, electrons and positrons. The Hamiltonian H_c describes the Coulomb interaction between electrons and positrons. In the Fourier representation of the functions $\mathbf{B}^{tr}(\mathbf{x})$ and $\psi_1(\mathbf{x})$ one has $|\mathbf{k}| < l, |\mathbf{p}| < l$, (see, however, items 1.4. and 1.4.1.), l being the cut-off parameter, V is large periodicity cube, V tends to infinity.

1. The Hamiltonian H_{QED} is expressed in terms of the gauge invariant quantities $\mathbf{B}^{tr}, \mathrm{q}(0)$ and ψ_1. If the functions $A_\mu(x)$ and $\psi(x)$ in the Lagrangian

$$L = -(\partial_\mu A_\nu - \partial_\nu A_\mu)^2/4 - \psi^*\gamma_4[\gamma_\mu(\partial_\mu - ieA_\mu) + m]\psi,$$

which gives rise to the Hamiltonian H_{QED}, undergo gauge transformation $A_\mu \to A_\mu + \partial\lambda(x)/\partial x_\mu$, $\psi \to e^{ie\lambda(x)}\psi$, the variables $\mathbf{B}^{tr}, \mathrm{q}(0)$ remain constant, and the function $\psi_1(\mathbf{x})$ acquires a spatially independent phase multiplier (see, e.g.[4]). Thus, my method of the cut-off does not break down the gauge invariance.

H_p is the zero momentum mode term of the QED Hamiltonian [1]. The discovery of it enabled F.Palumbo to prove the unboundedness of the Hamiltonian H_{QED} from below [1].

1.1. Here, I am going to show that not only the Hamiltonian H_{QED}, but also the Hamiltonian $h = H_{QED} + H_2$, H_2 being positive, see eq.(1.9), is unbounded from below if the positive quantity M^2 is not too large so that the inequality

$$M^2 < \gamma^2, \tag{1.10}$$

is fulfilled (see eq.(0.4)).We have $c(m/l,0) > 0$, thus $\gamma^2 > 0$ if $e^2 \ll 1$

independently of the sign of the function $d(m/l, 0)$. *Thus, for small values of e^2 my consideration gives stronger result than that by F.Palumbo.*

1.1.1. Omitting the Coulomb term in the present consideration, one gets an essentially analogous consideration for the massless Yukawa model. For this model also there holds the statement analogous to that of item 1.1 , where, however, one has to take d=0,so that the restriction $e^2 < e_0^2$ of the abstract disappears.

1.1.2. Let us note that one cannot prove the unboundedness from below of the scalar QED-s Hamiltonian via the method of this work: in case of the scalar QED the squared oscillator frequency in eq.(0.3) is positive if $e^2 \ll 1$(and has the order of magnitude $\sim e^2 l^2$) . Thus, combining the spinor field with several charged scalar fields, one hopes to construct the QED model whose Hamiltonian is bounded from below.

1.2 The Palumbo's proof of the unboundedness from below of the QED Hamiltonian (see eq. (0.2)) is essentially based on using zero momentum mode q(0) [1]. On the contrary, my consideration has little to do with the zero momentum mode term H_p. *The unboundedness from below of the spinor QED Hamiltonian (if $e^2 \ll 1$) is the consequence of the fact, that in any QFT model with a trilinear interaction $g \times fermion^* \times boson \times fermion$ the g^2 perturbation theory correction to the boson squared mass is negative.* The latter fact is common knowledge since long ago.

1.3. It is worth mentioning here that *the starting point of my key construction (2.7), (2.8)* was an attempt to get a variational estimate of the type of eq.(0.1) by using the trial function Ω_{pr},
$\Omega_{pr} \equiv exp(\kappa_0 + \kappa_1 e)|0>$ where the function Ω_0,

$$\Omega_0 \equiv exp(\sum_{n=0}^{\infty} \kappa_n e^n)|0> \equiv e^K|0> \tag{1.11}$$

is the ground state wave function of the Schroedinger equation

$$(H_{QED} - E)\Omega = 0, \tag{1.12}$$

. and $|0>$ is the state of the bare fermion vacuum.

1.3.1. The exponential representation (1.11), being substituted into the Schroedinger equation, enables one to recurrently find functions-operators $\kappa_n, n = 0, 1, 2, ...$ (were the ground state to exist). These functions-operators depend on the zero momentum mode variables $\mathbf{q}(0)$, on the photon variables $q(\mathbf{k}, \lambda), \mathbf{k} \neq 0$ and on the electron and positron creation operators. Of course, the exponential representation is equivalent to the straightforward linear representation

$$\Omega_0 = \sum_{n \geq 0} \Omega_{0n} e^n. \tag{1.13}$$

One should stress, however, that the representation of the type (1.11) is preferable to that of (1.13). This fact was first noticed by F. Coester and R. Haag [5]. I did systematically use the exponential representation to consider the boson models $g(\phi^4)_2, g(\phi^4)_3$, and $g((\phi^*\phi)^2)_2$ [6]. Later I have generalized the formalism to enable one to consider also fermions [7]. This generalization essentially boils down to substituting expression (1.11) of the ground state wave function into the Schroedinger equation, multiplying this equation by the operator e^{-K} and using eq. (2.10a) (see also comments after eq. (2.10a)).

1.4. The consideration of the present work is of any value only if one believes that the cut-off Schroedinger equation (1.12) governs the QED. Of course, the cut-off Schroedinger equation approach to QED, even if the Hamiltonian is bounded from below (item 1.1.2.) , gives rise to problems with the Lorentz invariance -cf., e.g., analogous approach to the $g(\phi^4)_4$ model.

I hope, these problemes can be solved via a properly chosen realistic regularization (cut-off). I do mean the introduction into the Fourier representation of the function $\mathbf{B}^{tr}(\mathbf{x})$, eq.(1.6), of a photon form-factor $F_{ph}(\mathbf{k}, l)$ and introduction into the analogous representation of the fermion operator $\psi_1(\mathbf{x})$ of a fermion form-factor $F_f(\mathbf{p}, l)$. These cut-off representations of the the vector potential and the fermion operators are to be used only in the interaction terms H_1, H_c and H_p.

1.4.1. Let us denote by m_n^2 the n-th order perturbation theory contribution to the squared fermion mass in the spinor QED. Obviously, one has $m_0^2 = m^2$. Using the form factors

$$F_{ph}(\mathbf{k}, l) = \sum_{n\geq 0} F_{nph} e^{-n|\mathbf{k}|/l}, \sum_{n\geq 0} F_{nph} = 1,$$

$$F_f(\mathbf{p}, l) = \sum_{n\geq 0} F_{nf} e^{-nE(\mathbf{p})/l}, \sum_{n\geq 0} F_{nf} = 1, \tag{1.14}$$

I was able to exhibit the momentum dependence of the quantity m_2^2:

$$m_2^2 = e^2(m^2 \ln(l/m) const_1 + \mathbf{p}^2 const_2 + o(1/l)). \tag{1.15}$$

Here, $o(x) \to 0$ as $x \to 0$. The second and third order perturbation theory contributions to the squared mass of the fundamental particle in the $g\phi_4^4$ model show the analogous momentum dependence. I hope, it is possible to eliminate the momentum dependence of the squared mass by a proper choice of the constants F_n in form factors and thus, to construct the Lorentz-invariant perturbation theory.

1.4.2. The textbook by W. Heitler [3] contains the calculation of the quantity m_{2F}^2 which is the Feynman perturbation theory contribution to the quantity m_2^2 ([6], Chapter 6, sec. 29, item 1., equation (29.14')). The value of m_{2F}^2 does not depend on $|\mathbf{p}|$.

1.4.3. The point, however, is that $m_2^2 - m_{2F}^2 \equiv \delta m_2^2 \neq 0$. The Hamiltonian H_{QED} without the Palumbo term H_p gives $m_2^2 = m_{2tr}^2 + m_{2c}^2$, where subscripts "tr" and "c" denote parts of the quantity m_2^2 which originate due to the exchange of transversal photons and due to the Coulomb interaction. As for the quantity m_{2F}^2, it can be represented as

$$m_{2F}^2 = \sum_{\mathbf{k},\mu} Trace(A(\mathbf{k},\mathbf{p})\gamma_\mu B(\mathbf{k},\mathbf{p})\gamma_\mu) = \sum_1^4 m_{2F\mu}^2$$

$$= m_{2Ftr}^2 + m_{2Flong}^2 + m_{2F4},$$

where $m_{2Ftr}^2 = m_{2tr}^2$. Thus, one has $\delta m_2^2 = m_{2c}^2 - m_{2Flong}^2 - m_{2F4}^2$. Straightfoward calculation gives

$$\delta m_2^2 = e^2 \int F_f(\mathbf{p}+\mathbf{k}, l) F_{ph}(\mathbf{k}, l) F_f(\mathbf{p}, l) \mathbf{pk}/|\mathbf{k}|^3 d\mathbf{k},$$

Using equations

$$|\mathbf{p}+\mathbf{k}| = |\mathbf{k}| + \mathbf{pk}/|\mathbf{k}| + ..., F_f(\mathbf{k}, l) \to \Phi_f(|\mathbf{k}|/l), F_{ph}(\mathbf{k}, l) \to \Phi_{ph}(|\mathbf{k}|/l)$$

as $l \to \infty$, here, $\Phi_f(z)$ and $\Phi_{ph}(z)$ are some functions, one gets

$$\delta m_2^2 = e^2 \mathbf{p}^2/3 \int_0^\infty \Phi_{ph}(z)((d/dz)\Phi_f(z))dz$$

as $l \to \infty$, cf. eq. (1.15) .

1.4.4. If the integral here equals zero, the second order perturbation theory consideration is compatible with the Lorentz invariance.

1.5. In principle, the term $O(1/V)$ in eq. (2.23) is able to reverse the result of my consideration. Let it be, e.g.,
$O(1/V) = const(\sum_{\lambda=1,2} q(-\mathbf{r},\lambda)q(\mathbf{r},\lambda))^2/V$ and $const > 0$. Then, the operator (2.23) will be bounded from below so that my consideration cannot

exclude possibility that the Hamiltonian H_{QED} possesses the ground state. Let us denote it by Ω_{0V}. Let us also denote by Ω_{0good} the ground state of the spinor QED, which it would possess, were the operator (2.23) without the term $O(1/V)$ be bounded from below. **The point is that these two vacua are as drastically different**, as for instance are the ground states of the quantum mechanical Hamiltonians $H_{1,0}$ and $H_{-1,1/V}$, $(V \rightarrow +\infty)$, $H_{a,b} = -(d/dz)^2 + az^2 + bz^4$.

SEC.II. THE PROOF OF THE STATEMENT OF THE ABSTRACT

I shall prove this statement in several steps.

2. At first, I shall average the Hamiltonian (1.1) over the normalized photon state Ω_{ph},

$$\Omega_{ph} = const\, \exp(-\omega[\mathbf{q}(0)^2 + \sum_{\mathbf{k}\neq 0, \mathbf{k}\neq\pm\mathbf{r}; \lambda=1,2} q(\mathbf{k},\lambda)q(-\mathbf{k},\lambda)]), r \equiv |\mathbf{r}|, r, \omega > 0, \tag{2.1}$$

i.e., I shall consider the transformation

$$H_{QED} \rightarrow H_{QED1} = \int \Omega^*_{ph} H_{QED} \Omega_{ph} dS(\mathbf{r}), \tag{2.2}$$

see eq. (0.3a), and analogous transformation $h \rightarrow h_1$. Then, one gets $H_p \rightarrow const$, $H_{0f} \rightarrow H_{0f}$, $H_c \rightarrow H_c$,

$$H_{0ph} \rightarrow H_{0ph1} \equiv \sum_{\lambda=1,2} [-\frac{\partial}{\partial q(\mathbf{r},\lambda)} \frac{\partial}{\partial q(-\mathbf{r},\lambda)} + r^2 q(\mathbf{r},\lambda)q(-\mathbf{r},\lambda)] + const, \tag{2.3a}$$

$$H_1 \rightarrow H_{11} \equiv \frac{e}{\sqrt{V}} \sum_{\mathbf{p},\mathbf{s};\mathbf{s}=\pm\mathbf{r};\lambda;\sigma;\tau} q(\mathbf{s},\lambda)[a^*(\mathbf{p}+\mathbf{s},\sigma)b^*(-\mathbf{p},\tau)\mathbf{A}(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)$$
$$+b(-\mathbf{p}-\mathbf{s},\sigma)a(\mathbf{p},\tau)\mathbf{D}(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)$$

$$+a^*(\mathbf{p}+\mathbf{s},\sigma)a(\mathbf{p},\tau)\mathbf{B}(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)$$

$$+b(-\mathbf{p}-\mathbf{s},\sigma)b^*(-\mathbf{p},\tau)\mathbf{C}(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)]\cdot\mathbf{e}(\mathbf{s},\lambda)$$

$$\equiv H_{11}(a^*b^*)+H_{11}(ba)+H_{11}(a^*a+bb^*), \tag{2.3b}$$

$$H_2\rightarrow M^2(r^2\sum_\lambda q(\mathbf{r},\lambda)q(-\mathbf{r},\lambda)+const)\equiv H_{21}, \tag{2.4}$$

$$H_{QED1}=H_{0f}+H_{0ph1}+H_{11}+H_c+const, \tag{2.5}$$

Here

$$\mathbf{A}(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)=u^*(\mathbf{p}+\mathbf{s},\sigma)\hat{\alpha}v(\mathbf{p},\tau),$$

$$\mathbf{B}(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)=u^*(\mathbf{p}+\mathbf{s},\sigma)\hat{\alpha}u(\mathbf{p},\tau),$$

$$\mathbf{C}(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)=v^*(\mathbf{p}+\mathbf{s},\sigma)\hat{\alpha}v(\mathbf{p},\tau)$$

$$\mathbf{D}(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)=v^*(\mathbf{p}+\mathbf{s},\sigma)\hat{\alpha}u(\mathbf{p},\tau). \tag{2.6}$$

2.1 Then, let us determine the function Ω_f and the operator K,

$$\Omega_f=e^K|0>, \tag{2.7}$$

$$K=\sum_{\mathbf{p},\mathbf{s},\sigma\tau;\mathbf{s}=\pm\mathbf{r}}K(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)a^*(\mathbf{p}+\mathbf{s},\sigma)b^*(-\mathbf{p},\tau)\equiv\sum K(\mathbf{p}+\mathbf{s},\mathbf{p}), \tag{2.8}$$

(here $|0>$ is the state of the fermion bare vacuum: $a(\mathbf{p},\sigma)|0>=b(\mathbf{p},\sigma)|0>=0$ for all values of $\mathbf{p}$ and σ) by the equation

$$(H_{0f}+H_{11}(a^*b^*))\Omega_f=0. \tag{2.9}$$

One easily gets

$$K(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)=-\sum_\lambda\frac{eq(\mathbf{s},\lambda)\mathbf{A}(\mathbf{p}+\mathbf{s},\mathbf{p};\sigma,\tau)\cdot\mathbf{e}(\mathbf{s},\lambda)}{\sqrt{V}(E(p)+E(|\mathbf{p}+\mathbf{s}|))}. \tag{2.10}$$

In order to derive eq. (2.10) from eq. (2.9), it is sufficient to multiply eq.(2.9) by the operator e^{-K} and apply the formula

$$e^{-K}Ae^{K} = A + [A, K] + \frac{1}{2}[[A, K], K] + ... \qquad (2.10a)$$

(where square brackets denote the commutator), to the operators $A_1 \equiv H_{0f}$ and $A_2 \equiv H_{11}(a^*b^*)$. For the second operator all the commutators in eq. (2.10a) disappear, analogously for the operator A_1 the decomposition in the r.h.s. of eq. (2.10a) reduces to its first two terms. Thus, equation (2.9) becomes trivial.

(Note that if the operator A were, e.g., bilinear in annihilation operators and not to contain derivatives with respect to boson variables, the series (2.10a) would reduce to its first three terms.)

It follows from eqs. (2.7) and (2.8) that

$$\Omega_f = \prod_{\mathbf{p}}(1 + \sum_{s=\pm\mathbf{r}} K(\mathbf{p}+\mathbf{s}, \mathbf{p}) + \frac{1}{2}(\sum_{s=\pm\mathbf{r}} K(\mathbf{p}+\mathbf{s}, \mathbf{p}))^2)|0> . \qquad (2.11)$$

Here, $\prod_{\mathbf{p}}$ denotes the product over all values of $\mathbf{p}$, $|\mathbf{p}| < l$, terms with $|\mathbf{p}+\mathbf{s}| > l$ have to be omitted.

Let us denote the quantity $\Omega_f^*\Omega_f$ by Q. Eqs. (2.8) and (2.11) give

$$Q = (\prod_{\mathbf{p}}(1 + \sum_{s=\pm\mathbf{r}} D_1(\mathbf{p}+\mathbf{s}, \mathbf{p}) + \sum_{s=\pm\mathbf{r}} D_2(\mathbf{p}+\mathbf{s}, \mathbf{p}) + D_3(\mathbf{p}, \mathbf{r})),$$

$$D_1(\mathbf{p}+\mathbf{s}, \mathbf{p}) =< 0|^* K(\mathbf{p}+\mathbf{s}, \mathbf{p})^* K(\mathbf{p}+\mathbf{s}, \mathbf{p})|0 >,$$

$$D_2(\mathbf{p}+\mathbf{s}, \mathbf{p}) =< 0|^*(K(\mathbf{p}+\mathbf{s}, \mathbf{p})^*)^2 K(\mathbf{p}+\mathbf{s}, \mathbf{p})^2|0 > /4,$$

$$D_3(\mathbf{p}, \mathbf{r}) =< 0|^* K(\mathbf{p}+\mathbf{r}, \mathbf{p})^* K(\mathbf{p}-\mathbf{r}, \mathbf{p})^* K(\mathbf{p}-\mathbf{r}, \mathbf{p}) K(\mathbf{p}+\mathbf{r}, \mathbf{p})|0 >,$$

$$D_1(\mathbf{p}+\mathbf{s}, \mathbf{p}) = O(1/V), D_2(\mathbf{p}+\mathbf{s}, \mathbf{p}) = O(1/V^2), D_3(\mathbf{p}, \mathbf{r}) = O(1/V^2). \qquad (2.12)$$

We shall introduce the quantities Q_1 and D_1,

$$Q_1 = e^{D_1}, D_1 = \sum_{\mathbf{p},\mathbf{s};\mathbf{s}=\pm\mathbf{r}} D_1(\mathbf{p}+\mathbf{s},\mathbf{p}). \tag{2.13}$$

2.1.1. The following important formula holds:

$$Q = Q_1(1 + O(1/V)). \tag{2.14}$$

2.1.2. Equations (2.8) and (2.10) result in the definition

$$K(\mathbf{p}+\mathbf{s},\mathbf{p}) \equiv \sum_\lambda q(\mathbf{s},\lambda)\dot{K}(\mathbf{p}+\mathbf{s},\mathbf{p},\lambda). \tag{2.15}$$

Here the function $K(\mathbf{p}+\mathbf{s},\mathbf{p},\lambda)$ does not depend on the vector potential variables $\mathbf{q}(0), q(\mathbf{k},\lambda)$. Eqs. (2.6), (2.9), (2.10), (2.14) and (2.15) give

$$\Omega_f^*(H_{0f} + H_{11})\Omega_f = \Omega_f^*(H_{11}(ba) + H_{11}(a^*a + bb^*))\Omega_f \equiv Z_1 + Z_2, \tag{2.16}$$

$$Z_1 = -Qe^2 \sum_\lambda q(\mathbf{r},\lambda)q(-\mathbf{r},\lambda)Z(r,m,l) + O(1/V)), \tag{2.17}$$

$$Z(r,m,l) = \frac{2}{(2\pi)^3}\int_{|\mathbf{p}|<l,|\mathbf{p}+\mathbf{r}|<l} \frac{E(\mathbf{p}+\mathbf{r})E(\mathbf{p}) + (\mathbf{pr})^2/r^2 + \mathbf{pr} - m^2}{E(\mathbf{p})E(\mathbf{p}+\mathbf{r})(E(\mathbf{p}) + E(\mathbf{p}+\mathbf{r}))} d\mathbf{p}. \tag{2.17a}$$

The quantity Z_2 evidently , equals zero:

$$Z_2 = 0. \tag{2.18}$$

Let us introduce the notation Ω_{f1}:

$$\Omega_{f1} \equiv \Omega_f/\sqrt{Q}, \quad \Omega_{f1}^*\Omega_{f1} = 1. \tag{2.19}$$

There hold the formulas (see Appendix A)

$$\Omega_{f1}^* \sum_\lambda \frac{\partial}{\partial q(\mathbf{r},\lambda)}\frac{\partial}{\partial q(-\mathbf{r},\lambda)}\Omega_{f1} = \sum_\lambda [\frac{\partial}{\partial q(\mathbf{r},\lambda)}\frac{\partial}{\partial q(-\mathbf{r},\lambda)}$$

$$+\sum_{\mathbf{s}=\pm\mathbf{r}} X(\mathbf{s},\lambda)\frac{\partial}{\partial q(\mathbf{s},\lambda)}] + Y, \tag{2.20}$$

$$X(\mathbf{s},\lambda) = O(1/V), Y = const + O(1/V). \tag{2.20a}$$

2.1.3. Now let us consider the quantity C,

$$C = \Omega^*_{f1} H_c \Omega_{f1}. \tag{2.21}$$

It is convenient to represent H_c in a normal form. We shall symbolically write down this representation as $H_c = const_1 + const_2(a^*a + b^*b) + const_3(a^*b^* + ba) + const_4(a^*a^*b^*b^* + a^*b^*ba + bbaa)$. Correspondingly, we shall represent C as $C = C_1 + C_2 + C_3 + C_4$. Then, C_1 does not depend on the variables $q(\mathbf{s},\lambda), \mathbf{s} = \pm\mathbf{r}$,, while C_2 and C_4 depend on these variables quadratically and $C_3 = 0$. Rotational invariance and dimensional considerations give

$$C = e^2 m f(m/l, r/l) - \sum_\lambda q(\mathbf{r},\lambda) q(-\mathbf{r},\lambda) e^4 d(m/l, r/l) l^2 + O(1/V). \tag{2.22}$$

Here, $f(x,y)$ and $d(x,y)$ are some functions.

2.2. Equations (2.3a), (2.3b), (2.5) and (2.16-22) prove the formula

$$\Omega^*_{f1} H_{QED1} \Omega_{f1} = -\sum_{\lambda=1,2} [\frac{\partial}{\partial q(\mathbf{r},\lambda)}\frac{\partial}{\partial q(-\mathbf{r},\lambda)} +$$

$$q(\mathbf{r},\lambda) q(-\mathbf{r},\lambda)(e^2 c(m/l, r/l) + e^4 d(m/l, r/l))l^2] + const + O(1/V), c(x,0) > 0, \tag{2.23}$$

-cf. the formulas (0.3) and (0.4). Equation (2.4) gives

$$\Omega^*_{f1} H_{2,1} \Omega_{f1} = M^2(\sum_{\lambda=1,2} q(\mathbf{r},\lambda) q(-\mathbf{r},\lambda) + const).$$

Last two formulas complete the task of this section.

2.2.1. The starting point of my consideration of the problem of unboundedness from below of the operators (2.23) and the like is the statement that the operator $-(d/dz)^2 - \gamma^2 z^2, \gamma^2 > 0$, is unbounded from below.

ACKNOWLEDGEMENT

I am deeply indebted to Professor M.Consoli whose remark enabled me to get at the construction of eq. (2.10) and thus to overcome the impass I was in. I am obliged very much also to Dr. Ch.Devchand for his kind interest in the work and to Drs. A.B.Govorkov and M.I.Shirokov for useful criticism and remarks.

APPENDIX A

Here I shall prove eq. (2.20a). Equations (2.19) and (2.20) give

$$X(\mathbf{s},\lambda) = \Omega_f^* \sum_{\mathbf{p}} K(\mathbf{p}+\mathbf{s},\mathbf{p},\lambda)\Omega_f/Q + \sqrt{Q}\frac{\partial}{\partial q(\mathbf{s},\lambda)}(1/\sqrt{Q}), \qquad (A1)$$

$$Y = \Omega_f^* \sum_{\mathbf{p1},\mathbf{p2},\lambda} K(\mathbf{p1}+\mathbf{r},\mathbf{p1},\lambda)K(\mathbf{p2}-\mathbf{r},\mathbf{p2},\lambda)\Omega_f/Q$$

$$+\Omega_f^* \sum_{\mathbf{p},\mathbf{s},\lambda;\mathbf{s}=\pm\mathbf{r}} K(\mathbf{p}-\mathbf{s},\mathbf{p},\lambda))\Omega_f/\sqrt{Q}\frac{\partial}{\partial q(\mathbf{s},\lambda)}(1/\sqrt{Q})$$

$$+\sum_{\lambda}\sqrt{Q}\frac{\partial}{\partial q(\mathbf{r},\lambda)}\frac{\partial}{\partial q(-\mathbf{r},\lambda)}(1/\sqrt{Q}) \equiv Y_1 + Y_2 + Y_3. \qquad (A2)$$

Equations (2.11)- (2.14) give

$$\Omega_f^* \sum_{\mathbf{p}} K(\mathbf{p}+\mathbf{s},\mathbf{p},\lambda)\Omega_f =$$

$$= \sum_{\mathbf{p}} < 0|^* K(\mathbf{p}+\mathbf{s},\mathbf{p})^* K(\mathbf{p}+\mathbf{s},\mathbf{p},\lambda)|0 > Q(1+O(1/V))$$

$$= \frac{1}{2}\frac{\partial}{\partial q(\mathbf{s},\lambda)}Q_1[1+O(1/V)], \qquad (A3)$$

$$Y_1 = \sum_{\mathbf{p1},\mathbf{p2},\lambda} < 0|^* K(\mathbf{p1}+\mathbf{r},\mathbf{p1})^* K(\mathbf{p1}+\mathbf{r},\mathbf{p1},\lambda)|0 >$$

$$< 0|^* K(\mathbf{p2}-\mathbf{r},\mathbf{p2})^* K(\mathbf{p2}-\mathbf{r},\mathbf{p2},\lambda)|0 > (1+O(1/V))$$

$$= (\frac{1}{2})^2 Q_1^{-2} \sum_\lambda \frac{\partial Q_1}{\partial q(\mathbf{r},\lambda)}\frac{\partial Q_1}{\partial q(-\mathbf{r},\lambda)}(1+O(1/V)). \qquad (A4)$$

It follows from equations (A2) and (A3) that

$$Y_2 = \frac{1}{2}\sum_{\mathbf{s}=\pm\mathbf{r};\lambda}(\sqrt{Q}\frac{\partial}{\partial q(\mathbf{s},\lambda)}[1/\sqrt{Q}])(\frac{\partial}{\partial q(-\mathbf{s},\lambda)}Q_1)/Q_1(1+O(1/V)). \quad (A5)$$

Now note that eqs. (2.10)-(2.15) result in the formula

$$Q = \exp(const(\sum_{\lambda=1,2} q(\mathbf{r},\lambda)q(-\mathbf{r},\lambda)))(1+O(1/V). \qquad (A6)$$

So, equations (A1)-(A6) and eq. (2.14) entail eq. (2.20a). This result completes the consideration of Appendix A.

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Received by Publishing Department
on March 25, 1993.