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ON THE GAUGE INDEPENDENCE OF ELASTIC ELECTRON-DEUTERON SCATTERING AMPLITUDE IN THE IMPULSE APPROXIMATION

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1 Introduction

It is well known that the gauge invariance principle imposes substantial constraints on the amplitudes of electromagnetic (EM) processes on compound systems. The continuity equation for the EM current density operator and the Ward–Takahashi (WT) identities for the corresponding Green functions follow from this principle to the first order in e. Here we shall consider the WT identity for the 5-point Green function and its consequences for the Mandelstam current that determines the electron scattering amplitude off the deuteron in the framework of the Bethe–Salpeter (BS) formalism [1, 2].

One should emphasize that the continuity equation for a primary (Noether) current and effective current operators, e.g., the Mandelstam current or conserved currents in nonrelativistic quantum mechanics, is insufficient to guarantee the gauge independence (GI) of the EM transition matrix elements (cf. [2, 3, 4]). In addition, the initial and final states must be consistent with the current.

As a rule the conserved deuteron EM current involves the two-body contributions associated with meson exchange (interaction) currents (MEC). At the same time it has been proved [5, 6] that the elastic e-d scattering amplitude in the impulse approximation (IA), i.e., only with the one-body free e-N scattering contribution included, may be gauge independent itself. This observation has been made using the BS formalism both with the OBE and separable potentials. At first sight the result does not agree with the commonly accepted standpoint.

The aim of our paper is to study this situation more thoroughly. We shall try to analyze it for other N-N interaction models. Applications to the elastic electron scattering on the pion (the two-body system at a quark level) will be given as well. In addition, the consideration for the two-body systems will be extended via some generalization [7, 8] of the WT identity for arbitrary system of charged particles.

2 Gauge invariance and gauge independence in the Bethe-Salpeter formalism

The Mandelstam current for a two-fermion system consists of the one-body, $\Lambda_\mu^{[1]}$, and two-body, $\Lambda_\mu^{[2]}$, parts:

$$\Lambda_\mu = \Lambda_\mu^{[1]} + \Lambda_\mu^{[2]} \tag{1}$$

which meet the following relationships [1] (see also [2]):

$$\begin{aligned} iq^\mu\Lambda_\mu^{[1]}(p,k;P,K) &= e_1\delta(p-k-\frac{q}{2}) \times \\ &\times \left[S^{(1)}(\frac{K}{2}+k)^{-1} - S^{(1)}(\frac{P}{2}+p)^{-1}\right] S^{(2)}(\frac{K}{2}-k)^{-1} + \\ e_2\delta(p-k+\frac{q}{2}) &\times \\ &\times \left[S^{(2)}(\frac{K}{2}-k)^{-1} - S^{(2)}(\frac{P}{2}-p)^{-1}\right] S^{(1)}(\frac{K}{2}+k)^{-1}, \end{aligned} \tag{2}$$

$$\begin{aligned} iq^\mu\Lambda_\mu^{[2]}(p,k;P,K) &= e_1V(p-\frac{q}{2},k;K) - V(p,k+\frac{q}{2};P)e_1 + \\ &+ e_2V(p+\frac{q}{2},k;K) - V(p,k-\frac{q}{2};P)e_2, \end{aligned} \tag{3}$$

where k and K (p and P) are the relative and total 4-momenta of virtual fermions involved in the initial and final states, $q=(\omega,\mathbf{q})$ is the momentum transfer, $P=K+q$, $S^{(i)}(p)$ is the dressed fermion propagator, and $V(s',s;P)$ is the kernel of the BS equation which depends on the relative momenta s', s and the total momentum P. Besides, in the isospin formalism

$$e_i = |e|\frac{1+\tau_z(i)}{2}, \tag{4}$$

where e is the electron charge, $\vec{\tau}(i)$ is the Pauli matrix, and $i=1,2$.

It should be noted that Eqs.(2) and (3) do not allow one to determine the current unambiguously. They only impose constraints on the longitudinal component of the current.

The elastic e-d scattering amplitude can be written as [1]

$$\mathcal{T}_{fi} = \varepsilon^\mu \int \bar{\chi}_P(p)\Lambda_\mu(p,k;P,K)\chi_K(k)\, d^4k d^4p \equiv \varepsilon^\mu M_\mu, \tag{5}$$

where $\chi_K(k)(\bar{\chi}_P(p))$ is the BS amplitude describing the initial (final) state, and ε^μ is the polarization vector of the virtual photon.

The GI condition

$$q^\mu M_\mu = 0 \tag{6}$$

will be fulfilled if the current satisfies the identities (2) and (3) and besides the BS amplitude χ_K (and $\bar{\chi}_P$) is a solution of the BS equation with the same kernel

$$S^{(1)}(\frac{K}{2}+k)^{-1}S^{(2)}(\frac{K}{2}-k)^{-1}\chi_K(k) + \int V(k,p;K)\chi_K(p)\, d^4p = 0. \tag{7}$$

3 Gauge independence in the impulse approximation

In the IA (see the figure) the amplitude takes the form

$$\begin{aligned} M_\mu^{[1]} = \int \bar{\chi}_P(p)\Big[&\Gamma_\mu^{(1)}(\frac{P}{2}+p,\frac{K}{2}+k)S^{(2)}(\frac{K}{2}-k)^{-1}\delta(p-k-\frac{q}{2}) + \\ &+\Gamma_\mu^{(2)}(\frac{P}{2}-p,\frac{K}{2}-k)S^{(1)}(\frac{K}{2}+k)^{-1}\delta(p-k+\frac{q}{2})\Big]\chi_K(k)\, d^4k d^4p, \end{aligned} \tag{8}$$

where $\Gamma_\mu(p',p)$ is the γNN vertex function (irreducible) which obeys the one-body WT identity [9]:

$$q^\mu\Gamma_\mu^{(i)}(p',p) = e_i\Big[S^{(i)}(p')^{-1} - S^{(i)}(p)^{-1}\Big]. \tag{9}$$

Here we shall not discuss the construction of the vertex which, in general, includes the nucleon off-mass-shell effects (see, e.g., [10]). Note that the WT identity for the on-mass-shell γNN vertex $\bar{\Gamma}_\mu(p',p)$, which can be

[1] We omit here the spinor and isospin indices.

expressed through the Dirac ($F_1(q^2)$) and Pauli ($F_2(q^2)$) form factors, reads

$$q^\mu \bar{\Gamma}^{(i)}_\mu(p',p) = F_1^{(i)}(q^2)\left[S_0^{(i)}(p')^{-1} - S_0^{(i)}(p)^{-1}\right], \quad (10)$$

where $S_0(p)$ is the free fermion propagator. Using these ingredients leads immediately to the standard IA.

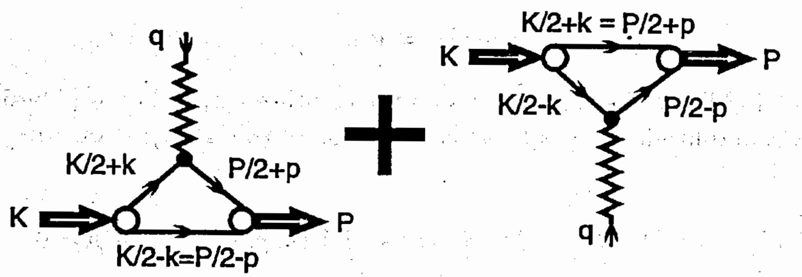

Figure : The amplitude for elastic electron-deuteron scattering in the impulse approximation

The GI condition of interest

$$q^\mu M^{[1]}_\mu = 0 \quad (11)$$

implies that

$$iq^\mu M^{[2]}_\mu = \int \bar{\chi}_P(p)\Big[e_1 V(p-\frac{q}{2},k;K) - V(p,k+\frac{q}{2};P)e_1 + $$
$$+ e_2 V(p+\frac{q}{2},k;K) - V(p,k-\frac{q}{2};P)e_2\Big]\chi_K(k)\, d^4k d^4p = 0. \quad (12)$$

It turns out that this relation is fulfilled for some models of the interaction kernel. At first, let us separate the isospin structure of V:

$$V(p,k;P) = \sum_{T=0,1} \Pi_T V_T(p,k;P), \quad (13)$$

where Π_T is the projector onto the state with the total isospin T and V_T is the corresponding component of the interaction. Choosing the ladder approximation

$$V_T(p,k;K) = v_T(p-k), \quad (14)$$

one can verify that Eq.(12) is satisfied identically. This result holds regardless of the value of the isospin of the initial (final) state, though, of course, in the case of the deuteron only the component V_0 contributes to Eq.(12). Note that the OBE interactions have the dependence (14) and, therefore, the corresponding amplitude will be gauge independent. This conclusion is in agreement with the statement of ref.[5].

The next example is the interaction of separable type. Many-rank separable BS kernels are being used in calculations of the properties of few-body nuclei [6, 11]. They have the structure

$$V(p,k;K) = \sum_{i,j} \lambda_{ij} g_i(p)\bar{g}_j(k), \quad (15)$$

where $g_i(p)$ and $\bar{g}_j(p)$ are the form factors, and $\lambda_{ij} = \lambda_{ji}$ are the constants of the model. The authors of [6] have proved the GI of the IA for the interaction (15). The proof was based on using the transformation properties of all the quantities in (12) with respect to the Lorentz boosts.

At this point one should emphasize that the result arises from the fact that the vertex function

$$\phi(p,P) = S^{(1)}(\frac{P}{2}+p)^{-1} S^{(2)}(\frac{P}{2}-p)^{-1}\chi_P(p) \quad (16)$$

for the model (15) is independent of the total momentum P. For example, in the case of the one-rank separable interaction one has

$$\phi(p,P) = \mathcal{N} g(p), \quad (17)$$

where $\mathcal{N}$ is the normalization factor calculated at $P^2 = M^2$ with M being the mass of the bound state

$$\mathcal{N} = \left[-\int \bar{g}(k)\frac{\partial}{\partial P^2} S^{(1)}(\frac{P}{2}+k) S^{(2)}(\frac{P}{2}-k) g(k)\, d^4k\right]^{-1/2} \quad (18)$$

Now let us consider the elastic pion form factor in the generalized Nambu–Jona-Lasinio model [12]. The $q\bar{q}$ interaction is chosen there in the form

$$V_{\alpha\beta;\delta\gamma}(p,k) = gf(p^2)f(k^2)\left[I_{\alpha\beta}I_{\delta\gamma} - (\gamma^5\vec{\tau})_{\alpha\beta}(\gamma^5\vec{\tau})_{\delta\gamma}\right], \tag{19}$$

where $f(k^2)$ is the model form factor, g is the coupling constant, I is a diagonal matrix, and the Greek indices are used for the Lorentz and flavor indices.[2] The $\pi q\bar{q}$ vertex function is

$$\phi(p,P) = \mathcal{N}\gamma^5 f(p^2)\chi_f, \tag{20}$$

where χ_f describes the pion isovector state. The corresponding pion BS amplitudes are written as

$$\chi_K(k) = S(k+\frac{K}{2})\phi(k,K)S(k-\frac{K}{2}), \tag{21}$$

$$\bar{\chi}_P(p) = S(p-\frac{P}{2})\bar{\phi}(p,P)S(p+\frac{P}{2}), \tag{22}$$

and

$$\bar{\phi}(k,K) = -\gamma_0\phi(k,K)^\dagger\gamma_0. \tag{23}$$

In this model the quantity $q^\mu M_\mu^{[2]}$ is equal to

$$\begin{aligned} q^\mu M_\mu^{[2]} = g\Sigma\left(K^2\right)\int d^4k f(k^2) Sp\left[S(k-\frac{P}{2})\gamma^5 S(k+\frac{P}{2})\gamma^5\right] \times \\ \times\left[e_1 f\left((k-\frac{q}{2})^2\right) + e_2 f\left((k+\frac{q}{2})^2\right)\right] - \\ -g\Sigma\left(P^2\right)\int d^4k f(k^2) Sp\left[S(k-\frac{K}{2})\gamma^5 S(k+\frac{K}{2})\gamma^5\right] \times \\ \times\left[e_1 f\left((k+\frac{q}{2})^2\right) + e_2 f\left((k-\frac{q}{2})^2\right)\right], \end{aligned} \tag{24}$$

$$\Sigma\left(K^2\right) = |\mathcal{N}|^2\int d^4k Sp\left[\gamma^5 S(k-\frac{K}{2})\gamma^5 S(k+\frac{K}{2})\right] f^2(k^2), \tag{25}$$

where μ_π is the pion mass, and $K^2 = P^2 = \mu_\pi^2$.

[2]The irrelevant dependence on the color degrees of freedom is omitted.

After calculating the traces in (24) we find

$$\begin{aligned} q^\mu M_\mu^{[2]} = 4g\Sigma\left(\mu_\pi^2\right)(e_1+e_2)\int d^4k(k^2+m^2-\frac{\mu_\pi^2}{4})f(k^2) \times \\ \times\left\{\frac{f\left((k-\frac{q}{2})^2\right)}{\left[(k-\frac{P}{2})^2-m^2\right]\left[(k+\frac{P}{2})^2-m^2\right]}\right. \\ \left. - \frac{f\left((k+\frac{q}{2})^2\right)}{\left[(k-\frac{K}{2})^2-m^2\right]\left[(k+\frac{K}{2})^2-m^2\right]}\right\} \end{aligned} \tag{26}$$

with m being the quark mass.

In order to see that the expression in the r.h.s. of Eq.(26) takes to zero we consider it in the Breit frame where

$$\begin{aligned} q = (0,0,0,q_B), \quad K = (E_B,0,0,-q_B/2), \quad P = (E_B,0,0,q_B/2), \\ E_B = \sqrt{\mu_\pi^2 + q_B^2/4}, \quad q^2 = -q_B^2. \end{aligned} \tag{27}$$

On replacing $\mathbf{k} \to -\mathbf{k}$ the product of the denominators in the second term in (26) reduces to that in the first term and

$$f\left((k+\frac{q}{2})^2\right) \to f\left(k_0^2 - (-\mathbf{k}+\frac{\mathbf{q}_B}{2})^2\right) = f\left((k-\frac{q}{2})^2\right). \tag{28}$$

Therefore $q^\mu M_\mu^{[2]} = 0$. As the quantity in question is a scalar, the result does not depend on the choice of reference frame.

Thus, the ladder and separable BS kernels lead to the gauge independent elastic scattering amplitudes. The common feature of these kernels is their independence of the total momentum of the pair. Note that nonrelativistic potentials have this property. In the next section we shall investigate the GI problem employing the identity [7, 8]. It will allow us to understand the origin of the above result from more general point of view.

4 Operator analogue of the Ward– Takahashi identity

Here we write down the Hamiltonian

$$H = K + V, \tag{29}$$

for a system of interacting particles. By definition, the operator H consists of the free (kinetic) part, K, and the interaction, V. According to [8] the 4-divergence of the EM current operator can be written as

$$q^\mu J_\mu(\mathbf{q}) = \omega\rho(\mathbf{q}) - [H, \rho(\mathbf{q})] = G(z_f)^{-1}\rho(\mathbf{q}) - \rho(\mathbf{q})G(z_i)^{-1}, \tag{30}$$

where $G(z)$ is the propagator

$$G(z) = (z - H)^{-1} \tag{31}$$

and the arbitrary parameters z_i and z_f satisfy the relation $z_f - z_i = \omega$. While obtaining Eq.(30) the continuity equation for the current has been used.

The GI condition for EM transition amplitudes between the eigenstates of H with the energies E_i and E_f follows from (30) if one chooses $z_i = E_i$ and $z_f = E_f$:

$$q^\mu (f|J_\mu(\mathbf{q})|i) = 0. \tag{32}$$

Separating the one-body contribution, $J_\mu^{[1]}(\mathbf{q})$, i.e., assuming that

$$J_\mu(\mathbf{q}) = J_\mu^{[1]}(\mathbf{q}) + J_\mu^{[r]}(\mathbf{q}), \tag{33}$$

where $J_\mu^{[r]}(\mathbf{q})$ denotes the two-body and more complex contributions, one gets

$$\mathbf{q}\mathbf{J}^{[1]}(\mathbf{q}) = [K, \rho^{[1]}(\mathbf{q})], \tag{34}$$

$$\mathbf{q}\mathbf{J}^{[r]}(\mathbf{q}) = [V, \rho^{[1]}(\mathbf{q})] + [H, \rho^{[r]}(\mathbf{q})], \tag{35}$$

It can be easily shown that

$$q^\mu J_\mu^{[1]}(\mathbf{q}) = G_0(z_f)^{-1}\rho^{[1]}(\mathbf{q}) - \rho^{[1]}(\mathbf{q})G_0(z_i)^{-1}, \tag{36}$$

$$q^\mu J_\mu^{[r]}(\mathbf{q}) = G(z_f)^{-1}\rho^{[r]}(\mathbf{q}) - \rho^{[r]}(\mathbf{q})G(z_i)^{-1} + [\rho^{[1]}(\mathbf{q}), V], \tag{37}$$

where $G_0(z) = (z - K)^{-1}$ is the free propagator.

We have from Eq.(37)

$$q^\mu (f|J_\mu^{[r]}(\mathbf{q})|i) = (f|[\rho^{[1]}(\mathbf{q}), V]|i). \tag{38}$$

Let us analyze the r.h.s. of this equation. Firstly, we employ the conservation of the 3-momentum:

$$q^\mu (f|J_\mu^{[r]}(\mathbf{q})|i) = (2\pi)^3\delta(\mathbf{q} + \mathbf{P}_i - \mathbf{P}_f) \times \\ \times[A_{M_iM_f}(\mathbf{P}_i, \mathbf{P}_f) - B_{M_iM_f}(\mathbf{P}_i, \mathbf{P}_f)], \tag{39}$$

$$A_{M_iM_f}(\mathbf{P}_i, \mathbf{P}_f) = (\mathbf{P}_f, M_f|\rho^{[1]}(0)V|\mathbf{P}_i, M_i);$$
$$B_{M_iM_f}(\mathbf{P}_i, \mathbf{P}_f) = (\mathbf{P}_f, M_f|V\rho^{[1]}(0)|\mathbf{P}_i, M_i), \tag{40}$$

where $\rho^{[1]}(0)$ is the density operator at the point $x = (t, \mathbf{x})$, and only the total momenta and spin projections are indicated in the states i and f.

Further, we restrict ourselves to the case of elastic transition. In laboratory frame one has $\mathbf{P}_i = 0$ and $\mathbf{P}_f = \mathbf{q}$. It is convenient to choose the quantization axis (the axis OZ) along $\mathbf{q}$. Obviously, the quantities defined by (40) are proportional to $\delta_{M_iM_f}$ and, therefore, it is sufficient to consider the case with $M_i = M_f$.

Now, using the transformation properties of $A_{M_iM_f}(\mathbf{P}_i, \mathbf{P}_f)$ and $B_{M_iM_f}(\mathbf{P}_i, \mathbf{P}_f)$ with respect to the time inversion, we find

$$B_{M_iM_i}(\mathbf{q}) = (0, -M_i|\rho^{[1]}(0)V| - \mathbf{q}, -M_i). \tag{41}$$

Finally, performing the rotation around the axis OY by 180° we obtain

$$B_{M_iM_i}(\mathbf{q}) = (0, M_i|\rho^{[1]}(0)V|\mathbf{q}, M_i). \tag{42}$$

Generally speaking it does not equal the quantity

$$A_{M_iM_i}(\mathbf{q}) = (\mathbf{q}, M_i|\rho^{[1]}(0)V|0, M_i). \tag{43}$$

However, they may coincide in some cases.

In fact, in nonrelativistic description the dependence of the state vector on the total momentum is separated as

$$|\mathbf{q}, M_i) = \exp(i\mathbf{q}\mathbf{R})|0, M_i), \tag{44}$$

where $\mathbf{R}$ is the center-of-mass coordinate operator. Since V depends only on the relative variables due to the Galilean invariance, and operator $\rho^{[1]}(0)$ commutes with $\mathbf{R}$, then

$$B_{M_iM_f}(\mathbf{q}) = A_{M_iM_f}(-\mathbf{q}) = A_{M_iM_f}(\mathbf{q}). \tag{45}$$

Note, that in deriving Eq.(45) the space inversion transformation has been used as well.

The GI condition

$$q^\mu(f|J_\mu^{[1]}(\mathbf{q})|i) = 0, \tag{46}$$

follows from Eqs.(32),(39), and (45).

In general case

$$|\mathbf{q}, M_i) = \exp(i\mathbf{w}\mathbf{N})|0, M_i), \tag{47}$$

where $\mathbf{w} = w\mathbf{n}_q$, $\tanh w = |\mathbf{q}|/\sqrt{\mathbf{q}^2 + M^2}$, and $\mathbf{N}$ is the boost operator[3]. In relativistic approaches the boost operator being a many-body operator commutes neither with V nor with $\rho^{[1]}(0)$. So, our previous proof for the nonrelativistic case, where $\mathbf{N} \propto M\mathbf{R}$, becomes invalid.

5 Concluding remarks

We have analyzed the elastic e-d scattering with special emphasis on the GI of the corresponding amplitude. Our consideration has been performed both within the BS formalism and the conventional nuclear approach. It has relied on the continuity equation for the deuteron (in general, two-fermion) EM current operator. The symmetry properties of the N-N interaction and the current with respect to the space-time transformations have been employed as well.

We proved that the one-body part of the conserved current gives a gauge independent contribution to the elastic scattering amplitude.

[3]Since the boost is performed along the axis OZ the spin projection is not affected.

It turns out that this result holds in the BS formalism with the two-body kernels of ladder or separable types. Common feature of these interactions is their independence of the total momentum of interacting pair.

It also was shown how one can extend the result to the elastic electron scattering off a nonrelativistic system with arbitrary interaction.

Of course, these observations do not mean that the MEC may be neglected in such situations. In fact, the results of papers [12, 13] for the separable $q\bar{q}$ interaction clearly demonstrate sizable influence of the two-body currents on the pion form factor (in particular, at high momentum transfers). Certainly, for any N-N interaction model the two-body and more complex EM currents should be included in a consistent way in calculations of the elastic form factors of nuclei.

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Буров В.В. и др. E2-93-467

О градиентной независимости амплитуды упругого электрон-дейтронного рассеяния в импульсном приближении

На основе тождеств Уорда — Такахаши сформулированы условия градиентной инвариантности и градиентной независимости для упругого рассеяния электронов дейтронами. Найдены условия, обеспечивающие градиентную независимость амплитуды рассеяния в импульсном приближении в формализме Бете — Солпитера и нерелятивистском подходе.

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Burov V.V. et al. E2-93-467

On the Gauge Independence of Elastic Electron-Deuteron Scattering Amplitude in the Impulse Approximation

Gauge invariance and gauge independence requirements for the elastic electron scattering off the deuteron are formulated using the Ward — Takahashi identity for the deuteron electromagnetic current. Conditions are found that ensure the gauge independence of the scattering amplitude in the impulse approximation both within the Bethe — Salpeter formalism and nonrelativistic description.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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