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QUANTUM GROUP GAUGE THEORIES  
AND COVARIANT QUANTUM ALGEBRAS

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# 1 Introduction

The quantum deformations of the non-abelian gauge field theories have been discussed recently in many papers [1]-[9]. The basic idea of some of these papers [2],[5]-[7] is to consider the gauge transformation as a coadjoint action of a quantum group  $\mathcal{G}_q$  on the comodule  $F_{ij}$

$$F_{ii} \longrightarrow (T_{ij}T_{ki}^{-1} \otimes F_{jk}) , \quad \bar{Z} \longrightarrow \mathcal{G}_q \otimes \bar{Z} , \quad (1.1)$$

where the  $N \times N$  matrix elements  $\{T_{ij}\}$  are the generators of the quantum group  $\mathcal{G}_q$  while the elements  $F_{ij}$  are the generators of some associative noncommutative algebra with unity  $\bar{Z}$ . Here, the quantum group  $\mathcal{G}_q$  and the comodule  $F_{ij}$  are interpreted as the noncommutative analogs of the gauge group and a curvature (gauge field strength) 2-form, respectively. In the framework of this interpretation one can try to rewrite the comodule  $F_{ii}$  as  $F_{ii}^{\mu\nu} dx_\mu \wedge dx_\nu$  using the basis of 1-forms  $dx_\mu$  where the space-time coordinates  $\{x_\mu\}$  could be treated as the usual coordinates of the classical (Minkowski) space (see [1]-[3], [5]-[9]) or as noncommutative coordinates (see [4]).

It is tempting, according to the classical case, to realize the field strength 2-forms  $F_{ij}$  by the square of the covariant differential  $\nabla_{ik} = \delta_{ik}d - A_{ik}$

$$F_{ij} \equiv -\nabla_{ik}\nabla_{kj} = d(A_{ij}) - A_{ik} \wedge A_{kj} , \quad (1.2)$$

where the gauge potential 1-forms  $A_{ik}$  are certain noncommutative objects. In our paper [2] (see also [6],[8],[9]), we have assumed that it is possible to represent  $A_{ij}$  as  $A_{ij}^\mu dx_\mu$  and to define the differential  $d$  as  $d = dx_\mu \partial^\mu$  where  $\partial^\mu$  are the usual derivatives over the classical space - time coordinates. Then, for the Yang-Mills curvature 2-form  $F_{ij}$  we obtained the expression

$$\begin{aligned} F_{ij}^{\mu\nu} dx_\mu dx_\nu &\equiv -\frac{1}{2} [\nabla^\mu, \nabla^\nu]_{ij} dx_\mu dx_\nu = \\ &= \frac{1}{2} \{ (\partial^\mu A^{j\nu} - \partial^\nu A^{j\mu}) \otimes \sigma_{ij}^I + (A^{i\mu} A^{j\nu} - A^{i\nu} A^{j\mu}) \otimes \sigma_{ik}^I \sigma_{kj}^J \} dx_\mu dx_\nu , \end{aligned} \quad (1.3)$$

where  $\nabla_{ik}^\mu = \delta_{ik} \partial^\mu - A^{I\mu}(x) \sigma_{ik}^I$ ,  $\{\sigma_{ij}^I\}$  is a basis of the classical gauge Lie algebra represented by  $N \times N$  matrices and we imply the wedge product in the multiplication of the differential forms (we will omit  $\wedge$  in all formulas below).

In all formulations [1]-[9] of the noncommutative gauge field theories important opened problems were to give the explicit representation of the noncommutative fields  $A_{ij}$ ,  $F_{ij}, \dots$  by some operator functions of  $x_\mu$  and to identify the algebras generated by these fields (see, however, some proposals in [5],[7]-[9]). In this paper, we would like to continue the investigations of the papers [1]-[9] in order to fill this gap. We will consider the special case when the quantum group  $\mathcal{G}_q$  is  $GL_q(N)$ . We present, for this case, the explicit definition of the noncommutative gauge fields

$A_{ij}, F_{ij}, \dots$  as the generators of the  $GL_q(N)$ -covariant quantum algebras [10] which have appeared in the bicovariant differential calculus on the  $GL_q(N)$  [11,12]. In particular, we obtain that the 1-forms  $A_{ij}$  (2-forms  $F_{ij}$ ) obey the  $q$ -deformed anti-commutation (commutation) relations typical of the Cartan 1-forms (invariant vector fields) defined on the  $GL_q(N)$ -group [10,12] while bosonic (fermionic or veilbein 1-form) matter fields  $e_i$  commute as coordinates of the quantum hyperplane. Finally, the closed algebra of the covariant noncommutative fields  $\{e_i, A_{ij}, F_{ij}\}$  is given by the following structure relations:

$$\begin{aligned}
R_{12}e_1e_2 &= ce_2e_1, \quad R_{12}A_1R_{21}A_2 + A_2R_{12}A_1R_{12}^{-1} = 0, \\
e_1A_2 &= (\pm)R_{21}A_2R_{12}e_1, \quad R_{12}F_1R_{21}F_2 = F_2R_{12}F_1R_{21}, \\
e_1F_2 &= R_{21}F_2R_{12}e_1, \quad R_{21}A_2R_{12}F_1 = F_1R_{21}A_2R_{12}.
\end{aligned} \tag{1.4}$$

Here, the indices 1, 2 enumerate the matrix (for  $A, F$ ) and vector (for  $e$ ) spaces,  $R_{12}$  is the  $R$ -matrix for the  $GL_q(N)$ ,  $c = q$ ,  $(\pm) = +1$  for the scalar bosons,  $c = -1/q$ ,  $(\pm) = +1$  for the fermions,  $c = -1/q$ ,  $(\pm) = -1$  for the veilbein 1-forms and  $q$  is a parameter of deformation. In the formulas (1.1), (1.2) and (1.4) we do not specify the space-time and we do not use the explicit expansion of the differential forms  $e, A$  and  $F$  in the basis of the 1-forms  $dx_\mu$ . As we will see below it is very difficult to use the classical space-time for the algebraic construction presented in this paper. Note, however, that the naive formulas (1.3) typical for the classical space-time are more attractive from the point of view of physical applications than their algebraic analog (1.2) considered here and appropriate for the formulation of the quantum group gauge models essentially based on the abstract theory of quantum groups [13,14].

In this paper, we use the  $R$ -matrix formalism [15] which is extremely convenient for the formulation of the bicovariant differential calculus on the quantum groups. In the next section we consider the abstract algebraic construction of the noncommutative gauge theories. Particularly, we build up the noncommutative analogs of the Chern-Simons Lagrangians. In Sect.3 we discuss unsolved problems such as the problem of the explicit realization of the quantum group generators  $T_{ij}$  via the operator functions of the space-time coordinates  $x_\mu$ .

## 2 $Z_2$ -Graded Extension of $GL_q(N)$ and Quantum Group Gauge Theories

Let us consider a  $Z_2$ -graded extension of the algebra of functions on the linear quantum group with  $GL_q(N)$  generators  $\{T_{ij}\}$  and additional new generators  $\{(dT)_{kl}\}$  ( $i, j, k, l = 1, 2, \dots, N$ ) satisfying the following commutation relations [10,12,16]:

$$R_{12}T_1T_2 = T_2T_1R_{12}, \tag{2.1}$$

$$R_{12}(dT)_1 T_2 = T_2(dT)_1 R_{21}^{-1}, \quad (2.2)$$

$$R_{12}(dT)_1 (dT)_2 = -(dT)_2 (dT)_1 R_{21}^{-1}. \quad (2.3)$$

where  $R_{12} = R_{j_1 j_2}^{i_1 i_2}$  is the  $GL_q(N)$   $R$ -matrix satisfying the Hecke relation

$$R_{12} = R_{21}^{-1} + (q - q^{-1}) P_{12}, \quad (2.4)$$

$P_{12}$  is the permutation matrix. Here and below we use the notation of Ref.[15]. The new generators  $(dT)_{ij}$  can be considered as the differential 1-forms on the quantum group [10]-[12]. It appears that the  $Z_2$ -algebra with the commutation relations (2.1)-(2.3) (we denote this  $Z_2$ -extension of  $Fun(GL_q(N))$  by  $\mathcal{G}$  and call it the  $Z_2$ -graded quantum group in what follows) is simply the Hopf algebra. Indeed, the comultiplication  $\Delta$ , the counit  $\epsilon$  and the antipode  $S$  are defined by

$$\begin{aligned} \Delta(T) &= T \otimes T, \quad \epsilon(T) = 1, \quad S(T) = T^{-1}, \\ \Delta(dT) &= dT \otimes T + T \otimes dT, \quad \epsilon(dT) = 0, \quad S(dT) = -T^{-1} dT T^{-1}, \end{aligned} \quad (2.5)$$

and one can check the following axioms:

$$\begin{aligned} (id \otimes \Delta)\Delta(\mathcal{G}) &= (\Delta \otimes id)\Delta(\mathcal{G}), \\ m(\epsilon \otimes id)\Delta(\mathcal{G}) &= m(id \otimes \epsilon)\Delta(\mathcal{G}) = \mathcal{G}, \\ m(S \otimes id)\Delta(\mathcal{G}) &= m(id \otimes S)\Delta(\mathcal{G}) = 1\epsilon(\mathcal{G}). \end{aligned} \quad (2.6)$$

For our next purposes we introduce also the  $Z_2$ -graded Zamolodchikov algebra (denoted by  $Z$ ) generated by the operators  $\{e_i, (de)_i\}$  ( $i = 1, 2, \dots, N$ ) with the following commutation relations:

$$\begin{aligned} R_{12} e_1 e_2 &= c e_2 e_1, \quad (\pm)c R_{12} (de)_1 e_2 = e_2 (de)_1, \\ R_{12} (de)_1 (de)_2 &= -\frac{1}{c} (de)_2 (de)_1. \end{aligned} \quad (2.7)$$

One can recognize in these relations (for  $(\pm) = +1$ ) the Mues-Zumino formulas of the covariant differential calculus on the bosonic ( $c = q$ ) and fermionic ( $c = -1/q$ ) quantum hyperplanes [14,15,17] where  $e_i$  are the coordinates of the quantum hyperplane while  $(de)_i$  are the associated differentials. The choice  $(\pm) = -1$  corresponds to the case when  $e_i$  are bosonic ( $c = -1/q$ ) and fermionic ( $c = q$ ) vielbein 1-forms.

Let us introduce the left-coaction  $g_i$  of the  $Z_2$ -graded quantum group  $\mathcal{G}$  (2.1)-(2.3) on the generators of the algebra  $Z$  by virtue of the following homomorphism:

$$e_i \xrightarrow{g_i} \bar{e}_i = T_{ij} \otimes e_j, \quad (2.8)$$

$$(de)_i \xrightarrow{g_i} (\bar{de})_i = (dT)_{ij} \otimes e_j + T_{ij} \otimes (de)_j, \quad (2.9)$$

or in the equivalent form

$$\begin{pmatrix} e \\ de \end{pmatrix} \xrightarrow{g_i} \begin{pmatrix} T & 0 \\ dT & T \end{pmatrix} \otimes \begin{pmatrix} e \\ de \end{pmatrix}. \quad (2.10)$$

The algebra  $Z = \{e, de\}$  becomes now a left-comodule of  $\mathcal{G}$  with respect to the coaction (2.8)-(2.10). One can verify that all axioms for the comodule are fulfilled. For example, we have the following identity:

$$(\Delta \otimes id)g_l(Z) = (id \otimes g_l)g_l(Z). \quad (2.11)$$

The algebra  $Z$  with the generators (2.7) has the following expansion  $Z = \bigoplus_{n=0} \Omega^n(Z)$  where  $\Omega^n(Z)$  denotes the subspace of the differential  $n$ -forms. Notice that there exists a similar expansion for the  $Z_2$ -graded quantum group (2.1)-(2.3):  $\mathcal{G} = \bigoplus_{n=0} \Omega^n(\mathcal{G})$ .

We have already mentioned that the generators  $(de)_i$  and  $(dT)_{ij}$  could be considered as differentials of the variables  $e_i$  and  $T_{ij}$ . One can show that it is possible to extend the action of the differential  $d$  uniquely to the whole algebra  $\mathcal{G} \otimes Z$  in such a way that  $d$  obeys the graded Leibnitz rule (e.g.  $d(g \otimes Z) = d(g) \otimes Z + (-1)^k g \otimes d(Z)$ , where  $g \in \Omega^k(\mathcal{G})$ ) and  $d^2 = 0$ .

We have introduced the left-coaction  $g_l$  (2.8)-(2.10) because we would like to interpret it as a quantum group gauge transformation where the matrix  $T_{ij}$  is a non-commutative analog of a gauge group element and  $e_i$  are analogs of the components of the bosonic or the fermionic (veilbein 1-form) matter fields. Now the operations  $id \otimes g_l$  or  $\Delta \otimes id$  in (2.11) can be interpreted as the second quantum group gauge transformations.

It is natural to consider the additional term  $(dT)_{ij} \otimes e_j$  presented in (2.9) as the effect of the noncovariance of the covector  $(de)_i$  under the gauge rotation (2.8). The analogous situation occurs in the classical gauge theory where derivatives of matter fields are transformed in the noncovariant way. Usually, in the classical case, we introduce additional compensating gauge fields which recover the covariance. We would like to repeat this trick in the noncommutative case and assume that the algebra  $Z$  can be extended to  $\bar{Z}$  by adding new elements  $A_{ij}$ . We also assume that the operator  $d$  can be extended (as a differential) onto the whole algebra  $\bar{Z}$  and hence again this algebra is decomposed as

$$\bar{Z} = \bigoplus_{n=0} \Omega^n(\bar{Z}). \quad (2.12)$$

In order to perform our construction we postulate, first, that the operators  $A_{ij}$  belong to the subspace  $\Omega^1(\bar{Z})$  and, second, the operator  $(\nabla e)_i \in \Omega^1(\bar{Z})$  defined as

$$(\nabla e)_i = (de)_i - A_{ij}e_j, \quad (2.13)$$

is transformed homogeneously under (2.8)-(2.10) as the left-comodule

$$(\nabla e)_i \xrightarrow{g_l} T_{ij} \otimes (\nabla e)_j = T_{ij} \otimes ((de)_j - A_{jk}e_k). \quad (2.14)$$

According to the classical case we interpret the compensating operator  $A_{ij}$  satisfying (2.14) as a quantum deformation of a gauge potential 1-form or as a noncommutative

analog of a connection 1-form. Using (2.8)-(2.10) and (2.14) one can deduce the noncommutative analog of the gauge transformation for this gauge potential as

$$A_{ik} \xrightarrow{g_i} \widetilde{A}_{ik} = T_{ij} T_{jk}^{-1} \otimes A_{ji} + dT_{ij} T_{jk}^{-1} \otimes 1, \quad \widetilde{A}_{ij} \in \mathcal{G} \otimes \bar{Z}. \quad (2.15)$$

It is natural to call the noncommutative connection  $A$ , satisfying the transformation rule (2.15), the  $q$ -deformed gauge comodule. One can justify this terminology taking into account the identity

$$(\Delta \otimes id)g_i(A) = (id \otimes g_i)g_i(A). \quad (2.16)$$

Here we stress again that it is possible to interpret the action  $\Delta \otimes id$  (or  $id \otimes g_i$ ) presented in Eq.(2.16) as the second noncommutative gauge transformation and it is exactly what has been assumed in [2]. To define the algebra  $\bar{Z}$  explicitly we have to deduce commutation relations of  $A_{ij}$  with the generators  $e_i$  and  $(de)_j$ . First of all, we note that the components  $A_{ij}$  of the  $q$ -deformed 1-form connection generate some closed algebra. In order to find structure relations for this algebra we remark that there is a trivial representation for the generators  $A_{ij}$ , namely  $A = dTT^{-1} \otimes 1$ , which is gauge equivalent to the trivial connection  $A_{ij} = 0$  corresponding to the "flat noncommutative geometry". Using this representation and the relations (2.1)-(2.3) we conclude that the generators  $A_{ij}$  have to satisfy the following  $q$ -deformed anticommutation relation:

$$R_{12}A_1R_{21}A_2 + A_2R_{12}A_1R_{12}^{-1} = 0. \quad (2.17)$$

Note that the same relations for the noncommutative gauge fields have been postulated also in [7,9]. Now one can show directly that if  $A_{ij}$  obey (2.17) then the gauge transformed operators  $\widetilde{A}_{ij}$  (2.15) satisfy (2.17) too. Hence, the transformation (2.15) is the homomorphism of the algebra (2.17). In order to find the commutation relations  $A_{ij}$  with the generators  $\{e_i, (de)_i\}$  we postulate that the coordinates of the covector (2.13) commute in the same way as the components of 1-forms  $(de)_i$  (see Eqs.(2.7))  $R_{12}(\nabla e)_1(\nabla e)_2 = -\frac{1}{c}(\nabla e)_2(\nabla e)_1$ . Using Eq.(2.17) one can show that this relation is equivalent to two covariant relations

$$e_1A_2 = (\pm)R_{21}A_2R_{12}e_1, \quad (2.18)$$

$$(\pm)(de)_1A_2 = -R_{21}A_2R_{12}(de)_1 + (q - q^{-1})R_{12}A_2R_{12}A_1e_1. \quad (2.19)$$

These are unique relations which are covariant under the gauge co-transformations (2.8)-(2.10), (2.15) and allow one to push the operators  $\{e_i, (de)_i\}$  through the operators  $A_{ki}$ . The concise covariant form for Eq.(2.19) is

$$(\nabla e)_1A_2 = -(\pm)R_{21}A_2R_{12}(\nabla e)_1. \quad (2.20)$$

Now applying the second covariant derivative  $\nabla$  to the expression (2.13) and using the  $Z_2$ -graded Leibnitz rule we can define the noncommutative analog of the field strength (curvature) 2-form  $F$ . As a result, we have

$$\nabla(\nabla e) = -\left(d(A) - A^2\right)e = -Fe. \quad (2.21)$$

The next action of the covariant derivative on the formula (2.21) yields the Bianchi identities which can be written in the classical form  $d(F) = [A, F]$ . It is clear that the quantum gauge transformation (2.15) for the curvature  $F$  is represented as the coadjoint action

$$F_{ij} \xrightarrow{g \text{ ad}} \tilde{F}_{ij} = \left(T_{ik}T_{lj}^{-1}\right) \otimes F_{kl}. \quad (2.22)$$

The commutation relations for the operators  $dA_{ij}$  and  $F_{ij} = dA_{ij} - A_{ik}A_{kj}$  can be deduced from the quantum hyperplane condition

$$R_{12}(Fe)_1(Fe)_2 = c(Fe)_2(Fe)_1. \quad (2.23)$$

Differentiating Eq.(2.18) and using (2.18), (2.19) and (2.23) one can derive the relations

$$e_1F_2 = R_{21}F_2R_{12}e_1, \quad P_{12}^-F_1R_{21}F_2P_{12}^+ = 0, \quad (2.24)$$

where we have introduced the projectors  $P_{12}^\pm = (R_{12} \mp q^{\pm 1}P_{12})/(q + q^{-1})$ . The last equality in (2.24) is the consequence (see [10]) of the  $q$ -deformed commutation relations

$$R_{12}F_1R_{21}F_2 - F_2R_{12}F_1R_{21} = 0 \quad (2.25)$$

which are the same for the invariant vector fields defined on the  $GL_q(N)$ -group (see [10]-[12]). Eqs.(2.25) are known as the reflection equations [18] and also are the structure relations for the braided algebras [19]. To complete the definition of the algebra  $\tilde{Z}$  we present the unique covariant commutation relations for  $F$  and  $A$

$$F_1R_{21}A_2R_{12} = R_{21}A_2R_{12}F_1. \quad (2.26)$$

The commutation relations (2.7), (2.17), (2.18)-(2.20) and (2.24)-(2.26) completely define the algebra  $\tilde{Z}$ . We stress that this algebra is covariant under the gauge group coactions (2.8)-(2.10) and (2.15). We note that the possible relation

$$R_{12}(dA)_1R_{21}(dA)_2 = (dA)_2R_{12}(dA)_1R_{21}$$

(noncovariant under the gauge co-transformations (2.15) and (2.22)) postulated in [7] is inconsistent with Eqs.(2.25) and (2.26) and is not fulfilled in our approach.

Our final aim is to define the noncommutative Lagrangians which describe the quantum group gauge theories. To write down the Lagrangians invariant under the co-transformations (2.8)-(2.10), (2.15) we extend  $\tilde{Z}$  by virtue of introducing the  $Z_2$ -graded contragradient comodule  $(\bar{e}_i, d\bar{e}_j)$  with the following commutation relations:

$$\bar{e}_1 \cdot \bar{e}_2 R_{12} = c\bar{e}_2 \bar{e}_1, \quad (d\bar{e})_1 \bar{e}_2 = (\pm)c\bar{e}_2 (d\bar{e})_1 R_{21}, \quad (2.27)$$

$$(d\bar{e})_1 (d\bar{e})_2 R_{12} = -\frac{1}{c}(d\bar{e})_2 (d\bar{e})_1.$$

The quantum group gauge transformation of the vector  $(\bar{e}_i, d\bar{e}_j)$  is expressed as the following homomorphism of the algebra (2.27):

$$(\bar{e}, d\bar{e}) \xrightarrow{g_r} (\bar{e}, d\bar{e}) \otimes \begin{pmatrix} T^{-1}, & -T^{-1}dT T^{-1} \\ 0, & T^{-1} \end{pmatrix}, \quad (2.28)$$

where the generators  $T_{ij}$  and  $dT_{kl}$  are the same as in Eqs.(2.1)-(2.3). The commutation relations of the contragradient generators  $\{\bar{e}_i, d\bar{e}_j\}$  with the former generators of  $\bar{Z}$  can be found using covariance of these relations under the gauge co-actions (2.8)-(2.10), (2.15) and (2.28). For example, one can deduce:

$$\begin{aligned} ce_1\bar{e}_2 &= \bar{e}_2 R_{21}^{-1} e_1, \quad (\pm)c(de)_2\bar{e}_1 = \bar{e}_1 R_{21}(de)_2, \\ A_1\bar{e}_2 &= (\pm)\bar{e}_2 R_{12} A_1 R_{21}, \quad F_1\bar{e}_2 = \bar{e}_2 R_{12} F_1 R_{21}. \end{aligned}$$

Now we define the invariant Lagrangian for the noncommutative fields  $e_i$ ,  $\bar{e}_i$  and  $A_{ij}$  as

$$\mathcal{L} = \bar{e}_i (de_i - A_{ij} e_j). \quad (2.29)$$

One can interpret this as the Lagrangian for the noncommutative version of various discrete gauge models (see e.g. [20]).

In order to write down other quantum group gauge invariants, it is possible to use the field strength 2-form  $F$  which transforms as the adjoint comodule (2.22). Indeed, let us consider the following Lagrangians:

$$L_{top}^{(k)} = Tr_q(F^k) = Tr(DF^k) = D_{ij} F_{ji} F_{j_1 j_2} \dots F_{j_{k-1} i}, \quad (2.30)$$

where we introduce the notion of the  $q$ -deformed trace [2,15,12,21] defined by the matrix  $D_{ij}$ . For the  $GL_q(N)$  case this matrix is the diagonal matrix  $D_{ij} = q^{2i} \delta_{ij}$ . Using the wellknown feature of the  $q$ -trace  $Tr_q(TET^{-1}) = Tr_q(E)$  where  $[T_{ij}, E_{kl}] = 0$ , one can obtain that the expressions (2.30) are invariant under the gauge coaction (2.22). Moreover, one can show that  $L_{top}^{(k)}$  is a closed  $2k$ -form and  $L_{top}^{(k)} = d(L_{CS}^{(k)})$  where we have introduced the  $(2k-1)$ -form

$$L_{CS}^{(k)} = Tr_q \left\{ A(dA)^{k-1} + \frac{1}{h_1^{(k)}} A^3 (dA)^{k-2} + \dots + \frac{1}{h_{k-1}^{(k)}} A^{2k-1} \right\}. \quad (2.31)$$

Notice that  $L_{CS}^{(k)}$  could be interpreted as the noncommutative Chern-Simons Lagrangians. The constants  $h_i^{(k)}$  in (2.31) depend on the choice of the quantum group  $\mathcal{G}_q$  and the parameter of deformation  $q$ . For example, in the case of the  $GL_q(N)$ -group,  $L_{CS}^{(2)}$  reproduces the noncommutative analog of the three-dimensional Chern-Simons term and we obtain

$$L_{CS}^{(2)} = Tr_q \left\{ AdA - \frac{1}{h_1^{(2)}} A^3 \right\}, \quad h_1^{(2)} = 1 + \frac{1}{q^2 + q^{-2}}. \quad (2.32)$$



Let us stress that we have not specified the space-time  $M$  which can be commutative as well as non-commutative. There are some arguments (see e.g. Conclusion) that it is rather difficult to use classical space-time for the quantum group gauge theories presented here. That is why we assume that the space-time  $M$  is the  $n$ -dimensional quantum hyperplane with coordinates  $\{x_\mu\}$ . One can define for such space  $M$  the metric tensor  $C_{\mu\nu}$  (e.g. the metric matrix of  $SO_q(n)$ ), the  $n$ -form  $\mathcal{E} = dx_1 \cdots dx_n$  which is the analog of the volume element and construct the dual isomorphism "  $*$  " of the differential forms on  $M$  (Hodge map). Then, we postulate that there exists a map  $\pi^{-1} : M \mapsto \bar{Z}$  and it is possible to expand the differential forms  $F_{ij}$  over the basis of the 1-forms  $dx_\mu$

$$F_{ij} = F_{ij}^{\mu\nu} dx_\mu dx_\nu, \quad *F_{ij} = dx_{\mu_1} \cdots dx_{\mu_{n-2}} \mathcal{E}_q^{\mu_1 \cdots \mu_{n-2}}{}_{\mu\nu} F_{ij}^{\mu\nu}.$$

The coefficients  $F_{ij}^{\mu\nu}, \dots$  of the differential forms are operator functions of  $\{x_\mu\}$ . The quantum group gauge invariant Lagrangian for the  $q$ -deformed Yang-Mills field theory ( $n = 4$ ) can be represented, following the line proposed in [2],[6] as

$$\mathcal{L} = Tr_q (F * F) \sim D_{ij} F_{jk}^{\mu\nu} \mathcal{E} F_{ki, \mu\nu}.$$

Another attractive possibility is the choice of the space-time  $M$  isomorphic to the space of the quantum group  $GL_q(N)$ . In this case it is tempting to explore monopole-instanton, type gauge potential 1-forms ( $[\bar{A}, T] = 0, \bar{A}(z)dT = dT\bar{A}(q^2z)$ )

$$A_{ij} = dT_{ik} \bar{A}_{kl}(z) T_{lj}^{-1} = dT_{ik} T_{lj}^{-1} \bar{A}_{kl}(z),$$

where  $z = det_q T$  and  $\bar{A}_1(q^2z) R_{12}^{-1} \bar{A}_2(z) R_{21}^{-1} - R_{12}^{-1} \bar{A}_2(q^2z) R_{21}^{-1} \bar{A}_1(z) = 0$ .

Along the same way as above one can formulate the noncommutative version of the Einstein theory of gravity (see another approach presented in [22]). For this purpose we have to take the underlying Zamolodchikov algebras (2.7), (2.27) in the form with  $c = -1/q$ ,  $(\pm) = -1$  and interpret  $\{e_i, \bar{e}_i = e_j C_{ji}\}$  as the noncommutative analogs of the vielbein 1-forms. Then, we assume the expansion of  $F_{ij}$  in the form:  $F_1 = \bar{e}_2 F_{1,2} e_2$ . In this case, the Lagrangian  $\mathcal{L} = e_{i_1} \cdots e_{i_n} \mathcal{E}_q^{i_1 \cdots i_n} F$ , where  $F = Tr_{q_1} Tr_{q_2} (F_{1,2} P_{12} R_{12})$  is a scalar curvature, describes the noncommutative Einstein gravity and the gauge quantum group  $\mathcal{G}_q$  must be the quantum Lorentz group considered in [23]. Note that the quantum trace plays the essential role in our formulation of the noncommutative gauge theories. At the end of this Sect. we stress that the transformations (2.8), (2.9) were also discussed in [24].

### 3 Conclusion

Our aim in this paper has been to present the algebraic formulation of the  $q$ -deformed non-abelian gauge theories discussed in our previous paper [2]. It appears that the noncommutative analogs of the gauge and matter fields generate various covariant quantum algebras specific for the covariant differential calculi on the quantum groups and quantum hyperplanes. After this, the next step must be the construction of

the explicit representations of these covariant algebras and especially the explicit realization of the maps

$$M \mapsto \bar{Z} \text{ and } M \mapsto \mathcal{G}, \quad (3.1)$$

where  $M$  is a classical (or quantum) space-time. The solution of this problems is extremely important for the realization of the noncommutative fields in terms of the classical fields in order to find the field theoretical interpretation to the algebraic construction presented in the previous section. For example, we stress that the Lagrangians (2.29)- (2.31) are in general the noncommutative elements of the algebra  $\bar{Z}$  and to obtain the usual Lagrangians we have to make an additional averaging over this algebra (see [2]). However, it is worth notice that the Lagrangian (2.29) is the central element for the subalgebra of  $\bar{Z}$  generated by the operators  $A_{ij}$ ,  $F_{ij}$ ,  $e_i$  and  $\bar{e}_i$ , while (2.30) is the central element for the subalgebra with the generators  $A_{ij}$  and  $F_{ij}$ . Unfortunately, these Lagrangians are not the central elements for the whole algebra  $\bar{Z}$ .

To conclude this paper, we illustrate the problem (related to the explicit construction of the maps (3.1)) by considering the map  $M \mapsto GL_q(2)$  where  $M$  is a classical space-time. Let us explore the following obvious realization of  $T_{ij}(x)$  for the  $GL_q(2)$ -group

$$T_{ij}(x) = \begin{pmatrix} \alpha(x, z)\mathbf{a} + \bar{\delta}(x, z, \mathbf{bc})(1/d) & \beta(x, z)\mathbf{b} \\ \gamma(x, z)\mathbf{c} & \delta(x, z)\mathbf{d} + (1/\mathbf{a})\bar{\alpha}(x, z, \mathbf{bc}) \end{pmatrix}, \quad (3.2)$$

where  $T_{ij} = \begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{pmatrix}$  is the standard notation for the  $GL_q(2)$  generators [15] ( $\mathbf{a}$  and  $\mathbf{d}$  are invertible generators),  $\bar{\alpha} = \sum_{i=0} \bar{\alpha}_i(\mathbf{bc})^i$ ,  $\bar{\delta} = \sum_{i=0} \bar{\delta}_i(\mathbf{bc})^i$ ,  $z = \det_q(T) = \mathbf{ad} - q\mathbf{bc}$  is the central element for  $GL_q(2)$  and  $\alpha$ ,  $\beta$ ,  $\gamma$ ,  $\delta$ ,  $\bar{\alpha}_i$  and  $\bar{\delta}_i$  are classical functions of  $x_\mu$  and  $z$  which could be considered as the parameters of the gauge quantum group. One can prove that  $C(x) = \det_q(T(x))$  is the central element for  $GL_q(2)$  and the operators (3.2) commute as in (2.1) only if

$$(qW + \kappa_1)z^2 + (\bar{\kappa}_1 z - q\bar{\kappa}_0) = 0, \quad q(qW + \kappa_1) + \kappa_2 z + \bar{\kappa}_2 = 0, \quad (3.3)$$

$$\kappa_i z + q\kappa_{i-1} + \bar{\kappa}_i = 0 \quad (i > 2),$$

where  $\kappa_i = \alpha\bar{\alpha}_i + \delta\bar{\delta}_i$ ,  $\bar{\kappa}_i = \sum_{j=0}^i \bar{\delta}_j \bar{\alpha}_{i-j}$  and  $W = \alpha\delta - \beta\gamma$ . From Eqs.(3.3) we obtain  $C(x) \equiv \det_q(T(x)) = \alpha\delta z + \kappa_0 + \bar{\kappa}_0/z$ . However if we would like to interpret generators  $dT_{ij}(x)$  as the usual differentials of (3.2) with respect to the space-time coordinates  $x_\mu$  then we obtain the contradiction. Indeed the  $dT_{ij}(x)$  can be written down as

$$dT_{ij}(x) = dx_\mu \partial^\mu \begin{pmatrix} \alpha(x)\mathbf{a} + \bar{\delta}_i(x)(\mathbf{bc})^i \frac{1}{d} & \beta(x)\mathbf{b} \\ \gamma(x)\mathbf{c} & \delta(x)\mathbf{d} + \bar{\alpha}_i(x) \frac{1}{\mathbf{a}}(\mathbf{bc})^i \end{pmatrix}. \quad (3.4)$$

From here we immediately obtain that (3.2) and (3.4) do not realize the algebra (2.1)-(2.3). For example, the equation  $T_{21}(x)dT_{11}(x) = q^{-1}dT_{11}T_{21}$  easily deduced from (3.2), (3.4) obviously contradicts the commutation relations (2.2) which take the following form for the  $GL_q(2)$  case:

$$\begin{aligned} T_{21}dT_{11} &= qdT_{11}T_{21}, \quad T_{i2}dT_{1i} = qdT_{1i}T_{i2}, \quad T_{ij}dT_{ij} = q^2dT_{ij}T_{ij}, \\ [T_{21}, dT_{12}] &= (q - \frac{1}{q})dT_{11}T_{22}, \quad [T_{22}, dT_{11}] = 0, \quad d(R_{12}T_1T_2) = d(T_2T_1R_{12}). \end{aligned} \quad (3.5)$$

This example shows that the obvious map  $M \mapsto \mathcal{G}$  (3.2) is inconvenient for the construction of the quantum group gauge theories discussed in the previous section. Moreover, it seems very difficult to construct the map (3.1) with  $M$  which is the classical space-time. The realization of the appropriate maps (3.1) for the quantum space-time  $M$  is in progress now.

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