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FINITE-TEMPERATURE SCALAR FIELD THEORY IN STATIC DE SITTER SPACE

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1 Introduction

The hypothesis that the early universe might have undergone an exponential expansion might explain a number of essential questions. Why, for example, the observed space is homogeneous and isotropic and energy density in it is so close to the critical value [1]. In the exponentially expanding epoch the universe has the de Sitter geometry with fixed radius. If the radius is sufficiently small, there may be interesting effects arising from the behavior of quantum field theories in such curved space. In this way gravitation can influence the properties of the effective potential and can change the symmetry-breaking pattern in gauge models.

In the one-loop approximation and assuming a de Sitter space-time this problem has been studied for scalar electrodynamics [2] and for the more realistic $SU(5)$ gauge theory [3],[4]. These papers show that gravitational effects change the phase structure of the theory, but analysis there was restricted to a particular choice for the quantum state of the system,i.e. to the state which is invariant under transformations of the de Sitter group [5]. All observers moving freely register it equally as a thermal equilibrium state at the same temperature $(2\pi a)^{-1}$ (Hawking temperature), with a the radius of the space [6].

Note that the thermal equilibrium state in de Sitter space-time is always possible in static coordinates where the external gravitational field does not depend on time. Thus, a natural question arises: how does symmetry breaking occur in the de Sitter universe if a given quantum field is in an arbitrary thermal equilibrium state different from the invariant one? For this purpose, the study of finite-temperature quantum field theory in a static de Sitter space-time is necessary.

This subject is also interesting by itself. Let us recall that a freely moving observer in this space has an event horizon separating from the whole space-time the region he can never see. The presence of horizons can have interesting consequences. It is known, for instance, that there is a close connection between event horizons and thermodynamics [6]. However, although the thermal properties of Green functions in the Rindler, de Sitter and Schwarzschild spaces were considered [7], finite-temperature effective potential and symmetry breaking in the static spaces with horizons were not investigated.

The present paper studies the quantum theory of a scalar field in the static de Sitter space-time at arbitrary temperature denoted by β^{-1}. The analysis of the scalar case turns out to be rather simple and can help to understand us the features specific of more realistic gauge theories.

The paper is organized as follows. Section 2 is devoted to the quantization of a scalar field ϕ in the static de Sitter space. The energy operator in that space can be introduced and divided into two commuting parts, defined in causally-disconnected regions. This enables one to formulate the functional integration formalism for the thermal averages in each region. It turns out that the integration here goes over the field configurations placed on the compact four-dimensional space O_β with an Euclidean signature. This space is the infinitely-sheeted along the "imaginary" time τ hypersphere S^4 of the radius a where points (τ, x^i) and $(\tau + \beta, x^i)$ are identified. At the Hawking temperature, when $\beta = 2\pi a \equiv \beta_H$, the space O_β becomes a four-sphere S^4. In the general case it has conic singularities where the Killing vector field generating translations along τ is null.

In Section 3 the finite-temperature effective potential $V(\phi, \beta)$ is introduced in the framework of the functional integration formalism for averages. Studying the spectrum of the Laplace operator on O_β we are able to find the expression of the one-loop effective potential as an expansion in β^{-1}. We use here zeta-function regularization [8], [9]. The suitable forms of $V(\phi, \beta)$ and of the average energy density $E(\phi, \beta)$ are given for the ground and de Sitter invariant states. It is shown that, in the limit of asymptotically small space-time curvature, they both coincide with the vacuum effective potential computed in Minkowski space.

The scaling properties of the theory in the conformally invariant case are considered in Section 4, where the stress tensor anomaly is obtained explicitly. Interestingly, it turns out to depend on temperature. At $\beta = \beta_H$ the standard value of the anomaly is recovered. The possible reasons of this circumstance are briefly discussed.

Finally, in Section 5 for a real selfinteracting scalar field we show the differences of the symmetry breaking pattern in the ground and de Sitter invariant states. It is shown that in the ground state a discrete symmetry of the classical theory is always spontaneously broken, whereas at the Hawking temperature it can be restored at a certain value of the space radius a_{cr}. Conclusions and remarks are then presented.

Technical details needed for the explicit evaluation of the zeta-function near $\beta = \beta_H$ and in the ground state are reported in Appendix A and B, respectively. The results of Appendix A can be used to estimate the temperature corrections to the potential near the de Sitter invariant state.

2 Static de Sitter Space-Time at Nonzero Temperatures

2.1 Quantization in the static de Sitter space

De Sitter space-time is a solution of the Einstein equations with a positive cosmological constant. In the static coordinates the line element can be written in the form

$$\begin{aligned} ds^2 &= \cos^2\chi dt^2 - a^2(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi \sin^2\theta d\xi^2) \\ &\equiv g_{tt}dt^2 - g_{ij}dx^i dx^j \quad (i,j = 1,2,3) \quad , \end{aligned} \tag{2.1}$$

and $-\infty < t < +\infty$, $-\pi \le \chi \le \pi$, $0 \le \theta, \xi \le \pi$, a is the radius of space. The properties of the static coordinates are discussed in [10]. One has to mention here that they cover only part of the space-time and that the regions $|\chi| < \pi/2$ and $|\chi| > \pi/2$ are separated by the surface $\mathcal{B} = S^2$ and are causally-disconnected.

We can always choose in de Sitter space a Killing vector field generating one-parameter group of isometries, a subgroup of $SO(1,4)$. The coordinates (2.1) correspond to the time-like part of a Killing vector field associated with translations along the time t. These coordinates are restricted by the bifurcate Killing horizon [11] on which the Killing vector field is null. It coincides with the event horizons of oservers with traectories being completely inside the static frame (2.1). The two-surface $\mathcal{B}$ is the bifurcation surface that is left unchanged under the action of the given one-parameter group.

The quantization procedure for a real scalar field in the curved space-time is given in terms of the commutation relations for the field variables [5]

$$[\hat{\phi}(x), \hat{\phi}(y)] = 0 \quad , \tag{2.2}$$

$$[\hat{\phi}_{,\mu}(x)d\sigma^\mu(x), \hat{\phi}_{,\nu}(y)d\sigma^\nu(y)] = 0 \quad , \tag{2.3}$$

$$\int_{\Sigma} f(y)[\hat{\phi}(x), \hat{\phi}_{,\mu}(y)] d\sigma^{\mu}(y) = i f(x) \quad , \tag{2.4}$$

where the points x and y belong to a space-like hypersurface Σ, such that the Cauchy data on Σ define uniquely a solution of the classical equation in the whole space-time. For the static spaces we can introduce the energy operator $\hat{H}$ which is associated with a generator of the unitary transformations of the field $\hat{\phi}$ under translations along the time coordinate t. In the static de Sitter space (2.1) $\hat{H}$ depends on the time component $\hat{T}_t{}^t$ of the energy momentum tensor

$$\hat{H} = \int_{t=const} \sqrt{-g}\; d^3x \hat{T}_t{}^t \quad . \tag{2.5}$$

It is splitted into two parts $\hat{H}_1$ and $\hat{H}_2$ depending on the field variables and acting in the regions $|\chi| \leq \pi/2$ and $|\chi| \geq \pi/2$ respectively (g is the determinant of the metric (2.1)). For the model of the real self-interacting scalar field with the action and the energy momentum tensor given respectively by

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) \quad , \tag{2.6}$$

and

$$T_{\mu\nu} = 2(-g)^{-1/2} \delta S / \delta g^{\mu\nu} \quad , \tag{2.7}$$

from (2.2)-(2.4) it follows

$$[\hat{H}_1, \hat{H}_2] = \frac{i}{2} \int_{\mathcal{B}} d\sigma^{\alpha} \sqrt{g_{tt}} (\partial_t \hat{\phi} \partial_\alpha \hat{\phi} + \partial_\alpha \hat{\phi} \partial_t \hat{\phi}) = 0 \quad , \tag{2.8}$$

where $d\sigma^{\alpha}$ is the surface element of $\mathcal{B}$; the operators $\hat{H}_1$ and $\hat{H}_2$ commute because the time component of the metric tensor g_{tt} vanishes on the bifurcation surface. In particular, the last equality shows explicitly that there is no energy exchange between the two causally-disconnected regions.

2.2 Functional integration formalism for the averages

We can choose now (in an oscillator approximation) the creation and annihilation operators of particles associated with the Hamiltonian (2.5), this allows us to construct the representation of the commutation relations (2.2)-(2.4) given on the corresponding Fock space.

Let us consider a canonical ensemble of such particles at temperature β^{-1} in one of the causally-connected regions, when $|\chi| < \pi/2$, for instance. The thermally averaged value of a physical variable $\hat{O}$ measured in this region reads

$$< \hat{O} >_\beta = Z_\beta^{-1} Tr(\hat{O} e^{-\beta \hat{H}_1}) \quad , \tag{2.9}$$

where Z_β is the partition function determined by the eigenvalues E_n of the operator $\hat{H}_1$

$$Z_\beta = Tr(e^{-\beta \hat{H}_1}) = \sum_n e^{-\beta E_n} \quad . \tag{2.10}$$

The parameter β^{-1} coincides with the local temperature measured by the observer being at the origin of the static coordinates at $\chi = 0$ and the average (2.9) does not depend on the behavior of the system in the rest of space.

To obtain the functional integral representation for the average values (2.9), let us make the coordinates x^i discrete (with intervals Δx^i) on the surface $t = const$. Then, in the causally connected region $|\chi| < \pi/2$, the transition amplitude from the state $|\phi' >$ to the state $|\phi >$ for the infinitesimal *imaginary-time* ϵ turns out to be

$$U_\epsilon(\phi, \phi') = < \phi | e^{-\epsilon \hat{H}_1} | \phi' > = \lim_{\Delta x^i \to 0} \prod_x \left(\frac{\sqrt{-g(x)} g^{tt}(x) \Delta x^1 \Delta x^2 \Delta x^3}{2\pi\epsilon} \right)^{1/2} e^{-S_\epsilon(\phi, \phi')} \quad , \tag{2.11}$$

where

$$S_\epsilon(\phi, \phi') = \frac{\epsilon}{2} \sum_x \sqrt{-g(x)} \Delta x^1 \Delta x^2 \Delta x^3 \left[g^{tt}(x) \left(\frac{\phi(x) - \phi'(x)}{\epsilon} \right)^2 + \right.$$
$$\left. + g^{ii}(x) \left(\frac{\phi(x) - \phi'(x + \Delta x^i))}{\Delta x^i} \right)^2 + V(\phi) \right] \quad . \tag{2.12}$$

According to this definition the functional $\Psi(\epsilon, \phi) = \int d\phi' U_\epsilon(\phi, \phi') \Psi(\phi')$ has the following properties:

$$\Psi(\epsilon, \phi)|_{\epsilon=0} = \Psi(\phi) \quad , \tag{2.13}$$

$$- \partial_\epsilon \Psi(\epsilon, \phi)|_{\epsilon=0} = \hat{H}_1 \Psi(\phi) \quad . \tag{2.14}$$

$\hat{H}_1$ is the Hamiltonian in the region $|\chi| < \pi/2$

$$\hat{H}_1 = \int_{|\chi| < \pi/2} \sqrt{-g}\, d^3x \left(\frac{1}{2} \hat{\Pi}^2 + \frac{1}{2} g^{ii} (\partial_i \hat{\phi})^2 + V(\hat{\phi}) \right) \quad , \tag{2.15}$$

connected with the stress tensor via the formula (2.5). On the surface $t = const$ (that is on S^3) $\hat{\Pi} = (g_{tt})^{-1/2}\partial_t\hat{\phi}$ is the quantity proportional to the canonical momentum in the coordinate representation

$$\hat{\Pi}(x) = \frac{1}{i(detg_{ij})^{1/2}}\frac{\delta}{\delta\phi(x)} \quad . \tag{2.16}$$

The transition amplitude for the final imaginary-time interval β is given by the integral

$$U_\beta(\phi,\phi') = \int D\tilde{\phi}e^{-S_\beta(\phi,\phi')} \quad , \tag{2.17}$$

where

$$D\tilde{\phi} = \lim_{\Delta\tau,\Delta x\to 0}\prod_{0\le\tau_i\le\beta}\prod_x\left(\frac{\sqrt{-g}g^{tt}\Delta x^1\Delta x^2\Delta x^3}{2\pi\Delta\tau}\right)^{1/2}d\tilde{\phi}(x,\tau_i) \quad , \tag{2.18}$$

$$S_\beta(\phi,\phi')_{\Delta\tau,\Delta x\to 0} = \int_{0\le\tau\le\beta}\sqrt{g_{tt}\ detg_{ij}}\ d\tau d^3x\left[\frac{1}{2}g^{tt}(\partial_t\tilde{\phi})^2 + \frac{1}{2}g^{ii}(\partial_i\tilde{\phi})^2 + V(\tilde{\phi})\right], \tag{2.19}$$

with the boundary condition $\tilde{\phi}(x,\tau=\beta) = \phi(x)$, $\tilde{\phi}(x,\tau=0) = \phi'(x)$. The representation for the average value of an operator $\hat{O}$ follows from (2.9) and (2.10)

$$<\hat{O}>_\beta = Z_\beta^{-1}\int D\phi O[\phi]e^{-S_\beta(\phi)} \quad , \tag{2.20}$$

$$Z_\beta = \int D\phi e^{-S_\beta(\phi)} \quad , \tag{2.21}$$

where $D\phi \equiv d\phi D\tilde{\phi}$ and $S_\beta(\phi) \equiv S_\beta(\phi,\phi)$. From the definition (2.19), the integration in (2.20) goes over the field variables placed on the compact space O_β with line element

$$ds^2 = \cos^2\chi d\tau^2 + a^2(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\xi^2) \quad , \tag{2.22}$$

which is the Euclidean form of the line element (2.1), and the periodic parameter τ ranging from 0 to β.

When $\beta = \beta_H$ the space O_β is the four-dimensional hypersphere S^4. The two point thermal Green function defined in agreement with (2.20) at $\beta = \beta_H$ coincides with the Green function of the de Sitter-invariant quantum state that also turns out in static coordinates to be periodic analitic function of the imaginary time [6] with period $2\pi a$. This state is the vacuum, but its field excitations, which are defined in a de Sitter invariant way [5], cannot be interpreted as particles of a certain energy. All observers moving freely register this state as a thermal equilibrium at the same temperature $\beta_H^{-1} = (2\pi a)^{-1}$ [6]. Let us point out that thermal equilibrium at the Hawking temperature only in the given part of static frame ($|\chi| < \pi/2$) does not mean the de Sitter-invariant vacuum because the quantum state of the system in the other casually independent part of space ($|\chi| > \pi/2$) can be quite arbitrary.

If $\beta = n^{-1}\beta_H$ ($n = 1, 2, ..$), the integration in the representation (2.20) for the averages goes over the fields on the hypersphere S^4 on which the points (τ, x^i) and $(\tau + \beta, x^i)$ are identified. Such space is an orbifold [12]. At zero temperature $O_\beta = O_\infty$ and is an infinitely-sheeted sphere S^4. For the arbitrary temperatures O_β is the factor space of O_∞ over the ciclic rotation group with period β leaving the two-surface $\mathcal{B}$ unchanged. In all the points out of $\mathcal{B}$ it has the geometry of an hypersphere but in the domain of $\mathcal{B}$, when $|\chi| \to \pi/2$, it looks like the product space $cone \otimes S^2$. The volume of O_β is $\beta\mathcal{V}$ where $\mathcal{V}$ is the volume of the spatial part of space-time ($\mathcal{V} = 4\pi a^3/3$).

3 The Effective Potential

3.1 Basic formalism

Phase transitions in curved spaces at arbitrary temperatures can be investigated as in the flat one applying the effective potential method. The effective potential $V(\varphi, \beta)$ in our case can be introduced via the path integral representation for the partition function (2.21). For this purpose, let us consider in (2.21) the "static" part $\varphi \equiv (\beta\mathcal{V})^{-1} \int_{O_\beta} \sqrt{g}\, d^4x \phi(x)$ of the field variables on O_β (g is the determinant of the metric (2.22))

$$Z_\beta = \int D\phi e^{-S_\beta(\phi)} = \int D(\varphi + \phi')e^{-S_\beta(\varphi+\phi')} =$$

$$= N \int d\varphi e^{-\beta\mathcal{V}V(\varphi,\beta)} \quad , \tag{3.1}$$

where N is a normalization constant. The potential V is defined by the integral

$$e^{-\beta\mathcal{V}V(\varphi,\beta)} \equiv \int D\phi' e^{-S_\beta(\varphi+\phi')} \quad , \tag{3.2}$$

over the fields obeying the condition

$$\int_{O_\beta} \sqrt{g}\, d^4x \phi'(x) = 0 \quad . \tag{3.3}$$

If $V(\varphi, \beta)$ is a known function of φ, the partition function can be found from (3.1) by the method of stationary phase. The points φ_i of minima of $V(\varphi, \beta)$ correspond to various field configurations with the average field strength in the considered volume $\mathcal{V}$ equal to φ_i in the one-loop approximation. The real part of $V(\varphi, \beta)$ is a sum of the classical potential energy $V(\varphi)$ and of the quantum corrections to it. If a field configuration φ_i is unstable, then $V(\varphi_i, \beta)$ has a nonvanishing imaginary part determining its decay-rate [13].

To calculate the one-loop effective potential, one has to expand the functional $S_\beta(\varphi + \phi')$ in (3.2) on ϕ', taking into account the condition (3.3), and to approximate it by the expression

$$S_\beta(\varphi + \phi') = (\mathcal{V}\beta)V(\varphi) + \frac{1}{2}\int_{O_\beta} \sqrt{g}\, d^4x\, \phi'(x)\hat{Q}(\varphi)\phi'(x) \quad , \tag{3.4}$$

where $\hat{Q}(\varphi) \equiv -\Box + V''(\varphi)$ ($\Box$ is the Laplace operator defined on O_β). The integration in (3.2) can be performed as usual if we use the completeness of the eigenfunctions $\psi_n(x)$ of $\Box$, so that the field $\phi(x)$ can be expanded as

$$\phi(x) = \sum_n \phi_n \psi_n(x) \quad , \tag{3.5}$$

where the eigenfunctions are normalized as follows

$$\int_{O_\beta} \sqrt{g}\, d^4x\, \psi_n(x)\psi_m(x) = \delta_{n,m} \quad , \tag{3.6}$$

and change the measure (2.18) by the measure $D\phi = \prod_n (2\pi)^{-1/2}\mu d\phi_n$ with μ a normalization constant . Integrating over ϕ_n we get from (3.2)

$$V(\varphi, \beta) = V(\varphi) + \frac{1}{2\beta\mathcal{V}}\left[\log(\det(\mu^{-2}\hat{Q})) - \log(\mu^{-2}V''(\varphi))\right] \quad . \tag{3.7}$$

According to (3.3), we eliminate from $V(\varphi, \beta)$ the contribution of zero mode of the Laplace operator. The last term is important for analitical properties of the effective potential when the space-time curvature is large.

If all field configurations are stable, the one-loop partition function can be derived from (3.1) considering the minima with the zero imaginary part $Im\ V(\varphi_i, \beta) = 0$. In the given approximation it turns out to be

$$Z_\beta = \sum_i \left(\frac{2\pi N^2}{V''(\varphi_i)\beta\mathcal{V}}\right)^{\frac{1}{2}} e^{-\beta\mathcal{V}V(\varphi_i,\beta)} \quad . \tag{3.8}$$

Taking into account the normalization of the zero mode φ in (3.1) and (3.5),(3.6) and (3.7) we can substitute N with $\mu(\beta\mathcal{V})^{1/2}$ and represent Z_β in the form

$$Z_\beta = \sum e^{-\beta\mathcal{V}\tilde{V}(\varphi_i,\beta)} \quad , \tag{3.9}$$

where

$$\tilde{V}(\varphi,\beta) = V(\varphi) + \frac{1}{2\beta\mathcal{V}} \log\left(\det(\mu^{-2}\hat{Q})\right) \quad . \tag{3.10}$$

It is obvious that in the flat-space limit, when the radius and volume of space tend to infinity, both the quantities $V(\varphi,\beta)$ and $\tilde{V}(\varphi,\beta)$ coincide.

The field average $< \hat{\phi}(x) >_\beta$ in the one-loop approximation can be found from (2.20) in a similar way, and is

$$< \hat{\phi}(x) >_\beta = \sum_i P_i(\beta)\,\varphi_i \quad , \tag{3.11}$$

where the coefficients

$$P_i(\beta) = \frac{e^{-\beta\mathcal{V}\tilde{V}(\varphi_i,\beta)}}{\sum_k e^{-\beta\mathcal{V}\tilde{V}(\varphi_k,\beta)}} \quad , \quad \sum_i P_i(\beta) = 1 \quad , \tag{3.12}$$

in the equilibrium state are the probabilities for a given field configuration φ_i to appear. From (3.9) we can also obtain the average energy as the sum

$$< \hat{H} >_\beta = -\frac{\partial}{\partial\beta}\log Z_\beta = \sum_i P_i(\beta)\,\mathcal{V}E(\varphi_i,\beta) \quad , \tag{3.13}$$

where the quantities

$$E(\varphi_i,\beta) = \frac{\partial}{\partial\beta}\left(\beta\tilde{V}(\varphi_i,\beta)\right) \quad , \tag{3.14}$$

are the energy densities of the configurations φ_i.

In the trivial case of a free scalar field the effective potential $V(\varphi,\beta)$ has only one minimum at $\varphi = 0$, as in the classical theory, because the determinant in (3.7) does not depend on the field φ.

3.2 Zeta-function

To regularize the determinant in Eq.(3.7), the zeta-function method [8] can be used because the eigenvalues $\lambda_{n,m}$ of the operator $-\Box + V''(\varphi)$ on O_β can be found exactly. They are characterized by two nonnegative numbers n and m and depend on the temperature β^{-1}

$$\lambda_{n,m}(\beta) = a^{-2}\left(n - m + (\beta_H/\beta)m\right)\left(n - m + (\beta_H/\beta)m + 3\right) + V''(\varphi) \quad , \tag{3.15}$$

$$n = 0,1,2,....; \quad m = 0,1,...,n;$$

The multiplicity $g_{n,m}$ of the eigenvalue $\lambda_{n,m}$ is $(n-m+1)(n-m+2)$ if $m \neq 0$, and $(n+1)(n+2)/2$ for $m=0$. In the case of the Hawking temperature β_H^{-1} this operator turns into the operator on the hypersphere S^4 with $\lambda_n = a^{-2}n(n+3) + V''(\varphi)$ and the multiplicity

$$g_n = \sum_{m=0}^{n} g_{n,m} = \frac{1}{6}(n+1)(n+2)(2n+3) \quad . \tag{3.16}$$

The renormalized $\log\det(\hat{Q}\mu^{-2})$, the effective potential $V(\varphi,\beta)$ and the average energy $E(\varphi,\beta)$ expressed in terms of the generalized zeta-function

$$\zeta(z,\beta) \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{n} g_{n,m}\left(a^2\lambda_{n,m}\right)^{-z} \quad . \tag{3.17}$$

now read

$$\log\left(\det(\mu^{-2}\hat{Q})\right) = -\left[\zeta'(0,\beta) + \log(\mu^2 a^2)\zeta(0,\beta)\right] \quad , \tag{3.18}$$

$$V(\varphi,\beta) = V(\varphi) - \frac{1}{2\beta\mathcal{V}}\left[\zeta'(0,\beta) + \log(\mu^2 a^2)\zeta(0,\beta) + \log\left(V''(\varphi)\mu^{-2}\right)\right] \quad , \tag{3.19}$$

$$E(\varphi,\beta) = V(\varphi) - \frac{1}{2\mathcal{V}}\frac{\partial}{\partial\beta}\left[\zeta'(0,\beta) + \log(\mu^2 a^2)\zeta(0,\beta)\right] \quad , \tag{3.20}$$

where $\zeta'(z,\beta) \equiv \frac{d}{dz}\zeta(z,\beta)$.

Let us find a more suitable form for the zeta-function (3.17). If we express $(a^2\lambda_{n,m})^{-z}$ as an expansion with parameter $\Delta \equiv 9/4 - a^2V''(\varphi)$ and point out that $\sum_{n=0}^{\infty}\sum_{m=0}^{n} f(m,n) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} f(m,n+m)$, we can immediately perform the summation on the n index in (3.17) obtaining

$$\begin{aligned}
\zeta(z,\beta) &= \sum_{k=0}^{\infty} C_k(z)\Delta^k \left\{ \sum_{m=0}^{\infty} \left[\zeta_R\left(2z+2k-2, (\beta_H/\beta)m + 3/2\right)\right.\right. \\
&- 2(\beta_H/\beta)m\zeta_R\left(2z+2k-1, (\beta_H/\beta)m + 3/2\right) \\
&+ \left.\left((m\beta_H/\beta)^2 - \frac{1}{4}\right)\zeta_R\left(2z+2k, (\beta_H/\beta)m + 3/2\right)\right] \\
&- \left.\frac{1}{2}\left[\zeta_R(2z+2k-2, 3/2) - \frac{1}{4}\zeta_R(2k+2z, 3/2)\right]\right\} \quad ,
\end{aligned} \tag{3.21}$$

where the coefficients $C_k(z)$ are defined by

$$C_k(z) \equiv \frac{\Gamma(z+k)}{k!\Gamma(z)} \quad . \tag{3.22}$$

With the following integral representation for the Riemannian ζ_R-function

$$\zeta_R(s,a) = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{y^{s-1}e^{-ay}}{1-e^{-y}}dy \quad , \tag{3.23}$$

we are able to sum up over m. For instance, one can get

$$\sum_{m=0}^{\infty} \zeta_R(s,(\beta_H/\beta)m+3/2) = \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{y^{s-1}}{1-e^{-y}} e^{-\frac{3}{2}y} \frac{1}{1-e^{-(\beta_H/\beta)y}} dy \quad , \tag{3.24}$$

for $Re\ s > 2$. Thus, inserting (3.24) in (3.21) we have

$$\zeta(z,\beta) = \sum_{k=0}^{\infty} C_k(z)\Delta^k \int_0^{\infty} dy\, \frac{y^{2z+2k-3}}{1-e^{-y}} e^{-\frac{3}{2}y} \left\{ \frac{\frac{1}{2}\coth\left(\frac{y\beta_H}{2\beta}\right)}{\Gamma(2z+2k-2)} \right.$$

$$\left. - \frac{\left(\frac{y\beta_H}{2\beta}\right)\sinh^{-2}\left(\frac{y\beta_H}{2\beta}\right)}{\Gamma(2z+2k-1)} + \frac{\left(\frac{y\beta_H}{2\beta}\right)^2 \coth\left(\frac{y\beta_H}{2\beta}\right)\sinh^{-2}\left(\frac{y\beta_H}{2\beta}\right) - \frac{y^2}{8}\coth\left(\frac{y\beta_H}{2\beta}\right)}{\Gamma(2z+2k)} \right\} . \tag{3.25}$$

Note that from (3.25), if the variable z is close to zero, $\zeta(z,\beta)$ is determined by the behavior of the integrand only in the vicinity of the lower limit of integration. In this case one can use in (3.25) the definition of the Bernoulli numbers B_n

$$\frac{x}{e^x-1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n \quad . \tag{3.26}$$

(valid for $|x| < \pi$), to get a representation for $\zeta(z,\beta)$ as a series of odd powers of the temperature

$$\begin{aligned}\zeta(z,\beta) \ &= \ \frac{\beta}{\beta_H}\sum_{k=0}^{\infty} C_k(z)\Delta^k \sum_{n=0}^{\infty} \frac{B_{2n}}{(2n)!}\left(\frac{\beta_H}{\beta}\right)^{2n} \times \\ \times \frac{\Gamma(2z+2k+2n-1)}{\Gamma(2z+2k)} &\left[\zeta_R(2z+2k+2n-3,3/2) - \frac{1}{4}\zeta_R(2z+2k+2n-1,3/2)\right] .\end{aligned} \tag{3.27}$$

This representation holds for z close to zero and can be applied to compute $\zeta(0,\beta) \equiv \lim_{z\to 0}\zeta(z,\beta)$ and its first derivative, obtaining for $\zeta(0,\beta)$ the exact simple expression

$$\zeta(0,\beta) = \frac{\beta}{\beta_H}\left[\frac{(51-60(\beta_H/\beta)^2-8(\beta_H/\beta)^4)}{2880} + \frac{(2(\beta_H/\beta)^2-3)}{24}\Delta + \frac{1}{12}\Delta^2\right] \quad . \tag{3.28}$$

At the Hawking temperature this result coincides with the expression obtained by other authors [2]. To compute the first derivative of (3.27) we observe that

$$\left.\frac{d}{dz}(z\zeta_R(z+n+1,a))\right|_{z=0} = \frac{(-1)^{n+1}}{n!}\frac{d^n}{da^n}\psi(a) \quad , \tag{3.29}$$

with n integer > 0. Unfortunately, as far as $\zeta'(0,T)$ is concerned, it can only be expressed in terms of an expansion

$$\zeta'(0,\beta) = -\frac{2}{T}\left[\zeta_R\left(-3,\frac{3}{2}\right) - \frac{1}{4}\zeta_R\left(-1,\frac{3}{2}\right)\right] - \frac{2}{T}\left[\zeta_R'\left(-3,\frac{3}{2}\right) - \frac{1}{4}\zeta_R'\left(-1,\frac{3}{2}\right)\right]$$

$$
+\frac{1}{T}\left(\Delta+\frac{T^2}{6}\right)\left[\zeta_R\left(-1,\frac{3}{2}\right)+\frac{1}{4}\psi\left(\frac{3}{2}\right)\right]+\frac{1}{T}\left(\frac{\Delta^2}{36}+\frac{\Delta}{4}+\frac{\Delta T^2}{12}-\frac{T^4}{120}\right)
$$

$$
-\frac{1}{T}\sum_{n=0}^{\infty}\left[\frac{\Delta^2}{12}B_{2n}+\Delta T^2\frac{B_{2n+2}}{(2n+2)(2n+1)}+2T^4\frac{B_{2n+4}}{(2n+4)(2n+3)(2n+2)(2n+1)}\right]\times
$$

$$
\times\frac{T^{2n}}{2n!}\left[(2n+2)(2n+1)\psi^{(2n)}\left(\frac{3}{2}\right)-\frac{1}{4}\psi^{(2n+2)}\left(\frac{3}{2}\right)\right]
$$

$$
-\frac{2}{T}\sum_{k=3}^{\infty}\sum_{n=0}^{\infty}B_{2n}\frac{(\sqrt{\Delta})^{2k}}{2k!}\frac{T^{2n}}{2n!}\left[(2n+2k-2)(2n+2k-3)\psi^{(2n+2k-4)}\left(\frac{3}{2}\right)\right.
$$

$$
\left.-\frac{1}{4}\psi^{(2n+2k-2)}\left(\frac{3}{2}\right)\right], \qquad (3.30)
$$

where $T \equiv \beta_H/\beta$. The last equations (3.28) and (3.30), once inserted into (3.19), define explicitly the effective potential as an expansion in the temperature β^{-1}. This expansion is especially useful to investigate the potential $V(\varphi,\beta)$ at the low temperatures. Another expansion of $V(\varphi,\beta)$ around the Hawking temperature can be found in Appendix A. However, in the most interesting cases we are going to consider, the potential can be written in a more suitable integral form.

3.3 Vanishing temperature and Hawking temperature

The effective potential for the space of radius a at the Hawking temperature β_H^{-1} can be found from (3.19) substituting in it the expressions of $\zeta'(0,\beta_H)$ obtained in [2], and of $\zeta(0,\beta_H)$ from (3.28). It reads

$$
V(\varphi,\beta_H)=V(\varphi)-\frac{3}{(4\pi)^2a^4}\left[-\frac{1}{3}\left(\int_{\frac{1}{2}}^{\frac{1}{2}+\sqrt{\Delta}}+\int_{\frac{1}{2}}^{\frac{1}{2}-\sqrt{\Delta}}\right)u(u-\frac{1}{2})(u-1)\psi(u)du\right.
$$

$$
\left.+\frac{1}{12}\Delta^2+\frac{1}{72}\Delta+\log(\mu^2a^2)\left(\frac{\Delta^2}{12}-\frac{\Delta}{24}-\frac{17}{2880}\right)+\log\left(V''(\varphi)\mu^{-2}\right)\right]+const, \qquad (3.31)
$$

where $\psi(u)$ is the psi-function. We can also derive the average energy density (3.14) in this state by an expansion of $\zeta(z,\beta)$ in powers of $(\beta_H-\beta)/\beta$ given in Appendix A

$$
E(\varphi,\beta_H)=V(\varphi)+\frac{3}{(4\pi)^2a^4}\left[-\frac{1}{8}\Delta^2+\frac{41}{144}\Delta-\frac{973}{5760}\right.
$$

$$
\left.+\frac{1}{12}\left(\frac{9}{4}-\Delta\right)\left(\frac{1}{4}-\Delta\right)\left(\psi(3/2+\sqrt{\Delta})++\psi(3/2-\sqrt{\Delta})-\log(\mu^2a^2)\right)\right]. \qquad (3.32)
$$

From (3.19),(3.20) and (3.27) the effective potential at zero temperature coincides with the vacuum energy density. A connection between zeta-functions at $\beta = \infty$ and $\beta = \beta_H$, described in Appendix B, implies

$$V(\varphi,\infty) = V(\varphi) - \frac{3}{(4\pi)^2 a^4}\left[\left(\int_{\frac{1}{2}}^{\frac{1}{2}+\sqrt{\Delta}} + \int_{\frac{1}{2}}^{\frac{1}{2}-\sqrt{\Delta}}\right) u\left(u - \frac{1}{2} - \sqrt{\Delta}\right)(u-1)\psi(u)du \right.$$
$$\left. + \frac{1}{36}\Delta^2 + \frac{7}{24}\Delta + \log(\mu^2 a^2)\left(\frac{\Delta^2}{12} - \frac{\Delta}{8} + \frac{17}{960}\right)\right] + const. \qquad (3.33)$$

At the points of minima the imaginary part $Im\ V(\varphi,\infty)$ gives the decay probability Γ of metastable vacuum configurations calculated in the quasiclassical approximation $\Gamma = -2Im\ V(\varphi,\infty)$. When $\sqrt{\Delta} \geq 3/2$ or if $V''(\varphi) \leq 0$, the integrand in (3.33) has the simple poles due to the psi-function and integration contour should be chosen so that $Im\ V(\varphi,\infty) \leq 0$. This can be achieved simply by changing $V''(\varphi)$ with $V''(\varphi) - i\epsilon/2$ ($\epsilon > 0$), which corresponds to go around the poles in the lower part of the complex plane.

A similar way to regularize the integral part of $V(\varphi,\beta)$ can be taken at $\beta = \beta_H$, but here the situation is different. The vacuum energy (3.33) is singular when $V''(\varphi) = 0$ where both $E(\varphi,\beta_H)$ and $V(\varphi,\beta_H)$ are finite. The singularity and imaginary part coming from the integral in (3.31) when $3/2 \leq \sqrt{\Delta} < 5/2$ are totally cancelled by the last term $\log(V''(\varphi)\mu^{-2})$. Consequently, in the vacuum state one has instability when $V''(\varphi) \leq 0$; whereas at the Hawking temperature, when $V''(\varphi) \leq -4a^{-2}$ (or $\sqrt{\Delta} \geq 5/2$).

Asymptotic expressions for $V(\varphi,\infty)$, $V(\varphi,\beta_H)$ and $E(\varphi,\beta_H)$ at the large radius a are written in the Appendices. One can thus show that all three quantities in the limit $a \to \infty$ coincide with the vacuum effective potential in Minkowski space

$$V_M(\varphi) = V(\varphi) + \frac{1}{64\pi^2}\left(V''(\varphi)\right)^2\left(\log(V''(\varphi)\mu^{-2}) - \frac{3}{2}\right). \qquad (3.34)$$

This property can be easily explained observing that the Hawking temperature ($(2\pi a)^{-1}$) vanishes in the flat-space limit. On the other hand, the effective potential calculated at $\beta = \beta_H$ coincides with the one in a de Sitter invariant state and can be turned, when $a \to \infty$, only into the potential in the Poincare-invariant vacuum state.

To complete the calculation of the renormalized $V(\varphi,\beta)$ we have to add to it finite counterterms and express the parameters through the measured quantities. It will be done for a particular model in Section 5.

4 Scaling and the Trace Anomaly

Let us consider the conformally invariant scalar field theory with the potential $V(\phi) = (R/12)\,\phi^2$, where R is the scalar curvature, $R = 12a^{-2}$ for de Sitter space-time. The energy operators $\hat{H}$ of conformally related static metrics $\tilde{g}_{\mu\nu}(x) = \alpha^2(x) g_{\mu\nu}(x)$ have the same eigenvalues [14]. In this case, the scale invariance of the unrenormalized partition function Z_β follows immediately from the definitions (2.18),(2.19) of the measure $D\phi$ and the Euclidean action $S_\beta(\phi)$ in (2.21).

In the conformally invariant scalar field theory, the logarithm of the renormalized partition function is defined by (3.9),(3.18) and reads

$$\log Z_\beta = \frac{1}{2}\ \left[\zeta'(0,\beta) + \log(\mu^2 a^2)\zeta(0,\beta)\right] \quad . \tag{4.1}$$

For the constant scale transformations of the metric $\tilde{g}_{\mu\nu}(x) = \alpha^2 g_{\mu\nu}(x)$ we have $\tilde{\lambda}_{n,m} = \alpha^{-2}\lambda_{n,m}$,$\tilde{\zeta}(z,\beta) = \alpha^{2z}\zeta(z,\beta)$ and therefore the following equality for the partition function, as a function of $g_{\mu\nu}$ and the renormalization parameter μ, holds

$$Z_\beta(\alpha^2 g_{\mu\nu}, \alpha^{-1}\mu) = Z_\beta(g_{\mu\nu}, \mu) \quad . \tag{4.2}$$

In static space-times the thermally averaged energy momentum tensor does not depend on time and can be determined by functionally differentiating the free energy $F(\beta) = -\beta^{-1}\log Z_\beta$ [14]

$$T_{\mu\nu}(\beta, x) = -\frac{2}{\sqrt{-g}}\frac{\delta F(\beta)}{\delta g^{\mu\nu}(x)} \tag{4.3}$$

(x^i are three spatial coordinates). Thus, one can write for the integral of its trace $(T_\nu{}^\nu(\beta))$ over the spatial volume $\mathcal{V}$ the following equation

$$\int_{\mathcal{V}} d^3x\ \sqrt{-g}T_\sigma{}^\sigma(\beta, x) = \frac{2}{\beta}\int_{\mathcal{V}} d^3x\ \frac{\delta \log Z_\beta}{\delta g^{\mu\nu}(x)} g^{\mu\nu}(x) = -\beta^{-1}\frac{\partial}{\partial\alpha}\log Z_\beta(\alpha^2 g_{\mu\nu}, \mu)|_{\alpha=1} \quad . \tag{4.4}$$

Finally, Eqs.(4.1) and (4.2) give

$$\int_{\mathcal{V}} d^3x\ \sqrt{-g}T_\sigma{}^\sigma(\beta, x) = -\beta^{-1}\frac{\partial}{\partial\alpha}\log Z_\beta(g_{\mu\nu}, \alpha\mu)|_{\alpha=1} = -\beta^{-1}\zeta(0,\beta) \quad . \tag{4.5}$$

Substituting here the derived expression (3.28) for $\zeta(0,\beta)$ in the conformal case ($\Delta = 1/4$) we get the trace anomaly at the temperature β^{-1}

$$\mathcal{V}^{-1}\int_{\mathcal{V}} d^3x\ \sqrt{-g}T_\sigma{}^\sigma(\beta, x) = \frac{1}{960\pi^2 a^4}\left(3 + (\beta_H/\beta)^4\right) \quad . \tag{4.6}$$

Remarkably, it is a function of β^{-1} and leads at the Hawking temperature $\beta_H^{-1} = (2\pi a)^{-1}$ to the correct trace anomaly and energy-momentum tensor of the de Sitter-invariant state

$$T_{\mu\nu}(\beta_H, x) = (960\pi^2 a^4)^{-1} g_{\mu\nu}(x) \quad . \tag{4.7}$$

It is ordinary believed that the trace anomaly does not depend on the quantum state in which is the system [15] because it is determined by the ultraviolet divergences and is sensible only to the space-time geometry and to the possible boundaries.

The general finite-temperature quantum field theory in static space-times has been investigated in [14]. It has been shown there that infinities, renormalization, and the trace anomaly are the same as at zero temperature. However, the effects of horizons that can be crucial for our analysis were ignored in that work.

The divergences arising in the case of the static de Sitter space can be investigated for the thermal two-point Green function. Considered as a function of the imaginary time ranging from zero to β, it is given on the compact space O_β (see (2.22)) with the conic singularities near two-surface B, which may effect its unusual thermal properties at short distances. Analogous thermal Green functions, corresponding to the Rindler and Schwarzschild metrics, are defined on the spaces with the same conical structure near the horizons. This is probably true for the case of every space-time with the bifurcate Killing vector field. However a detailed analysis of the dependence of the anomaly on the thermal state, and of the role of the horizon is outside the aim of the present work.

We should also mention the calculation of the average energy according to (3.13) in terms of the renormalized function $E(\varphi, \beta_H)$, Eq.(3.32). In the conformal case we are interested in , it is simply equal to $\mathcal{V}E(0, \beta_H)$, with $\Delta = 1/4$ and the average value of the field $\varphi = 0$. *The energy thus obtained does not depend on the scale parameter* μ. There is a discrepancy between it and the quantity $< \hat{H} >_\beta \equiv \int d^3x \sqrt{-g}\, T_t{}^t(\beta_H, x)$ defined through the anomalous energy momentum tensor (4.7). However, whereas the first one, $E(0, \beta_H)$, is defined up to finite renormalization terms, the quantity $< \hat{H} >_\beta$ is totally anomalous and consequently is of a pure geometrical character and independent of the renormalization procedures [15].

5 The Model

We study here, as an example, the model of a real quantum scalar field with symmetrical potential

$$V(\phi) = -\frac{1}{2}\sigma^2\phi^2 + \frac{\lambda}{4}\phi^4 \quad , \tag{5.1}$$

$(\sigma^2, \lambda > 0)$ and compute the effective potential in the ground and de Sitter-invariant quantum states.

The discrete symmetry $\phi \to -\phi$ inherent in the classical model (5.1) is known to be broken in the ground state in flat space-time: in this case the zero-field configurations are unstable. The symmetrical phases correspond to the configurations with zero field strength and their relevance at nonzero space-time curvature may be found from the results derived in Section 3.

From these results we draw immediately the conclusion that there cannot be stable symmetrical phases in the ground state at any curvature because $V''(0) = -\sigma^2 < 0$ and the effective potential has a non-zero imaginary part at $\phi = 0$. On the other hand, symmetry can be restored at the Hawking temperature β_H^{-1} at a certain value of a if the following conditions hold:

$$V'(0, \beta_H) = 0, \quad V''(0, \beta_H) \geq 0, \quad V''(0) > -4a^{-2}. \tag{5.2}$$

The first condition is always true for this model as far as $V(\varphi)$ depends only on the square of the field. To investigate the second one we have to fix the meaning of the constants σ and λ in terms of the measurable quantities, obtained for instance in flat-space.

Following the standard renormalization procedure we can eliminate the scale parameter μ from $V(\varphi, \beta_H)$, Eq.(3.31), by absorbing it into the definition of the finite counterterms that should be added to the effective potential. These counterterms have the same structure as the initial potential (5.1). Thus, the renormalized $V(\varphi, \beta_H)$ turns out to be

$$V(\varphi, \beta_H) = V(\varphi) - \frac{3}{(4\pi)^2 a^4}\left[-\frac{1}{3}\left(\int_{\frac{1}{2}}^{\frac{1}{2}+\sqrt{\Delta}} + \int_{\frac{1}{2}}^{\frac{1}{2}-\sqrt{\Delta}}\right) u(u-\frac{1}{2})(u-1)\psi(u)du \right.$$

$$\left. +\frac{1}{12}\Delta^2 + \frac{1}{72}\Delta + \log\left(V''(\varphi)a^2\right)\right] + A\varphi^2 + B\varphi^4 + const, \tag{5.3}$$

In the limit of asymptotically small curvature ($a \to \infty$) (5.3) takes the form

$$V_M(\varphi) = V(\varphi) + \frac{1}{64\pi^2}\left(V''(\varphi)\right)^2\left[\log(V''(\varphi)a^2) - \frac{3}{2}\right] + A\varphi^2 + B\varphi^4 + const, \qquad (5.4)$$

and the renormalization conditions for it can be chosen as

$$V'_M(\varphi)|_{\varphi^2=\sigma^2/\lambda} = 0, \qquad V''_M(\varphi)|_{\varphi^2=\sigma^2/\lambda} = 2\sigma^2 \equiv m^2. \qquad (5.5)$$

They just define the positions of minima of the asymptotically flat $V(\varphi, \beta_H)$ and the physical mass m of the field as in the classical theory (5.1). Moreover, they fix the values for the constants A and B

$$A = \frac{\lambda\sigma^2}{32\pi^2}\left(3\log(2\sigma^2 a^2) + 6\right), \quad B = -\frac{9\lambda^2}{64\pi^2}\log(2\sigma^2 a^2). \qquad (5.6)$$

The flat-space potential (5.4) so obtained recovers the already known result reported in [1]

$$V(\varphi, \beta_H)|_{a\to\infty} = -\frac{1}{2}\sigma^2\varphi^2 + \frac{\lambda}{4}\varphi^4 + \frac{(3\lambda\varphi^2 - \sigma^2)^2}{64\pi^2}\log\left(\frac{3\lambda\varphi^2 - \sigma^2}{2\sigma^2}\right) +$$
$$+ \frac{21\lambda\sigma^2\varphi^2}{64\pi^2} - \frac{27\lambda^2\varphi^4}{128\pi^2} + const. \qquad (5.7)$$

The same renormalization conditions (5.5) and constants (5.6) can be chosen at zero temperature because $V(\varphi, \infty)$ and $V(\varphi, \beta_H)$ have the same flat-space limit.

We can now investigate the second derivative $V''(0, \beta_H)$ that follows from (5.3),(5.6) and takes particularly simple form at sufficiently large curvature, when $a^2 << \sigma^{-2}$,

$$V''(0, \beta_H) = -\sigma^2 + \frac{\lambda}{16\pi^2 a^2}(1 + 6\gamma) + \frac{6\lambda\sigma^2}{32\pi^2}\left(2 + \log(2\sigma^2 a^2)\right) \quad , \qquad (5.8)$$

where $\gamma \neq 0,577...$ is the Euler constant. As one can see $V''(0, \beta_H)$ changes sign and becomes positive at some critical value of the radius $a = a_{cr}$. It can be found neglecting the last term in (5.8) with respect to the second one and reads

$$a_{cr}^2 = \frac{(1 + 6\gamma)}{8\pi^2}\frac{\lambda}{m^2} \quad . \qquad (5.9)$$

The third condition (5.2) holds if $m^2 a_{cr}^2 < 8$, which is true for not very large values of λ.

As a conclusion, we have shown in this paragraph that, while in the ground state the symmetry is always spontaneously broken, the stable symmetrical phases can appear at the Hawking temperature at some finite values of the space-time curvature. The nature of the given phase transition can be understood by considering the global structure of the effective potential with the help of the expressions (5.3),(5.6).

6 Conclusions and remarks

We have evaluated the finite-temperature effective potential for a scalar field theory in de Sitter space-time. The expression found enables one to study the symmetry breaking in two of the most interesting cases: at low temperature, and at a temperature close to the Hawking one. The analysis is explicitly performed for the bare scalar potential reported in (5.1) and shows how strongly the presence of the temperature affects the phase transition of the system.

It is well known that in Minkowski space-time the classical symmetry of a scalar potential under the discrete transformation $\phi \to -\phi$ is spontaneously broken by the quantum effects. Remarkably, at low temperatures the symmetrical phase under this transformation is instable for every value of the radius a, whereas at the Hawking temperature, this symmetry can be recovered for some finite value of a.

For a generalization of these results to more realistic gauge theory, one has to find the eigenvalues and multiplicities of the corresponding wave operators of the bosonic and fermionic fields on the compact space O_β, which appear in the integral representation for thermal averages.

Finally, we also study the stress tensor anomaly for the conformally invariant case and find that it is a function of the thermal quantum state of the system. The reason of this interesting fact and the possible role of the horizon here will be investigated separately.

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Appendix A. Zeta-function at $\beta \simeq \beta_H$

To discuss the expression of the effective potential near the Hawking temperature, it is useful to represent $\zeta(z,\beta)$ as an expansion in powers of $(\beta_H-\beta)/\beta$. In fact from (3.15) and (3.17), we can write

$$\zeta(z,\beta) = \zeta(z,\beta_H) + \frac{1}{3}\sum_{k=0}^{\infty}\Delta^k\frac{\Gamma(z+k)}{\Gamma(z+1)k!}\left\{z\zeta_R\left(2z+2k-3,\frac{3}{2}\right)\right.$$

$$\left.-\frac{1}{4}z\zeta_R\left(2z+2k-1,\frac{3}{2}\right)\right\}+\sum_{p=1}^{\infty}\sum_{r=0}^{p}\sum_{k=0}^{\infty}\Delta^k(-1)^p2^{p-r}\binom{p}{r}\left(\frac{\beta_H-\beta}{\beta}\right)^{p+r}\frac{\Gamma(z+p+k)}{\Gamma(z+1)p!k!}\times$$

$$\times\left\{\sum_{t=0}^{p+r+3}\frac{(p+r)!}{(p+r+3-t)!}\frac{(t-1)(t-2)}{t!}(-1)^t z\zeta_R\left(2z+2k+t-3,\frac{3}{2}\right)\times\right.$$

$$\left(t2^{1-t}-\left(1-2^{1-t}\right)B_t\right)+z\zeta_R\left(2z+2k+p+r-2,\frac{3}{2}\right)\left(-\frac{B_{p+r+1}}{(p+r+1)}\right)$$

$$+z\zeta_R\left(2z+2k+p+r-1,\frac{3}{2}\right)\left(\frac{2B_{p+r+2}}{(p+r+2)}\right)$$

$$+z\zeta_R\left(2z+2k+p+r,\frac{3}{2}\right)\left(-\frac{B_{p+r+3}}{(p+r+3)}+\frac{1}{4}\frac{B_{p+r+1}}{(p+r+1)}\right)$$

$$\left.-\frac{\frac{1}{4}}{(p+r+1)}\sum_{t=0}^{p+r+1}\binom{p+r+1}{t}(-1)^t z\zeta_R\left(2z+2k+t-1,\frac{3}{2}\right)\left(t2^{1-t}-\left(1-2^{1-t}\right)B_t\right)\right\} \quad . \tag{A.1}$$

where we have used the well known relation

$$\sum_{m=1}^{n} m^p = \frac{1}{p+1}\left(B_{p+1}(n+1)-B_{p+1}\right) \quad , \tag{A.2}$$

where $B_n(x)$ (B_n) are the Bernoulli polynomials (numbers). It is worth pointing out that the expression (3.28) for $\zeta(0,\beta)$, derived from another expansion (3.27), can be also obtained from (A.1).

The zeta-function at $\beta=\beta_H$ was found in [2] and is given by the series

$$\zeta(z,\beta)|_{\beta=\beta_H=1} \equiv \zeta_H(z,\Delta) =$$

$$=\frac{1}{3}\sum_{k=0}^{\infty}C_k(z)\left(\sqrt{\Delta}\right)^{2k} \times \left[\zeta(2z+2k-3,3/2)-\frac{1}{4}\zeta(2z+2k-1,3/2)\right] \quad . \tag{A.3}$$

Its derivative has the following integral representation

$$\zeta'(0,\Delta) = -\frac{1}{3}\left(\int_{\frac{1}{2}}^{\frac{1}{2}+\sqrt{\Delta}} + \int_{\frac{1}{2}}^{\frac{1}{2}-\sqrt{\Delta}}\right) u\left(u-\frac{1}{2}\right)\left(u-1\right)\psi(u)du + \frac{1}{12}\Delta^2 + \frac{1}{72}\Delta + \frac{2}{3}\left[\zeta_R'(-3,3/2) - \frac{1}{4}\zeta_R'(-1,3/2)\right] \quad . \qquad \text{(A.4)}$$

From (A.1) it is quite easy to obtain the approximate expressions for $\zeta(0,\beta)$ and $\zeta'(0,\beta)$ for $\beta \approx \beta_H$, in fact we have

$$\zeta(0,\beta) \approx \frac{\Delta^2}{12} - \frac{\Delta}{24} - \frac{17}{2880} + (1-\beta_H/\beta)\left(\frac{\Delta^2}{12} - \frac{5\Delta}{24} + \frac{3}{64}\right) \quad , \qquad \text{(A.5)}$$

$$\zeta'(0,\beta) \approx \left\{-\frac{1}{3}\left(\int_{\frac{1}{2}}^{\frac{1}{2}+\sqrt{\Delta}} + \int_{\frac{1}{2}}^{\frac{1}{2}-\sqrt{\Delta}}\right) u(u-\frac{1}{2})(u-1)\psi(u)du + \frac{\Delta^2}{12} + \frac{\Delta}{72} + \frac{2}{3}\left[\zeta_R'(-3,\frac{3}{2}) - \frac{1}{4}\zeta_R'(-1,\frac{3}{2})\right]\right\} + (\beta_H/\beta - 1)\left[\frac{41}{144}\Delta - \frac{\Delta^2}{8} - \frac{973}{5760} + \frac{1}{192}\left(16\Delta^2 - 40\Delta + 9\right)\left(\psi(\frac{3}{2}+\sqrt{\Delta}) + \psi(\frac{3}{2}-\sqrt{\Delta})\right)\right] \quad . \qquad \text{(A.6)}$$

Inserting (A.5) and (A.6) in (3.19),(3.20) we obtain the expressions for the one-loop effective potential and energy density at a temperature approaching the Hawking value. The next temperature corrections can be also estimated.

The asymptotic behavior of $V(\varphi,\beta_H)$ and $E(\varphi,\beta_H)$ at large a when $-\Delta \approx a^2V''(\varphi) \gg 1$ can be found from (3.19) and (3.20) by the asymptotic form of the psi-function [16]. For instance,

$$Re\ \left(\psi(1/2+iu)\right)|_{u\to\infty} = \log u - \frac{1}{24}u^{-2} - \frac{7}{960}u^{-4} + O(u^{-8}). \qquad \text{(A.7)}$$

One can thus obtain

$$V(\varphi,\beta_H) = V(\varphi) + \frac{3}{(4\pi)^2a^4}\left[\log\left(|\Delta|(a\mu)^{-2}\right)\left(\frac{\Delta^2}{12} - \frac{\Delta}{24} - \frac{17}{2880}\right) - \log\left(V''(\varphi)\mu^{-2}\right) - \frac{\Delta^2}{8} + \frac{\Delta}{24}\right]. \qquad \text{(A.8)}$$

Appendix B. Zeta-function in the ground state

The expression for the effective potential in the ground state follows from (3.19)

$$V(\varphi,\infty)=V(\varphi)-\frac{3}{(4\pi)^2a^4}\left[f'(0,\Delta)+\log(\mu^2)f(0,\Delta)\right] \quad , \tag{B.1}$$

where $f(z.\Delta)\equiv\lim_{\beta\to\infty}(\beta^{-1}\zeta(z,\beta))$. Using (3.27) one can see that

$$f(z,\Delta)=\sum_{k=0}^{\infty}C_k(z)\frac{\left(\sqrt{\Delta}\right)^{2k}}{2z+2k-1}\left[\zeta(2z+2k-3,3/2)-\frac{1}{4}\zeta(2z+2k-1,3/2)\right] \quad , \tag{B.2}$$

since $f(0,\Delta)$ can be found using (3.28) we only need to compute the derivative $\frac{d}{dz}f(z=0,\Delta)$. Remarkably, $f(z,\Delta)$ turns out to be connected with the zeta-function $\zeta_H(z,\Delta)$ in the de Sitter invariant state. Comparing (B.2) and (A.3) one can see that

$$\frac{d}{d\sqrt{\Delta}}\left[f(z,\Delta)\left(\sqrt{\Delta}\right)^{2z-1}\right]=3\Delta^{z-1}\zeta_H(z,\Delta) \quad , \tag{B.3}$$

and consequently

$$\frac{d}{d\sqrt{\Delta}}\left(\frac{1}{\sqrt{\Delta}}f'(0,\Delta)\right)=\frac{1}{\Delta}\left(3\zeta_H'(0,\Delta)-2f(0,\Delta)\right) \quad . \tag{B.4}$$

This equation has the following solution

$$\begin{aligned} f'(0,\Delta) \;&=\; \sqrt{\Delta}\int_0^{\sqrt{\Delta}}\frac{dy}{y^2}\left[3\left(\zeta_H'(0,y^2)-\zeta_H'(0,0)\right)-2\left(f(0,y^2)-f(0,0)\right)\right] \\ &-\; \left(3\zeta_H'(0,0)-2f(0,0)\right) \quad . \end{aligned} \tag{B.5}$$

To find $f(0,y^2)$ we take into account (3.28) and (A.5), so obtaining

$$\begin{aligned} f'(0,\Delta)=&\left(\int_{\frac{1}{2}}^{\frac{1}{2}+y}+\int_{\frac{1}{2}}^{\frac{1}{2}-y}\right)u\left(u-\frac{1}{2}-\sqrt{\Delta}\right)(u-1)\psi(u)du \\ &+\frac{1}{36}\Delta^2+\frac{7}{24}\Delta+\frac{17}{480}-2\left[\zeta_R'(-3,3/2)-\frac{1}{4}\zeta_R'(-1,3/2)\right] \quad , \end{aligned} \tag{B.6}$$

where in (B.6) we have integrated by parts to eliminate one integration. This result can be inserted into (B.1) and $V(\varphi,\infty)$ takes the form (3.33). The asymptotic form of $V(\varphi,\infty)$ can be obtained using (A.7). It is

$$V(\varphi,\infty)=V(\varphi)+\frac{3}{(4\pi)^2a^4}\left[\log\left(|\Delta|(a\mu)^{-2}\right)\left(\frac{\Delta^2}{12}-\frac{\Delta}{8}-\frac{17}{960}\right)-\frac{\Delta^2}{8}+\frac{\Delta}{8}\right], \tag{B.7}$$

where $-\Delta\approx a^2V''(\varphi)\gg 1$.

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