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GENERAL RELATIVITY AND DARK MATTER

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1 Introduction

At the present time, the problems of missing mass [1],[2] and of the existence of the so-called fifth force [3],[4] are widely discussed. It will be shown here that both the problems are mutually connected and have a positive solution in the framework of General Relativity that predicts the existence of new forces and unknown particles which can be interpreted as particles of dark matter.

2 The preliminaries

General Relativity was originally formulated as theory valid for mass distributions on a macroscopic scale, as opposed to atomic scale, and for classical electromagnetic fields. The Einstein equations in the presence of moving matter have the form [5]:

$$R^{ij} - \frac{1}{2}g^{ij}R = -8\pi\rho v^i v^j,$$

where v^i is a field of velocity of the matter. We will compare the Einstein equations with the Maxwell one for the continuously distributed charged matter [5]:

$$F^{ij}{}_{;i} = 4\pi\sigma v^j,$$

where v^i is the field of velocity of a charged matter. As it is known [6], the right hand side of the Maxwell equations has got quite a new interpretation on the quantum level. When one is founded on the deep likeness between the phenomena of gravity and electromagnetism, which is impossible to deny , it is natural to assume that the right hand side of the Einstein equations admits another representation similar to the representation given by the Dirac theory of the electron for the right hand side of the Maxwell equations. In atomic phenomena the gravitational forces are quite unimportant, and besides, the spinor fields are not linked with the group of diffeomorphism ,the group of symmetry of gravitational interactions [7]. Thus,the new content of the right hand side of Einstein equations is apparently not connected with known elementary particles . Thus ,in order to find new possibilities, it is natural to appeal to General Relativity itself.

3 Formulation of the Problem

A development of the differential geometry, caused mainly by the requirements of General Relativity led to the production and exact formulation of two fundamental notions :a differentiable manifold M and a vector space $T_p(M)$ tangent to M at a point p. Today these notions compose the reliable foundation not only for differential geometry but also General Relativity [7].The analyzis of these notions shows that the full group of symmetry of General Relativity must include not only the group of diffeomorphisms, mentioned above, but also the group of transformations of the such kind

$$\bar{V}^i = S^i_j V^j,$$

where $\bar{V}^i$ and V^i are components of the vector fields, S^i_j are components of the tensor field of type (1,1) ,det$(S^i_j) \neq 0$. The group thus defined is the group of gauge symmetry of General Relativity. It must be noted that in General Relativity the group of diffeomorphisms plays the role of the group of space-time symmetry. It can be shown ,that the diffeomorphism group is the group of external automorphisms of the gauge group of General Relativity,i.e. the gauge group is invariant under the transformations of the group Diff(M). Thus, we have nontrivial unification of space-time and gauge symmetres.

The tensor fields can transform under the gauge transformations in different ways. We will say that a tensor field T of the type (m, n) has the gauge type (p, q) if under the transformations of the gauge group there is the correspondence

$$\bar{T} = \underbrace{S \ldots S}_{p} T \underbrace{S^{-1} \ldots S^{-1}}_{q},$$

where $0 \leq p \leq m$, $0 \leq q \leq n$, and S^{-1} is the transformation inverse to S, $S^{-1} = (T^i_j)$, $S^i_k T^k_j = \delta^i_j$. From the equations $R_{ij} = 0$ which express the Einstein law of gravity , one can find that the Einstein gravitational potentials g_{ij} have the gauge type (0,0). Hence, the gauge symmetry is connected with the right hand side of the Einstein equations and, if following the analogy with electrodynamics, defines its structure.

So, it is natural to suppose that there are unknown gravitating elementary particles with which a new representation is linked of the right hand side of the Einstein equations like new representations of the right hand side of the Maxwell equations are connected with electrons and the

Dirac wave function. Hence, we should to find the type of the field which corresponds to the new particles and derive the simplest equations of the field with the help of the gauge group of General Relativity. It must be emphasized that from the Cartan theory of spinors [8] it follows that spinors are not connected with the group of gauge symmetry of General Relativity too.

4 Basis fields

From the theory of linear vector spaces and for the reasons of symmetry and simplicity it follows that there is a single quantity which can be put in correspondence with a hypothetical particles. This quantity is a tensor field of the type (1,1), Ψ^i_j, and the gauge type (1,1). Since under the action of the gauge group a tensor field Ψ is transformed as follows

$$\bar{\Psi} = S\Psi S^{-1},$$

then the scalars $\Psi^i_i = Tr\Psi$ and $\Psi^i_j \Psi^j_i = Tr(\Psi\Psi)$ are evidently invariants of the gauge group of GR. It is known from the theory of linear operators that there also exist other invariants, but in what follows we will only use the invariant $Tr(\Psi\Psi)$.

To derive the nontrivial gauge invariant equations for Ψ, one can consider the properties of such an important notion as the covariant derivative from the point of view the gauge symmetry. The covariant derivative of Ψ with respect to the affine connection $\Gamma_i = (\Gamma^j_{ik})$ can be written in the form

$$\nabla_i \Psi = \partial_i \Psi + [\Gamma_i, \Psi].$$

The use of the matrix notation is rather evident and does not require special explanations. Let $\bar{\Gamma}_i = (\bar{\Gamma}^j_{ik})$ is another affine connection on M and $\bar{\nabla}_i$ denotes the covariant derivative with respect to this connection. Then

$$\bar{\nabla}_i \Psi = \nabla_i \Psi + [\delta\Gamma_i, \Psi],$$

where $\delta\Gamma_i = \bar{\Gamma}_i - \Gamma_i$. Substituting, into the above relation, $\bar{\Psi} = S\Psi S^{-1}$ instead of Ψ, we get

$$\bar{\nabla}_i \bar{\Psi} = S(\nabla_i \Psi)S^{-1} + [\delta\Gamma_i - S\nabla_i S^{-1}, \Psi].$$

From this it follows that

$$\bar{\nabla}_i \bar{\Psi} = S(\nabla_i \Psi)S^{-1} \tag{1}$$

if $\delta\Gamma_i = S\nabla_i S^{-1}$ or that the same,

$$\bar{\Gamma}_i = \Gamma_i + S\nabla_i S^{-1} \quad (2)$$

Let

$$(B_{ij}{}^k{}_l) = B_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i + [\Gamma_i, \Gamma_j]$$

be the Riemann tensor of the affine connection Γ_i, then from (2) it follows that

$$\bar{B}_{ij} = SB_{ij}S^{-1}, \quad (3)$$

where $\bar{B}_{ij}$ is the Riemann tensor of the connection $\bar{\Gamma}_i$. Thus, from (2) and (3) it follows that in the framework of the gauge group of General Relativity one can consider the affine connection as the gauge field and the tensor B_{ij} as the strength tensor of this field. According to (1) a tensor field $\nabla_i\Psi$ has the same gauge type as Ψ, but this is not true for the covariant derivative of $\nabla\Psi$ or B_{ij}. For this reason it is necessary to introduce the important notion of the gauge covariant derivative.

Let T be a tensor field (tensor density) of the gauge type (1, 1), then by definition

$$D_iT = \partial_iT + [\Gamma_i, T]$$

is the gauge covariant derivative. For example, for the Riemann tensor, we have

$$D_iB_{jk} = \partial_iB_{jk} + [\Gamma_i, B_{jk}].$$

As it must be , for the field Ψ the gauge covariant derivative coincides with the standard covariant derivative, $D_i\Psi = \nabla_i\Psi$. In the general case the operator D_i is not general covariant, since D_iT will not always be a tensor field together with T. However, the commutator $[D_i, D_j]$ is always general covariant, because

$$[D_i, D_j]T = [B_{ij}, T].$$

From this we get the important relation for the Riemann tensor

$$[D_i, D_j]B_{kl} = [B_{ij}, B_{kl}] \quad (4)$$

Thus, the gauge symmetry shows that not only the Einstein gravitational potentials g_{ij} but also the tensor field Ψ and gauge field (affinity) Γ_i are to be treated as primary fields. Before writing the simplest gauge invariant equations, that express the law of interaction of these fields, it

must be noted that all the three fields have the geometrical interpretation. For the field g_{ij} it is well known, so we dwell only on the geometrical interpretation of the fields Ψ and Γ.

Let $\delta V^i = dV^i + \Gamma^i_{jk}\dot{x}^jV^kdt$ be an infinitesimal change of the vector field V on a curve $\gamma(t)$ and $S^i_j = \delta^i_j + \Psi^i_jdt$ be an infinitesimal gauge transformation. We assume that at every moment of time t an infinitesimal change of a vector V along a curve $\gamma(t)$ is equal to an infinitesimal linear transformation of the vector field V induced by the gauge group, $\delta V^i = \Psi^i_jV^idt$. Thus, we obtain a system of ordinary linear homogeneous differential equations for $V^i(t)$

$$\frac{dV^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}V^k = \Psi^i_jV^j.$$

from which it follows that the composite geometrical object (Ψ, Γ_i) defines the general law of parallel displacement of a vector field along the given curve $\gamma(t)$.

5 Field Equations

The simplest gauge invariant Lagrangian of the field Ψ has the form

$$L_\Psi = -\frac{1}{2}Tr(D_i\Psi D^i\Psi - m^2\Psi\Psi), \quad (5)$$

where m is a constant, $D_i = g^{ij}D_j$. From (5) by the variation with respect to Ψ we obtain the following equations

$$D_i(\sqrt{|g|}D^i\Psi) + m^2\sqrt{|g|}\Psi = 0, \quad (6)$$

where $|g|$ is the absolute value of the determinant of the matrix (g_{ij}). When deriving (6) one should take into account that $Tr(D_i\Psi) = \partial_i(Tr\Psi)$. In accordance with (6) one can consider m as the mass of a particle defined by the field Ψ. The simplest Lagrangian of the gauge field Γ is a direct consequence of (3)

$$L_\Gamma = -\frac{1}{4}Tr(B_{ij}B^{ij}), \quad (7)$$

where $B^{ij} = g^{ik}g^{jl}B_{kl}$. Varying the Lagrangian $L = L_\Psi + L_\Gamma$ with respect to Γ with the help of the relation $\delta B_{ij} = D_i\delta\Gamma_j - D_j\delta\Gamma_i$ we obtain the following equations of the gauge field Γ

$$D_i(\sqrt{|g|}B^{ij}) = \sqrt{|g|}J^j, \quad (8)$$

the right hand side of which contains the tensor field of the third rank

$$J^i = [\Psi, D^i\Psi]. \quad (9)$$

This field obviously has the gauge type (1,1). The tensor current J^i has to satisfy the equation

$$D_i(\sqrt{|g|}J^i) = 0, \quad (10)$$

as in accordance with (4) , $D_iD_j(\sqrt{|g|}B^{ij}) \equiv 0$. From (6) and (9) it follows that J^i really satisfies equation (10) and thus the system of equations (6) , (8) is consistent.

Varying the Lagrangian $L = L_\Psi + L_\Gamma$ with respect to g^{ij} we obtain the so-called metric tensor of energy-momentum of the considered system of interacting fields

$$T_{ij} = Tr(D_i\Psi D_j\Psi) + Tr(B_{ik}B_j{}^k) + g_{ij}L, \quad (11)$$

where $B_j{}^k = B_{jl}g^{kl}$. If the fields Ψ and Γ satisfy equations (6) and (8), then one can show that the metric tensor of the energy-momentum satisfies the well-known equations

$$T^{ij}{}_{;i} = 0$$

where the semicolon denotes as usual the covariant derivative with respect to the Levi-Civita connection belonging to the field g_{ij}

$$\{^i_{jk}\} = \frac{1}{2}g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}).$$

It is evident that the metric tensor energy-momentum is gauge invariant.

Now we can write down the full action for the fields g_{ij}, Ψ, Γ

$$S = -\frac{c^3}{G}\int R\sqrt{|g|}d^4x - \frac{\hbar}{2}\int Tr(D_i\Psi D^i\Psi + m^2\Psi\Psi)\sqrt{|g|}d^4x - -\frac{\hbar}{4}\int Tr(B_{ij}B^{ij})\sqrt{|g|}d^4x, \quad (12)$$

where R is the scalar curvature, G is the Newton gravitational constant and $\hbar$ is the Planck constant. From the geometrical interpretation of the fields Ψ and Γ it follows that they have the dimension sm^{-1}. As all coordinates can be considered have the dimension sm, the action S has the correct dimension. It is necessary to substantiate only why we have introduced the Planck constant $\hbar$into the full action S and not , say, the constant of interaction ε with the gauge field Γ ,similar to the electric charge of the electron e.

Consider the infinitesimal transformations of the diffeomorphism group , $\bar{x}^i = x^i + K^i(x)dt$. If under such transformations the gravitational potentials g_{ij} do not vary, i.e. the vector field $K^i(x)$ satisfies the Killing equations

$$K^i\partial_i g_{jl} + g_{il}\partial_j K^i + g_{ji}\partial_l K^i = K_{j;l} + K_{l;j} = 0,$$

then the vector field $P^i = T^{ij}K_j$ will satisfy the equation $P^i{}_{;i} = 0$. Integrating this equation we obtain the conservation law. The Killing equations impose severe constraints on the gravitational potentials. Thus, the Killing equations are quite integrable if the tensor of curvature of the metric g_{ij} satisfies the equations [9]:

$$R_{ijkl} = \frac{R}{12}(g_{ik}g_{jl} - g_{il}g_{jk}).$$

In the general case the Killing equations have no solutions at all and there are no conservation laws. This result allows us to understand the absence of the constant of interaction of matter fields with the gravitational one,similar to the electric charge e and why the Newton gravitational constant G does not enter into the equations of matter fields. It is impossible to ,,switch on,, or ,,switch off,, the gravitational field. It does not admit the existence of the gravitational screen.

Now consider infinitesimal gauge transformations $S^i_j = \delta^i_j + \Omega^i_j dt$. If under such transformations the gauge field Γ_i does not vary, i. e. the tensor field Ω^i_j satisfies the equations

$$\partial^i\Omega^j_k + \Gamma^j_{il}\Omega^l_k - \Omega^j_l\Gamma^l_{ik} = \nabla_i\Omega^j_k = 0,$$

then the vector field $Q^i = Tr(J^i\Omega)$, where J^i is a tensor current (9), will satisfy the equation $Q^i;i = 0$. Integrating this equation we obtain the conservation law as usual. The equation $\nabla_i\Omega^j_k = 0$, like the Killing equations , imposes severe constraints , in the given case, on the gauge field Γ_i. Thus, the equation $\nabla_i\Omega^j_k = 0$ is quite integrable, if the strength tensor B_{ij} satisfies the equations $B_{ij}{}_l{}^k = 1/4B_{ij}{}_m{}^m\delta^k_l$. So, in the considered case ,generally speaking, there are no conservation laws too. From this fact it is natural to conclude ,that there is no special coupling constant

(like the electric charge) of the gauge field Γ with the field of matter Ψ. The gauge field Γ is impossible to screen. In the known sense ,the Planck constant $\hbar$ is analogous to the Newton gravitational constant G.

6 General Physical Interpretation

Varying the full action (12) with respect to g^{ij} we derive the Einstein equations

$$R_{ij} - \frac{1}{2}g_{ij}R = l^2 T_{ij},$$

where $l = \sqrt{\hbar G/c^3}$ is the Planck length and T_{ij} is the metric tensor of energy-momentum (11). Thus, it is shown that the problem formulated above has the solution. As it is seen , the right hand sides of the Einstein equations on the quantum level are determined by the gauge symmetry as a matter of fact uniquely. This is in full analogy with the right hand side of the Maxwell equations under the same conditions.

A tensor field of the second rank Ψ^i_j describes unknown gravitating particles which, as it was shown earlier ,are a single source of the gauge field Γ that is known as the affine connection in geometry.

It is not difficult to show that for the field Ψ there is no nontrivial gauge invariant equations of the first order. Let S be an element of the gauge group ; obviously, $(-S)$ also belongs to that group. Further, under the gauge transformation $\bar{\Psi} = S\Psi S^{-1}$ the same transformation of the field Ψ corresponds to different elements S and $(-S)$ of the gauge group.We have remarked these properties because they characterize the bose particles.

The equations of motion for a particle in external gravitational and gauge fields have the form

$$\frac{d^2x^i}{ds^2} + \{^{\,i}_{jk}\}\frac{dx^j}{ds}\frac{dx^k}{ds} + \frac{\hbar}{mc}\omega^i{}_j\frac{dx^j}{ds} = 0,$$

where $\omega_{ij} = Tr(Bij) = \partial_i Q_j - \partial_j Q_i$ and $Q_i = Tr(\Gamma_i) = \Gamma^k_{ik}$. An additional gauge force acting on the particle has the quantum nature and the same form that the Lorentz force. However , the new force has one essential feature which can be seen from a rather general consideration. Taking the trace of equations (8) and taking into account that in accordance with (9), $Tr(J^i) \equiv 0$, one obtains that if Γ satisfies equations (6) and (8), then ω_{ij} will satisfy the equations $\omega^{ij}{}_{;i} = 0$. These equations are the same as the Maxwell equations without sources. From this one can conclude that the new force connected with the gauge field Γ, is neither central, nor Coulomb.

7 Conclusion

1. It is shown that General Relativity predicts the existence of a new type of particles which practically do not enteract with the known elementary particles, because the spinor fields are connected neither with the space-time symmetry group nor with the gauge symmetry group of General Theory of Relativity . However, these particles, as it is established, interact gravitationally and emit the quanta of the unknown longe-range field responsible for the existence of the fifth force. It is natural to suppose that these particles can explain the existence of the so-called dark matter.

2. The formulated theory containts one free parameter , the mass of a particle. It is not difficult to understand that the experimental discovery of these particles will be a very important confirmation of the Einstein General Relativity.

3. If the suggested physical interpretation of the theory is true, then the evidences of the existence of the dark matter , obtained from the astronomical observations , show that the macroscopic quantum effects can appear not only on the scale of solid bodies (superconductivity),but also on the scales of the galaxies.

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Общая теория относительности и скрытая масса

Показано, что из первых принципов общей теории относительности следует существование нового вида взаимодействий, тесно связанных с гравитационными. Носителями неизвестного взаимодействия являются частицы, которые не взаимодействуют с известными частицами ни электромагнитно, ни сильно, ни слабо. Физика этих частиц определяется планковскими масштабами.

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General Relativity and Dark Matter

It is shown that from the first principles of General Relativity it follows that there exists a new type of interactions which are tightly connected with the gravitational interactions. New particles representing a new form of interactions do not interact electromagnetically, strongly and weakly with the known elementary particles. Physics of the new particles is defined by the Planck scales.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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