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# COVARIANT DIFFERENTIAL COMPLEXES ON QUANTUM LINEAR GROUPS*

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# 1 Introduction

Since Woronowicz formulated the general scheme for constructing differential calculi on quantum matrix groups [1], the most publications on this theme have appealed more or less to it (see e.g. [3]-[17]). This scheme has the following structure: the first order differential calculus is defined in axiomatic way and, once it is fixed, the higher order differential calculus can be constructed uniquely. The underlying quantum group structure is taken into account by the bicovariance condition.

The principal problem of the Woronowicz's approach that has been mentioned already in [1] but still remains unsolved is that the scheme possesses a variety of differential calculi for each quantum group, and there is no criterion to choose the most appropriate one.

On the other hand, the $R$-matrix formalism (see [2] and references therein), initially motivated by the quantum inverse scattering method, appears to be an extremely useful tool in dealing with quantum groups and essentially with differential calculus on them. So, it is not surprising that there have appeared some papers relating the Woronowicz's scheme and $R$-matrix formalism [3]-[5]. Based on the differential calculus on a quantum hyperplane and using the $R$-matrix formulation one may hope to construct finally the most natural differential calculus on quantum group (see [6]-[9]). This program has been realized for the $GL_q(N)$-case in [10]-[15], but being restricted to $SL_q(N)$, the calculus obtained reveals some unfavourable properties (see the discussion in Section 5), which forces us to search for other possibilities. Thus, the classification of differential calculi on linear quantum groups remains an actual problem up to now.

In the present paper we make an attempt to approach this problem from an opposite direction, i.e. to construct firstly the higher order differential calculus. Here, the key role is played by the conditions:

a.) Cartan's 1-forms realize the adjoint representation of $GL_q(N)$;

b.) all higher order invariant forms, being polynomials of the Cartan's 1-forms, can be ordered (say, lexicographically) uniquely.

The paper is organised as follows: all the preliminary information and notation are collected in Section 2. In Section 3, developing the ideas of Ref.[14] we consider $GL_q(N)$-*covariant* quantum *algebras* (CA). Arranging them into two classes, the $q$-symmetrical (SCA) and $q$-antisymmetrical (ACA) ones, we then concentrate on studying the homogeneous ACA's, that could be interpreted as the external algebras of Cartan's 1-forms. We find four one-parametric families of such algebras. Section 4 is devoted to the construction of differential complexes on the homogeneous ACA's. In doing so we admite the deformation of the Leibnitz rule and, thus, extend the class of the permitted complexes. We conclude the paper by considering how the known $GL_q(N)$-differential calculi are included into this scheme and by discussing the problems of the $SL_q(N)$-reduction.

## 2 Notation

We consider the Hopf algebra $Fun(GL_q(N))$ generated by elements of the $N \times N$ matrix $T = ||T_{ij}||$, $i,j = 1,\dots,N$ obeying the following relations:

$$\mathbf{RTT'} = \mathbf{TT'R}\,. \tag{2.1}$$

Here $\mathbf{T} \equiv T_1 \equiv T \otimes I$, $\mathbf{T'} \equiv T_2 \equiv I \otimes T$, $I$ is $N \otimes N$ identity matrix, $\mathbf{R} \equiv \hat{R}_{12} \equiv P_{12}R_{12}$, $P_{12}$ is the permutation matrix and $R_{12}$ is the $GL_q(N)$ $R$-matrix [1] satisfying the Yang-Baxter equation and Hecke condition, respectively,

$$\mathbf{RR'R} = \mathbf{R'RR'}\,, \tag{2.2}$$

$$\mathbf{R}^2 - \lambda\mathbf{R} + \mathbf{1} = 0\,, \tag{2.3}$$

where $\lambda = q - q^{-1}$, $\mathbf{R'} \equiv \hat{R}_{23} \equiv P_{23}R_{23}$ and $\mathbf{1}$ is the $N^2 \times N^2$ identity matrix. In accordance with (2.3), for $q^2 \neq -1$ the matrix $\mathbf{R}$ decomposes as

$$\begin{aligned} \mathbf{R} &= qP^+ - q^{-1}P^-\,, \\ P^\pm &= (q+q^{-1})^{-1}\{q^{\mp 1}\mathbf{1} \pm \mathbf{R}\}\,, \end{aligned} \tag{2.4}$$

where the projectors $P^+$ and $P^-$ are quantum analogues of the antisymmetrizer and symmetrizer, respectively.

The comultiplication for the algebra $Fun(GL_q(N))$ is defined as $\Delta T_{ij} = T_{ik} \otimes T_{kj}$, and the antipode $S(.)$ [2] obeys the conditions $S(T_{ij})T_{jl} = T_{ij}S(T_{jl}) = \delta_{il}1$, so in what follows we use the notation $T^{-1}$ instead of $S(T)$.

## 3 $GL_q(N)$-covariant Quantum Algebras

Consider the $N^2$-dimensional adjoint $Fun(GL_q(N))$-comodule $\mathcal{A}$. We arrange its basic elements into the $N \times N$ matrix $A = ||A_{ij}||$, $i,j = 1,\dots,N$. The adjoint coaction is

$$A^i_j \to T^i_{i'}S(T)^{j'}_j \otimes A^{i'}_{j'} \equiv (TAT^{-1})^i_j\,, \tag{3.1}$$

where the last part of the formula (3.1) is the standard notation to be used below.

The comodule $\mathcal{A}$ is reducible, and the irreducible subspaces in $\mathcal{A}$ can be extracted by using the so called quantum trace ($q$-trace) [2, 19] (see also [5, 13, 20]). In the case of $Fun(GL_q(N))$ it has the form:

$$Tr_qA \equiv Tr(\mathcal{D}A) \equiv \sum_{i=1}^N q^{-N-1+2i}A^i_i\,, \quad \mathcal{D} \equiv diag\{q^{-N+1}, q^{-N+3}, \dots, q^{N-1}\}, \tag{3.2}$$

[1]For the explicit form of $GL_q(N)$ $R$-matrix see Refs.[18, 2].

[2]Strictly speaking, to define the antipodal mapping on $Fun(GL_q(N))$, we must add one more generator $(det_qT)^{-1}$ to the initial set $\{T_{ij}\}$ (see [2]).



and possesses the following invariance property:

$$Tr_q(TAT^{-1}) = Tr_q(A)\,,$$

i.e. $Tr_q(A)$ is a scalar part of the comodule $\mathcal{A}$, while the $q$-traceless part of $A$ forms the basis of $(N^2-1)$-dimensional irreducible $Fun(GL_q(N))$-adjoint comodule. Let us note also the following helpful formulae:

$$Tr_{q(2)}(\mathbf{RAR}^{-1}) = Tr_{q(2)}(\mathbf{R}^{-1}\mathbf{AR}) = Tr_qA\, I_{(1)}\,,$$

$$Tr_{q(2)}\mathbf{R}^\pm = q^{\pm N} I_{(1)}, \quad Tr_qI = [N]_q\,,$$

where $\mathbf{A} \equiv A_1 \equiv A \otimes I$, $[N]_q = \frac{q^N - q^{-N}}{q-q^{-1}}$, and by $X_{(i)}$ we denote quantities (operators) $X$ living (acting) in the $i$-th space.

Consider now the associative unital $\mathbf{C}$-algebra $\mathbf{C}< A_{ij} >$ freely generated by the basic elements of $\mathcal{A}$. As a vector space, $\mathbf{C}< A_{ij} >$ naturally carries the $Fun(GL_q(N))$-comodule structure. Now we introduce $GL_q(N)$- *covariant* quantum *algebra* (CA) as the factoralgebra of $\mathbf{C}< A_{ij} >$, possessing the following properties [14]:

(A) The multiplication in this algebra is defined by a set $\{\alpha\}$ of polynomial identities quadratic in $A_{ij}$:

$$C^\alpha_{ijkl}A_{ij}A_{kl} = C^\alpha_{ij}A_{ij} + C^\alpha. \tag{3.3}$$

In other words, CA is the factor algebra of $\mathbf{C}< A_{ij} >$ by the bi-ideal generated by (3.3).

(B) Considered as a vector space CA is a $GL_q(N)$-adjoint comodule, so the coefficients $C^\alpha_{ijkl}$ in (3.3) are $q$-analogues of the Clebsh-Gordon coefficients coupling two adjoint representations, and the set of the relations (3.3) is divided into several subsets corresponding to different irreducible $Fun(GL_q(N))$-comodules in $\mathcal{A} \otimes \mathcal{A}$. Parameters $C^\alpha_{ij}$ are not equal to zero when $C^\alpha_{ijkl}$ couple $\mathcal{A} \otimes \mathcal{A}$ into the adjoint $GL_q(N)$-comodule again, while $C^\alpha \neq 0$ only if $C^\alpha_{ijkl}A_{ij}A_{kl}$ are scalars.

(C) All the monomials in CA can be ordered lexicographically due to (3.3).

(D) All the nonvanishing ordered monomials in CA are linearly independent and form a basis in CA.

Now we recall that for the classical case ($q = 1$) the dimensions of the irreducible $Fun(GL(N))$-subcomodules in $\mathcal{A} \otimes \mathcal{A}$ are given by Weyl formula [21]:

$$dim\ \mathcal{A} \otimes \mathcal{A} = [(N^2-1)+1]^2 = 2 \cdot [1] \oplus (3 + \theta_{N,2}) \cdot [N^2-1] \oplus \tag{3.4}$$

$$2\theta_{N,2} \cdot \left[\frac{(N^2-1)(N^2-4)}{4}\right] \oplus \left[\frac{N^2(N+3)(N-1)}{4}\right] \oplus \theta_{N,3} \cdot \left[\frac{N^2(N+1)(N-3)}{4}\right],$$

where $\theta_{N,M} = \{1\ for\ N > M;\ 0\ for\ N \leq M\}$. Thus, $\mathcal{A} \otimes \mathcal{A}$ splits into 2 scalar subcomodules, 4 (3 for $N = 2$) adjoint (traceless) subcomodules and 4 (1 for $N = 2$ and 3 for $N = 3$) higher-dimensional mutually inequivalent subcomodules. In the quantum case according to the results of Ref. [22] the situation generally is not changed (the exception is for $q$ being a root of unity). Below we employ the $q$-(anti)symmetrization projectors $P^\pm$ and $q$-trace to extract the irreducible subcomodules in $\mathcal{A}\otimes\mathcal{A}$, thus, supposing from the beginning that $q \neq -1$ and $Tr_q I = [N]_q \neq 0$.

First, we shall obtain the sets of combinations quadratic in $A_{ij}$ that correspond to the left hand side of (3.3) and contain four higher dimensional $Fun(GL_q(N))$-subcomodules (see (3.4)). Let us start with the $N^2 \times N^2$ matrix $\mathbf{ARA}$ containing all the $N^4$ independent combinations quadratic in $A_{ij}$ and having convenient comodule transformation properties:

$$\mathbf{ARA} \to (\mathbf{TT}')\mathbf{ARA}(\mathbf{TT}')^{-1}\,. \tag{3.5}$$

From (2.1), (2.4) it follows that $P^\pm\mathbf{TT}' = \mathbf{TT}'P^\pm$, hence we can split $\mathbf{ARA}$ into four independently transforming (for $N \geq 3$) parts:

$$X^{\pm\pm} \equiv P^\pm\mathbf{ARA}P^\pm\,,\quad X^{\pm\mp} \equiv P^\pm\mathbf{ARA}P^\mp\,. \tag{3.6}$$

Namely the $q$-traceless (in both 1st- and 2nd- spaces) parts of $X^{++}$, $X^{--}$ and $X^{\pm\mp}$ are the four higher dimensional subcomodules in $\mathcal{A}\otimes\mathcal{A}$ with dimensions: $\frac{N^2(N+3)(N-1)}{4}$, $\frac{N^2(N-3)(N+1)}{4}$ and $\frac{(N^2-1)(N^2-4)}{4}$ respectively.

Now acting on $X$'s by $Tr_q$-operation we obtain (for $N \neq 2$ and $q$ not being a root of unity) four independent combinations transforming as adjoints:

$$A^2\,,\quad (Tr_qA)A\,,\quad A(Tr_qA)\,,\quad A*A \equiv Tr_{q(2)}(\mathbf{R}^{-1}\mathbf{ARAR}^{-1})\,. \tag{3.7}$$

The $q$-traceless parts of these combinations correspond to the irreducible adjoint subcomodules in $\mathcal{A}\otimes\mathcal{A}$. Applying $Tr_q$ to Eqs.(3.7) once again we arrive at two independent expressions

$$(Tr_qA)^2\,,\quad Tr_q(A^2)\,, \tag{3.8}$$

corresponding to the scalar subcomodules. We refer to the expressions (3.6), (3.7) and (3.8) as higher-dimensional, adjoint and scalar terms respectively.

As it was argued in [14], to satisfy the condition (C) for CA, the left hand side of the relations (3.3) must contain independently either $X^{++}$ with $X^{--}$, or $X^{+-}$ with $X^{-+}$. One can combine these pairs into single expressions:

$$(q+q^{-1})(X^{++}-X^{--}) = \mathbf{RARA} + \mathbf{ARAR}^{-1}\,, \tag{3.9}$$

$$(q+q^{-1})(X^{-+}-X^{+-}) = \mathbf{RARA} - \mathbf{ARAR}\,. \tag{3.10}$$

The way of combining the quantities (3.6) is not important. We choose the concise forms (3.9), (3.10) because in the classical limit they are nothing but the anticommutator $[A_2, A_1]_+$ and commutator $[A_2, A_1]_-$. So it is natural to call (3.9) and (3.10) the $q$-anticommutator and $q$-commutator, respectively. In view of this all the CA's with the defining relation (3.3) are classified into two types depending on whether their defining relations contain the $q$-anticommutator or $q$-commutator. The first will be called *antisymmetric* CA (ACA) and the last – *symmetric* CA (SCA).



At the moment we still fix the higher-dimensional terms in a quadratic part of the relations (3.3), but there remains an uncertainty in the choice of the adjoint and scalar terms. Let us show this explicitly. First of all we employ simple dimensional considerations. To satisfy the ordering condition (C) at a quadratic level, we must include at least $\frac{N^2(N^2-1)}{2}$ independent relations into (3.3) (e.g. for the classical case of $gl(N)$ this corresponds to the number of commutators $[A_{ij}, A_{kl}]$). Since the h-dimensional terms (3.10) for SCA contain $\frac{(N^2-1)(N^2-4)}{2}$ independent combinations, we must add to them at least $2\cdot(N^2-1)$ independent combinations, i.e. two $q$-traceless adjoint terms. Actually, the estimation is precise: including any other additional adjoint or scalar terms into (3.3) would result in a linear dependence of quadratic ordered monomials and, thus, would contradict (D). As regards the ACA's, from the $\frac{N^2(N^2-3)}{2}$ independent combinations, contained in h-dimensional terms (3.9), the $N^2$ combinations lead to the relations (3.3) of the type: $A_{ij}^2 = 0$ $(i \neq j)$, $A_{ii}^2 = \sum_{kl} f_i^{kl} A_{kl}A_{lk}$ (where $f_i^{kl}$ are some constants), i.e. they are useless in the ordering procedure. Hence we have a deficit of the $2N^2$ independent quadratic combinations in $N \geq 3$ case (5 combinations for $N=2$) and are forced to include in (3.3) 2(1 for $N=2$) independent q-traceless adjoint terms and a pair of scalar terms. With this inclusion ACA's are defined by the set of $\frac{N^2(N^2+1)}{2}$ relations.

Thus, we have determined the number of independent adjoint and scalar terms in symmetric and antisymmetric CA's. Note that the $q$-commutator and $q$-anticommutator themselves contain the true number of adjoints and scalars, which is demonstrated by the following symmetry properties:

$$P^\pm\{\mathbf{RARA} \;-\; \mathbf{ARAR}\}P^\pm = 0\,, \tag{3.11}$$

$$P^\pm\{\mathbf{RARA} \;+\; \mathbf{ARAR}^{-1}\}P^\mp = 0\,. \tag{3.12}$$

But there is an opportunity to change the form of quadratic adjoint terms in the left-hand side of Eq.(3.3) without changing their number. Indeed, consider the quantities

$$\Delta_\pm(U_{ad}(A)) = \mathbf{R}U_{ad}(A)\mathbf{R}^{\pm1} \pm U_{ad}(A)\,, \tag{3.13}$$

$$\begin{aligned} U_{ad}(A) \;=\; & u^1(\mathbf{R})\cdot\mathbf{A}^2 + (u^2-e)(\mathbf{R})\cdot(Tr_qA)\mathbf{A} \\ & + (u^3-e)(\mathbf{R})\cdot\mathbf{A}(Tr_qA) + u^4(\mathbf{R})\cdot(A*A)\,, \end{aligned} \tag{3.14}$$

where $u^a(\mathbf{R}) = u_1^a + u_2^a\mathbf{R}$, $a = 1,2,3,4$, and $e(\mathbf{R}) = \frac{1}{[N]_q}(u^1(\mathbf{R}) + q^{-N}u^4(\mathbf{R}) - 1)$. We make the $e(\mathbf{R})$-shift of the parameters $u^2(\mathbf{R})$ and $u^3(\mathbf{R})$ for the sake of future convenience. Expressions $\Delta_\pm$ are the most general covariant combinations which contain only adjoint and scalar (for $\Delta_+$) terms and satisfy symmetry properties

$$P^\pm\Delta_-P^\pm = P^\pm\Delta_+P^\mp = 0\,. \tag{3.15}$$

Therefore we may use $\Delta_+$ and $\Delta_-$ in varying the quadratic part of the defining relations (3.3) for ACA's and SCA's, respectively. Note that in principal one could add to the r.h.s. of (3.14) the scalar combination $U_{sc} = h(\mathbf{R})Tr_q(A^2) + g(\mathbf{R})(Tr_qA)^2$, where $h$

and $g$ are arbitrary functions of $\mathbf{R}$. This addition, obviously, does not affect $\Delta_-$. As concerns $\Delta_+$, remember that defining relations for ACA must contain a pair of independent quadratic scalars represented generally as:

$$Tr_q(A^2) = C_1 Tr_q(A) + C_2\ , \quad (Tr_q A)^2 = C_3 Tr_q(A) + C_4\ . \tag{3.16}$$

Here $C_i$ are some constants. Thus, even changing the form of $\Delta_+$, the term $U_{sc}$ cannot change the content of the bilinear part of defining relations for ACA and we will omit this term in further considerations.

Now we shall concentrate on studying the homogeneous (pure quadratic) ACA's, which possess the natural $Z_2$-grading and may be interpreted as external algebras of the invariant forms on $GL_q(N)$. To emphasize this step, we change the notation from $A$ to $\Omega$. All the other cases can be considered following the same lines.

As we have shown, the general defining relations for homogeneous ACA look like

$$\mathbf{R}\Omega\mathbf{R}\Omega + \Omega\mathbf{R}\Omega\mathbf{R}^{-1} = \Delta_+\ . \tag{3.17}$$

These relations contain 8 random parameters $u_i^a$, $(a = 1,2,3,4;\ i = 1,2)$, but actually this parametrization of the whole variety of homogeneous ACA's is redundant. To minimize the number of parameters in (3.17), let us pass to the new set of generators:

$$\Omega \to \left\{ \begin{array}{lcl} \omega & = & Tr_q\Omega\ , \\ \tilde{\Omega} & = & \Omega - \frac{\omega}{[N]_q} I\ , \quad Tr_q\tilde{\Omega} = 0\ . \end{array} \right. \tag{3.18}$$

Using these new variables one can extract the first scalar relation $\omega^2 = 0$ and (3.17) is changed slightly to

$$\begin{aligned} \mathbf{R}\tilde{\Omega}\mathbf{R}\tilde{\Omega} \ &+\ \tilde{\Omega}\mathbf{R}\tilde{\Omega}\mathbf{R}^{-1} = \Delta_+(U(\tilde{\Omega}))\ , & (3.19) \\ \omega^2 \ &=\ 0\ , & (3.20) \end{aligned}$$

where $\Delta_+(U) = \mathbf{R}U\mathbf{R} + U$ and

$$U(\tilde{\Omega}) = u^1(\mathbf{R})\tilde{\Omega}^2 + u^2(\mathbf{R})\omega\tilde{\Omega} + u^3(\mathbf{R})\tilde{\Omega}\omega + u^4(\mathbf{R})(\tilde{\Omega} * \tilde{\Omega})\ . \tag{3.21}$$

Here, as usual, $\tilde{\boldsymbol{\Omega}} \equiv \tilde{\Omega}_1 \equiv \tilde{\Omega} \otimes I$.

Applying the operations $Tr_{q(2)}[...]$, $Tr_{q(2)}[\mathbf{R}^{-1}...]$, and then $Tr_{q(1)}[...]$ to Eq.(3.19) we extract adjoint relations and then obtain the second scalar relation

$$Tr_q(\tilde{\Omega}^2) = q^{-N} Tr_q(\tilde{\Omega} * \tilde{\Omega}) = 0\ , \tag{3.22}$$

The adjoint relations are represented in the form:

$$v^1(\mathbf{R})\tilde{\boldsymbol{\Omega}}^2 + v^2(\mathbf{R})\omega\tilde{\boldsymbol{\Omega}} + v^3(\mathbf{R})\tilde{\boldsymbol{\Omega}}\omega + v^4(\mathbf{R})(\tilde{\Omega} * \tilde{\Omega}) = 0\ , \tag{3.23}$$

where

$$\begin{aligned} v^a(\mathbf{R}) = v_1^a + v_2^a\mathbf{R} \ &=\ x(\mathbf{R})u^a(\mathbf{R}) - \delta_{a,1}q^N\mathbf{R}^2 - \delta_{a,4}\mathbf{1}\ , & (3.24) \\ x(\mathbf{R}) = x_0 + x_1\mathbf{R} \ &=\ (q^N + q^{-N}) + ([N]_q + \lambda q^N)\mathbf{R}\ . & (3.25) \end{aligned}$$



Here we arrange the pair of adjoints into a single matrix relation. Expanding (3.23) in a power series of $\mathbf{R}$ one can obtain both the adjoint relations explicitly.

Now we can reduce the number of coefficients parametrizing ACA's. Namely, we use Eqs. (3.23) to represent some pair of adjoint terms (3.7) as linear combinations of the other two adjoints. Let us denote $2 \times 2$ minors of the system (3.23) as

$$\gamma^{ab} = det \begin{vmatrix} v_1^a & v_1^b \\ v_2^a & v_2^b \end{vmatrix}\ . \tag{3.26}$$

Note, if $\gamma^{34} = \gamma^{24} = 0$, then we get from (3.23) that $\tilde{\Omega}^2$ must be proportional to either $\omega\tilde{\Omega}$, or $\tilde{\Omega}\omega$, which contradicts the condition (D). Hence, there are only two variants of solving (3.23) with respect to either $\tilde{\Omega} * \tilde{\Omega}$ and $\tilde{\Omega}\omega$ (if $\gamma^{34} \neq 0$), or $\tilde{\Omega} * \tilde{\Omega}$ and $\omega\tilde{\Omega}$ (if $\gamma^{24} \neq 0$). Both the choices are quite natural since, first, we eliminate the cumbersome expression $\tilde{\Omega} * \tilde{\Omega}$ from further considerations and, second, we fix the order of quantities $\omega$ and $\tilde{\Omega}$ in their monomials (turning $\omega$, respectively, to the left, or to the right). In fact, as we shall see further (see remark 3 to Theorem 1), both these variants are equivalent and conditions $\gamma^{34} \neq 0$, $\gamma^{24} \neq 0$ are necessary in obtaining consistent ACA's. So, we suppose from the beginning that both $\gamma^{34}$ and $\gamma^{24}$ are not equal to zero, and choose solving (3.23) w.r.t. $\tilde{\Omega} * \tilde{\Omega}$ and $\tilde{\Omega}\omega$. The result is

$$\tilde{\Omega} * \tilde{\Omega} = \delta\tilde{\Omega}^2 + \tau\omega\tilde{\Omega}\ , \quad \tilde{\Omega}\omega = -\rho\omega\tilde{\Omega} + \sigma\tilde{\Omega}^2\ , \tag{3.27}$$

where

$$\delta = \frac{\gamma^{13}}{\gamma^{34}}\ , \quad \tau = \frac{\gamma^{23}}{\gamma^{34}}\ , \quad \rho = \frac{\gamma^{24}}{\gamma^{34}} \neq 0\ , \quad \sigma = -\frac{\gamma^{14}}{\gamma^{34}}$$

are namely that minimal set of parameters, we are searching for. In this parametrization the defining relations for ACA look like

$$\mathbf{R}\tilde{\Omega}\mathbf{R}\tilde{\Omega} + \tilde{\Omega}\mathbf{R}\tilde{\Omega}\mathbf{R}^{-1} = \bar{x}(\mathbf{R})\left\{(\delta + q^N\mathbf{R}^2)(\mathbf{R}\tilde{\boldsymbol{\Omega}}^2\mathbf{R} + \tilde{\boldsymbol{\Omega}}^2) + \tau\omega(\mathbf{R}\tilde{\boldsymbol{\Omega}}\mathbf{R} + \tilde{\boldsymbol{\Omega}})\right\}\ , \tag{3.28}$$

$$\tilde{\Omega}\omega = -\rho\omega\tilde{\Omega} + \sigma\tilde{\Omega}^2\ , \tag{3.29}$$

$$\omega^2 = 0\ , \tag{3.30}$$

where

$$\begin{aligned} \bar{x}(\mathbf{R}) \ &\equiv\ \{x(\mathbf{R})\}^{-1} = \frac{1}{[N+2]_q[N-2]_q}\{-x_2 + x_1\mathbf{R}\}\ , & (3.31) \\ x_2 \ &\equiv\ x_0 + \lambda x_1 = q^N(q^{-2} + q^2)\ . \end{aligned}$$

To get the relations (3.28) we solve (3.24) with respect to $u^a(\mathbf{R})$, substitute the resulting expressions into (3.21),(3.19), and then use (3.23) and the first of Eqs.(3.27). The systems of relations (3.28-3.30) and (3.19-3.21) are equivalent if the matrix $x(\mathbf{R})$ is invertible (i.e. if $[N+2]_q[N-2]_q \neq 0 \ \Leftrightarrow\ [N]_q \neq \pm[2]_q \ \Leftrightarrow\ q^{2N\pm 4} \neq 1$[3]). Further

[3]Cf. with the remark in the brackets above Eq. (3.7).

we shall consider this nonsingular case. The case $N=2$ will be treated in detail in the next Section.

Now let us discuss the symmetry properties of Eq.(3.28). Consider the following transformation

$$\left\{ \begin{array}{lll} q & \to & q^{-1}, \quad hence\,, \quad \mathbf{R}_q \to \mathbf{R}_{\frac{1}{q}}\,, \quad x_q(\cdot) \to x_{\frac{1}{q}}(\cdot)\,; \\ \tilde{\Omega} & \to & \tilde{\Omega}' \equiv \tilde{\Omega}_2 \equiv I \otimes \tilde{\Omega}\,. \end{array} \right. \tag{3.32}$$

Here in the lower indices we write values of the quantization parameter for the considered quantities. Note, that using the symmetry property of $GL_q(N)$ $R$-matrix:

$$\mathbf{R}_{\frac{1}{q}} = P_{12}\mathbf{R}_q^{-1}P_{12}$$

one can find that (3.32) is a product of two symmetries: the involution transformation of the operators $\mathbf{B} \to P_{12}\mathbf{B}P_{12}$ and discrete symmetry

$$\left\{ \begin{array}{lll} q & \to & q^{-1}, \quad x_q(\cdot) \to x_{\frac{1}{q}}(\cdot)\,, \quad but \quad \mathbf{R}_q \to \mathbf{R}_q^{-1}\,; \\ \tilde{\Omega} & \longrightarrow & remains \quad unchanged\,. \end{array} \right. \tag{3.33}$$

It must be stressed that the replacement $q \to q^{-1}$ doesn't concern the definition of $\omega$, i.e. of the quantum trace. Otherwise, we would obtain an algebra with different covariance properties, namely, the algebra of left-invariant (w.r.t. transitions in underlying quantum group $GL_q(N)$) objects.

Now using the identity

$$x_q(\mathbf{R}_q)\mathbf{R}_q^{-1} = \frac{q^N\mathbf{R}_q^2 - q^{-N}\mathbf{R}_q^{-2}}{\lambda}$$

we deduce the following properties of the matrix function $\bar{x}(\mathbf{R})$:

$$\begin{array}{rcl} \bar{x}_q(\mathbf{R}_q) & = & \mathbf{R}_q^{-2}\bar{x}_{\frac{1}{q}}(\mathbf{R}_q^{-1})\,, \\ q^N\mathbf{R}_q^2\bar{x}_q(\mathbf{R}_q) & = & q^{-N}\mathbf{R}_q^{-2}\bar{x}_q(\mathbf{R}_q) + \lambda\mathbf{R}_q^{-1}\,. \end{array}$$

and, then, it is no hard to check that Eq. (3.28) is invariant under the substitution (3.33) and, therefore, under (3.32). In the classical limit $q=1$ this transformation reduces to the identity, but in the quantum case we get the discrete $Z_2$-group of symmetries of Eq. (3.28). Namely this symmetry produces the doubling of differential calculi on $GL_q(N)$ observed by many authors ( see e.g. [13]).

Thus, the most general form for the algebras which admit an ordering for any quadratic monomial in their generators is (3.28)-(3.30). The next step in finding out the consistent defining relations for homogeneous ACA's is to consider the ordering of cubic polynomials. Let us present the result of our considerations in



**Theorem 1:** *For general values of the quantization parameter $q$ there exist four one-parametric families of homogeneous ACAs. The defining relations for the first pair of them look like*

$$\left\{ \begin{array}{lll} \mathbf{R}\tilde{\Omega}\mathbf{R}\tilde{\Omega} + \tilde{\Omega}\mathbf{R}\tilde{\Omega}\mathbf{R}^{-1} & = & \kappa_q\left(\tilde{\Omega}^2 + \mathbf{R}\tilde{\Omega}^2\mathbf{R}\right)\,, \\ \omega^2 & = & 0\,, \end{array} \right. \tag{3.34}$$

*and*

$$type\ I: \quad \tilde{\Omega}\omega \;\dot{=}\; -\rho\omega\tilde{\Omega}\,, \quad \rho \neq 0\,; \tag{3.35}$$

$$type\ II: \; [\tilde{\Omega},\omega]_+ \;=\; \sigma\tilde{\Omega}^2\,, \quad \sigma \neq 0\,. \tag{3.36}$$

*Here $\kappa_q = \frac{\lambda q^N}{[N]_q + \lambda q^N}$, and $q \neq -1$, $[N]_q \neq \{0,\ -\lambda q^N,\ -\lambda[2]_q q^{N\pm1},\ \pm[2]_q\}$. For both cases the following remarkable relation holds:*

$$\mathbf{R}\tilde{\Omega}^2\mathbf{R}\tilde{\Omega} - \tilde{\Omega}\mathbf{R}\tilde{\Omega}^2\mathbf{R} = 0\,. \tag{3.37}$$

*The remaining pair of families can be obtained from the first one by the involution (3.32) or (3.33).*

*Finally in the classical limit ($q=1$) there exists one more family of homogeneous ACAs:*

$$\left\{ \begin{array}{lll} [\tilde{\Omega}_1,\tilde{\Omega}_2]_+ & = & \tau'\left(P_{12} - \frac{2}{N}\right)\omega\left(\tilde{\Omega}_1 + \tilde{\Omega}_2\right)\,, \\ [\tilde{\Omega},\omega]_+ & = & 0\,, \\ \omega^2 & = & 0\,. \end{array} \right. \tag{3.38}$$

*where $\tau' = \frac{\tau N}{N^2-4} \neq 0$ (see Eqs. (3.27),(3.28)).*

**Proof:** We shall prove the Theorem for type I and II algebras. The results for the second pair of algebras are obviously obtained by applying transformation (3.33) to all the formulae below.

To check the ordering at a cubic level, it is enough to consider two monomials: $(\mathbf{R}\tilde{\Omega})^2\omega$ and $(\mathbf{R}'\mathbf{R}\tilde{\Omega})^3$. In the classical limit these combinations become $\tilde{\Omega}_2\tilde{\Omega}_1\omega$ and $\tilde{\Omega}_3\tilde{\Omega}_2\tilde{\Omega}_1$, respectively, and for the ordinary external algebra of invariant forms on $GL(N)$ the procedure of their ordering looks like $\tilde{\Omega}_2\tilde{\Omega}_1\omega \to \omega\tilde{\Omega}_1\tilde{\Omega}_2$ and $\tilde{\Omega}_3\tilde{\Omega}_2\tilde{\Omega}_1 \to \tilde{\Omega}_1\tilde{\Omega}_2\tilde{\Omega}_3$.

Before establishing the quantum analog of this procedure we have to choose the basis of "ordered" cubic monomials. Here the notion "ordered" is given in quotation-marks since we can't achieve true lexicographic ordering of monomials without loosing the compact matrix form of our considerations and passing to cumbersome calculations in $\Omega_{ij}$-components. Such in-component calculations, based on the use of the Diamond Lemma (see [23]), were carried out for the case $N=2$ in Refs. [9, 10, 16] and it seems doubtful that they could be repeated for general $N$. So, we use the basis of quasi-ordered cubic combinations which are convenient for our matrix manipulations. In this way, we cannot prove that we have exhausted all the possible types of ACAs, but the algebras obtained surely satisfy all the conditions for ACA and our conjecture is that Theorem 1 gives all the possible ACA's.

Let us define some new symbols:

$$
\begin{array}{lll}
(A\circ B)_{12} = \mathbf{A}\mathbf{R}\mathbf{B}\mathbf{R}^{-1}, & (A\circ B)_{13} = \mathbf{R}'(A\circ B)_{12}\mathbf{R}', & (A\circ B)_{23} = \mathbf{R}\mathbf{R}'(A\circ B)_{12}\mathbf{R}'\mathbf{R}, \\
(A)_1 = \mathbf{A}, & (A)_2 = \mathbf{R}\mathbf{A}\mathbf{R}, & (A)_3 = \mathbf{R}'\mathbf{R}\mathbf{A}\mathbf{R}\mathbf{R}'.
\end{array}
$$

Here, as usual, $\mathbf{B} \equiv B_1 \equiv B \otimes I$. The lower indices in thise notation originate from the analogy with the classical case ($q = 1$), where $(A\circ B)_{12} = A_1B_2$, $(A\circ B)_{13} = A_1B_3$ etc.

We choose the following basic set of cubic matrix combinations:

$$
(\tilde{\Omega}^2\circ\tilde{\Omega})_{ij}\,,\ (\tilde{\Omega}\circ\tilde{\Omega}^2)_{ij}\,,\ \omega(\tilde{\Omega}\circ\tilde{\Omega})_{ij}\,,\ (\tilde{\Omega}^3)_i\,,\ \omega(\tilde{\Omega}^2)_i\,, \tag{3.39}
$$

where $i < j$ and $i,j = 1,2,3$. We also imply that these basic combinations can be multiplied from the left by any matrix function $f(\mathbf{R},\mathbf{R}')$, but expressions produced from the combinations (3.39) by multiplication from the right are to be ordered yet.

Now in the quantum case we order monomials $(\mathbf{R}\tilde{\Omega})^2\omega$ and $(\mathbf{R}'\mathbf{R}\tilde{\Omega})^3$ in the following way:

$$
\left\{
\begin{array}{lcl}
(\mathbf{R}\tilde{\Omega})^2\omega & \to & -\omega\tilde{\Omega}\mathbf{R}\tilde{\Omega}\mathbf{R}^{-1} + \dots \\
(\mathbf{R}'\mathbf{R}\tilde{\Omega})^3 & \to & -\tilde{\Omega}\mathbf{R}\mathbf{R}'\tilde{\Omega}\mathbf{R}\mathbf{R}'^{-1}\tilde{\Omega}\mathbf{R}^{-1}\mathbf{R}'^{-1} + \dots,
\end{array}
\right. \tag{3.40}
$$

where by dots we denote some additional terms to be expressed in terms of the basic combinations (3.39). The point is that such an ordering can be performed in two different ways, depending on whether we first permute the left pair of the generators, or the right one. According to the condition (D) both results must be identical, i.e. the additional terms in (3.40) calculated in two ways must coincide, otherwise the ordered cubic monomials would not be linearly independent. Checking this condition for the combination $(\mathbf{R}\tilde{\Omega})^2\omega$ we get the following relation:

$$
\begin{array}{l}
\sigma\left[(\tilde{\Omega}\circ\tilde{\Omega}^2)_{12} + (\tilde{\Omega}\circ\tilde{\Omega}^2)_{21} - \rho((\tilde{\Omega}^2\circ\tilde{\Omega})_{12} + (\tilde{\Omega}^2\circ\tilde{\Omega})_{21})\right] = \\
\sigma\bar{x}(\mathbf{R})\left[(1-\rho)(\delta + q^N\mathbf{R}^2)((\tilde{\Omega}^3)_1 + (\tilde{\Omega}^3)_2) + \tau\omega((\tilde{\Omega}^2)_1 + (\tilde{\Omega}^2)_2)\right]\,,
\end{array} \tag{3.41}
$$

where

$$
(\tilde{\Omega}^2\circ\tilde{\Omega})_{21} \equiv \mathbf{R}\tilde{\Omega}^2\mathbf{R}\tilde{\Omega},\quad (\tilde{\Omega}\circ\tilde{\Omega}^2)_{21} \equiv \mathbf{R}\tilde{\Omega}\mathbf{R}\tilde{\Omega}^2 \tag{3.42}
$$

are the combinations to be expressed in terms of the basic ones (3.39). In doing so one can start with the relation

$$
\mathbf{R}\tilde{\Omega}^2\mathbf{R}\tilde{\Omega} - \tilde{\Omega}\mathbf{R}\tilde{\Omega}^2\mathbf{R} = \mathbf{R}(\tilde{\Omega}\Delta_+ - \Delta_+\tilde{\Omega})\mathbf{R} \tag{3.43}
$$

that directly follows from (3.28). Here $\Delta_+$ is the shorthand notation for the r.h.s. of (3.28). Omitting the straightforward but rather tedious calculations we present the 'ordered' expressions for $(\tilde{\Omega}^2\circ\tilde{\Omega})_{21}$ and $(\tilde{\Omega}\circ\tilde{\Omega}^2)_{21}$ in Appendix A (see (A.4)-(A.6)). Substituting (A.4),(A.5) into (3.41) and carefully considering the conditions



for vanishing consequently $\omega(\tilde{\Omega}\circ\tilde{\Omega})_{12}$, $(\tilde{\Omega}\circ\tilde{\Omega}^2)_{12}$, $(\tilde{\Omega}^2\circ\tilde{\Omega})_{12}$, $\omega(\tilde{\Omega}^2)_{1,2}$ and $(\tilde{\Omega}^3)_{1,2}$ -terms there we conclude that (3.41) is satisfied iff

$$
\begin{array}{ll}
a.)\ \sigma = 0\,; & \\
b.)\ \sigma \neq 0 \text{ and } \tau = 0,\ \rho = 1\,. &
\end{array} \tag{3.44}
$$

Now we repeat these considerations for $(\mathbf{R}'\mathbf{R}\tilde{\Omega})^3$. Performing the ordering of this expression in two different ways we obtain the following condition

$$
\begin{array}{lllll}
\Delta_+\mathbf{R}'\mathbf{R}\tilde{\Omega}\mathbf{R}^{-1}\mathbf{R}'^{-1} & - \ \mathbf{R}'\mathbf{R}\tilde{\Omega}\mathbf{R}\mathbf{R}'\Delta_+ & + \ \mathbf{R}'\Delta_+\mathbf{R}'\mathbf{R}\tilde{\Omega}\mathbf{R}^{-1} & - & \\
\mathbf{R}\tilde{\Omega}\mathbf{R}\mathbf{R}'\Delta_+\mathbf{R}'^{-1} & + \ \mathbf{R}\mathbf{R}'\Delta_+\mathbf{R}'\mathbf{R}\tilde{\Omega} & - \ \tilde{\Omega}\mathbf{R}\mathbf{R}'\Delta_+\mathbf{R}'^{-1}\mathbf{R}^{-1} & = & 0\,.
\end{array} \tag{3.45}
$$

Considering $(\tilde{\Omega}^3)_{1,2,3}$-terms in the decomposition of (3.45) over the basic set (3.39) (here formulae (A.4),(A.5) are to be used) we get the condition on the parameter $\delta$

$$
\delta = -\frac{q^N[N]_q - \lambda}{[N]_q + \lambda q^N} \quad \Leftrightarrow \quad \bar{x}(\mathbf{R})(\delta + q^N\mathbf{R}^2) = \frac{\lambda q^N}{[N]_q + \lambda q^N}\mathbf{1} = \kappa_q\mathbf{1}\,, \tag{3.46}
$$

where $[N]_q \neq -\lambda q^N$ is implied. Further restrictions on possible values of the quantization parameter $q$ follow from the condition of invertibility of the matrix $E(\mathbf{R})$ (see (A.6)), the inverse power of which enters, through the formulae (A.4),(A.5), into all our calculations. These restrictions are

$$
\kappa_q \neq 1\,,\ \ \mathbf{1} + \kappa_q\mathbf{R}^2 \not\sim P^\pm \quad \Leftrightarrow \quad [N]_q \neq 0\,,\ \ [N]_q \neq -\lambda[2]q^{N\pm1}\,.
$$

And finally, analyzing the condition (3.45) for $\omega(\tilde{\Omega}\circ\tilde{\Omega})_{12,13,23}$-terms we obtain further restrictions on parameters for case a.) (3.44):

$$
\begin{array}{lcl}
a1.)\ \sigma & = & 0 \text{ and } \tau = 0\,; \\
a2.)\ \sigma & = & 0 \text{ and } \tau \neq 0,\ \rho = 1\,, \lambda = 0\,.
\end{array}
$$

Checking the remaining terms of Eq. (3.45) doesn't lead to further restrictions.

Thus, we prove the ordering conditions for cubic monomials for the algebras (3.34)-(3.36), (3.38). To conclude the proof of the Theorem, we note that if the ordering condition is checked at a cubic level, then in accordance with the Manin's general remark [7], it automatically follows for all the higher power monomials. Finally, the relation (3.37) follows directly from (A.4), (A.5) under the obtained restrictions on $\rho$, $\tau$, $\sigma$, $\delta$. **Q.E.D.**

In conclusion of the Section we make few comments on the Theorem:

1. The parameters $\sigma \neq 0$ for the type II algebra and $\tau \neq 0$ for the nonstandard classical algebra are unessential. They can be removed from the defining relations by simple rescalings of generators $\omega$ or $\tilde{\Omega}$.

2. Note that reproducing explicit formulas for certain ordering prescriptions from the covariant relations (3.34)-(3.36) and (3.38) one may obtain some additional limitations on the values of the parameters $\rho$, $\sigma$, $\tau$ (e.g. see below the $N=2$ case).

3. One can directly check the requirements (3.44) assuming the following natural condition:
$$Tr_q(\tilde{\Omega}^3) \neq 0 \tag{3.47}$$
(this is true, e.g., for the classical case $q=1$). Then, Eqs.(3.27) lead to the relations
$$\begin{aligned} &[\tilde{\Omega}, \tilde{\Omega} * \tilde{\Omega}] = \tau \left(\sigma\tilde{\Omega}^3 - (1+\rho)\omega\tilde{\Omega}^2\right), \\ &\tilde{\Omega}^2\omega - \rho^2\omega\tilde{\Omega}^2 = \sigma(\rho-1)\tilde{\Omega}^3. \end{aligned} \tag{3.48}$$
Applying the operation $Tr_q(.)$ to them and using (3.47), (3.22) we deduce
$$\sigma(\rho-1) = 0 = \tau\sigma$$
which is equivalent to (3.44).

4. From the relation (3.37) one can deduce that operators $\tilde{\Omega}^2$ commute as generators of the reflection equation algebra (see e.g. [24]):
$$\mathbf{R}\tilde{\Omega}^2\mathbf{R}\tilde{\Omega}^2 = \tilde{\Omega}^2\mathbf{R}\tilde{\Omega}^2\mathbf{R}\,.$$

5. Finally, we present the defining relations for homogeneous ACA's in terms of $\Omega$'s (see (3.18))
$$\mathbf{R\Omega R\Omega + \Omega R\Omega R^{-1}} = \kappa_q\left(\mathbf{R\Omega^2R + \Omega^2}\right) + \left[\begin{array}{ll} \text{type I}: & \frac{(1-\rho)(1-\kappa_q)}{[N]_q}\omega\left(\mathbf{R\Omega R + \Omega}\right) \\ \text{type II}: & \frac{(1-\kappa_q)\sigma}{[N]_q+\sigma}\left(\mathbf{R\Omega^2R + \Omega^2}\right) \end{array}\right]. \tag{3.49}$$
It should be mentioned that the condition $\rho \neq 0$ appears to be important just here, since the relations (3.49) for the type I algebra contain both scalar terms only under this restriction.

# 4 Differential Complexes of Invariant Forms.

As we argued before, among the algebras presented in Theorem 1 there exists the true algebra (or maybe a set of such algebras) of invariant differential forms on $GL_q(N)$. To make the connection with the differential calculi on quantum groups more clear, we shall supply the homogeneous ACA's listed in Theorem 1 with a grade-1 nilpotent operator $d$ of the external derivation. The definition of $d$ must respect the covariance properties (3.1) of Cartan 1-forms, i.e. $d$ must commute



with the adjoint $GL_q(N)$-coaction on $\Omega$. Hence, the following general ansatz is allowed:
$$\left\{\begin{array}{lcl} d\cdot\tilde{\Omega} &=& x\tilde{\Omega}^2 + y\omega\tilde{\Omega} - z\tilde{\Omega}\cdot d \\ d\cdot\omega &=& -t\omega\cdot d\,. \end{array}\right. \tag{4.1}$$
Here $x$, $y$, $z$ and $t$ are some parameters to be fixed below. We stress that the last term in the right hand side of (4.1) defines the deformed version of Leibnitz rules for differential forms. The ordinary Leibnitz rules are restored in the limit $z=t=1$. Note that the different deformations of the Leibnitz rules for the case of quantum groups have been proposed earlier by L.D.Faddeev [25]. For the case of quantum hyperplanes they are considered in [26].

Now it is straightforward to obtain

**Theorem 2:** *Under the restrictions of Theorem 1 there exist two distinct covariant differential complexes for type I algebras, defined by*
$$typeIA: \left\{\begin{array}{lcl} d\cdot\tilde{\Omega} &=& \tilde{\Omega}^2 - \tilde{\Omega}\cdot d, \\ d\cdot\omega &=& -\rho\omega\cdot d; \end{array}\right. \tag{4.2}$$
$$typeIB: \left\{\begin{array}{lcl} d\cdot\tilde{\Omega} &=& \omega\tilde{\Omega} - z\tilde{\Omega}\cdot d, \\ d\cdot\omega &=& -\omega\cdot d. \end{array}\right. \tag{4.3}$$
*The differential complexes for type II and the nonstandard classical algebras are defined uniquely:*
$$typeII: \left\{\begin{array}{lcl} d\cdot\tilde{\Omega} &=& \tilde{\Omega}^2 - \tilde{\Omega}\cdot d, \\ d\cdot\omega &=& -\omega\cdot d; \end{array}\right. \tag{4,4}$$
$$\begin{array}{c} nonstandard \\ classical\ case: \end{array} \left\{\begin{array}{lcl} d\cdot\tilde{\Omega} &=& \omega\tilde{\Omega} - \tilde{\Omega}\cdot d, \\ d\cdot\omega &=& -\omega\cdot d. \end{array}\right. \tag{4.5}$$
*Here all unessential parameters are removed by $\omega$- and $\tilde{\Omega}$-rescalings.*

**Proof:** These restrictions are easily obtained by demanding $d^2=0$ and checking the compatibility of ansatz (4.1) with the algebraic relations (3.34)-(3.36). We would like only to mention that the relation (3.37) plays an important role when elaborating the type I and II cases. **Q.E.D.**

Let us discuss which of the differential complexes listed in Theorem 2 can be treated as $q$-deformations of the complex of right-invariant forms on $GL(N)$. Comparing the formulae (3.34-3.36) and (4.2-4.4) with the conventional classical relations:
$$[\tilde{\Omega}, \omega]_+ = [\tilde{\Omega}_1, \tilde{\Omega}_2]_+ = 0\,, \quad d\cdot\tilde{\Omega} = \tilde{\Omega}^2 - \tilde{\Omega}\cdot d\,, \quad d\cdot\omega = -\omega\cdot d\,, \tag{4.6}$$
we conclude that there are two different possibilities to deform the complex of $GL(N)$-invariant differential forms. The first is realized by the type IA differential complexes with the additional restriction on the parameter $lim_{q\to 1}\rho = 1$. Note

that in this case the Leibnitz rules are deformed under quantization (for $\rho \neq 1$). The second possibility is realized by the type II differential complexes with $lim_{q\to 1}\sigma = 0$. Here the Leibnitz rules take their conventional form. We would like to mention that all the other types of differential complexes listed in Theorem 2 also may be interesting as examples of 'exotic' differential complexes on $GL(N)$ and $GL_q(N)$, but this subject lies beyond the scope of the present paper. [4]

Now let us treat the $SL_q(N)$-case. The $q$-traceless generators $\tilde{\Omega}_{ij}$ can naturally be identified with the $(N^2-1)$-dimensional basis of right-invariant 1-forms on $SL_q(N)$. These generators form a closed algebra under external multiplication given in Theorem 1 (see (3.34)), and, remarkably, the algebra of these generators doesn't contain any random parameters. As Theorem 2 states, the action of the external derivative on these generators can be only defined like in the classical case: $d\cdot\tilde{\Omega} = \tilde{\Omega}^2 - \tilde{\Omega}\cdot d$ (see (4.2), or (4.4)). So, we conclude that the complex of $SL(N)$-invariant differential forms possesses the unique $q$-deformation.

In the classical Lie-group theory the differential complex of invariant forms serves as suitable basis in the whole de-Rahm complex of all the differential forms on the group manifold. So, to get the full differential calculi on the linear quantum groups, we have to supply the algebras obtained with the suitable cross-multiplication rules for $T_{ij}$ and $\Omega_{ij}$, and to define additionally the action of the external derivative on $T_{ij}$. Note that in the Woronowicz's sheme [1] these questions are to be solved in the first place, when constructing the first order differential calculus. Not trying to solve the problem in general we present here one example of such construction, and establish the correspondence between our homogeneos ACA's and the existing examples of $GL_q(N)$-bicovariant differential calculi.

For the matrix group $GL_q$ of a general rank $N$ two versions of differential calculus have been considered. They were obtained first in the local coordinate representation, where the differential algebra is generated by the coordinate functions $T_{ij}$, their differentials $dT_{ij}$, and derivations $D_{ij}$ $\left(\text{means } \frac{\partial}{\partial T_{ji}}\right)$. We present here the full set of relations between such generators:

$$\begin{aligned} \mathbf{R\,T\,T'} &= \mathbf{T\,T'R}, \\ \mathbf{R\,dT\,dT'} &= -\mathbf{dT\,dT'R^{-1}}, \\ \mathbf{R\,dT\,T'} &= \mathbf{T\,dT'R^{-1}}, \end{aligned} \tag{4.7}$$

$$\begin{aligned} \mathbf{R\,D'D} &= \mathbf{D'D\,R}, \\ \mathbf{D\,R\,T} &= 1+\mathbf{T'R^{-1}D'}, \\ \mathbf{D\,R\,dT} &= \mathbf{dT'R^{-1}D'}. \end{aligned} \tag{4.8}$$

Here, as usual, $\mathbf{D} = D\otimes I$, $\mathbf{D'} = I\otimes D$, $\mathbf{dT} = dT\otimes I$, $\mathbf{dT'} = I\otimes dT$. This algebra is checked to possess unique ordering for any quadratic and cubic monomials.

[4]For $N=2$ such 'exotic' complexes have been considered in [16].



The relations (4.7) were obtained in [10] and in the $R$-matrix formulation in [11, 12]. The first two of relations (4.8) have appeared in [14, 15]. Note that the algebra (4.7),(4.8) implies the commutativity of the derivations $D$ and external derivative $d$. The defining relations for the second version of differential calculus can be obtained from (4.7), (4.8) by the symmetry transformation $\mathbf{R}\leftrightarrow\mathbf{R}^{-1}$ of the type (3.33).

The right-invariant 1-forms and vector fields are then constructed as

$$\Omega = dT\cdot T^{-1} \quad ; \quad V = T\cdot D, \tag{4.9}$$

and they possess the following algebra

$$\mathbf{R\Omega RT} = \mathbf{T\Omega'}, \quad \mathbf{RVRT} = \mathbf{TV'}+\mathbf{RT}, \tag{4.10}$$

$$\mathbf{R\Omega R\Omega} = -\mathbf{\Omega R\Omega R^{-1}}, \tag{4.11}$$

$$\mathbf{RVRV} = \mathbf{VRVR}+\mathbf{RV}-\mathbf{VR}, \tag{4.12}$$

$$\mathbf{R\Omega RV} = \mathbf{VR\Omega R^{-1}}+\mathbf{R\Omega}. \tag{4.13}$$

Here Eqs.(4.11) are the commonly used commutation relations for $GL_q(N)$-invariant differential forms (see [12]-[15]). Comparing (4.11) with (3.49) we see that $\Omega$'s (4.9) realize the special case of type II external algebra with $\sigma = -\kappa_q[N]_q$. Eqs.(4.12) are the well known commutation relations for $GL_q(N)$-invariant vector fields [3]-[5], but in a slightly different notation. To obtain these relations in the conventional form, we have to pass to a new basis of generators $\mathbf{Y} = 1-\lambda\mathbf{V}$. In this basis Eqs.(4.13),(4.12) look like

$$\mathbf{RYRY} = \mathbf{YRYR}, \tag{4.14}$$

$$\mathbf{R\Omega RY} = \mathbf{YR\Omega R^{-1}}. \tag{4.15}$$

Note that our commutation relations of $V$'s with $\Omega$'s or $Y$'s (4.13), (4.15) are different from those presented in [13, 15] for invariant 1-forms and Lie derivatives.

The operator of external derivation in (4.7),(4.8) admits the following explicit representation

$$d = Tr_q(\Omega VY^{-1}) \equiv Tr_q(dTD(1-\lambda V)^{-1}), \tag{4.16}$$

which surprisingly differs from the expected formula $Tr_q(dTD) = Tr_q(\Omega V)$. The operator (4.16) satisfies the nilpotence condition and the ordinary Leibnitz rules. The form of relation (4.16) suggests us an idea of changing the definition (4.9) of invariant vector fields. Indeed, consider the new set of generators $U_{ij}$ obtained from the old $V$'s by the nonlinear invertible transformation:

$$U = \frac{V}{I-\lambda V}, \quad V = \frac{U}{I+\lambda U}. \tag{4.17}$$

With a little algebra one can check that the commutation relations (4.10),(4.12), (4.13) can be concisely rewritten in terms of $U$'s

$$\begin{aligned}
\mathbf{R}^{-1}\mathbf{U}\mathbf{R}^{-1}\mathbf{T} &= \mathbf{T}\mathbf{U}' + \mathbf{R}^{-1}\mathbf{T}\,, && (4.18)\\
\mathbf{R}^{-1}\mathbf{U}\mathbf{R}^{-1}\mathbf{U} &= \mathbf{U}\mathbf{R}^{-1}\mathbf{U}\mathbf{R}^{-1} + \mathbf{R}^{-1}\mathbf{U} - \mathbf{U}\mathbf{R}^{-1}\,, && (4.19)\\
\mathbf{R}\Omega\mathbf{R}^{-1}\mathbf{U} &= \mathbf{U}\mathbf{R}\Omega\mathbf{R} + \mathbf{R}\Omega\,. && (4.20)
\end{aligned}$$

Now, if we consider $U_{ij}$ instead of $V_{ij}$ as invariant vector fields on $GL_q(N)$; then the formula for external derivative takes its standard form: $d = Tr_q(\Omega U)$.

Finally, let us consider the simplest case of $GL_q(2)$-covariant differential calculus in more detail. Note that while the proof of Theorem 1 does not work for $N = 2$, the resulting formulae are applicable to this case as well. This can be directly checked by using only the general properties of $\mathbf{R}$-matrix, namely the Yang-Baxter equation, Hecke condition, and the $q$-trace formula. The failure of the general proof of Theorem 1 is due to the different structure of $Ad^{\otimes 2}$ decomposition in the case $N = 2$ and is not fatal.

Denote the components of matrix $\Omega$ as $\begin{pmatrix} \theta_1 & \theta_2 \\ \theta_3 & \theta_4 \end{pmatrix}$. Then from the covariant expressions (3.34-3.36) the following explicit ordering prescriptions can be extracted:

$$\begin{aligned}
\textit{type I}:\quad \theta_2^2 &= \theta_3^2 = 0\,, \qquad \theta_3\theta_2 = -\theta_2\theta_3\,,\\
\theta_1^2 &= \frac{1}{q^{-2}+\rho}\{q\lambda\rho\theta_2\theta_3 + (\rho-1)\theta_1\theta_4\}\,,\\
\theta_4^2 &= \frac{1}{q^{-2}+\rho}\{q^{-3}\lambda\theta_2\theta_3 - q^{-2}(\rho-1)\theta_1\theta_4\}\,,\\
\theta_4\theta_1 &= -\frac{1}{q^{-2}+\rho}\{(1+q^{-2}\rho)\theta_1\theta_4 + q^{-1}\lambda(1+\rho)\theta_2\theta_3\}\,,\\
\theta_3\theta_1 &= \frac{1}{1+\rho}\{-\rho(1+q^2)\theta_1\theta_3 + (\rho-q^2)\theta_3\theta_4\}\,,\\
\theta_4\theta_3 &= \frac{1}{1+\rho}\{-(1+q^{-2})\theta_3\theta_4 + (1-q^{-2}\rho)\theta_1\theta_3\}\,,\\
\theta_2\theta_1 &= \frac{1}{q^{-2}+q^2\rho}\{-(1+q^{-2})\rho\theta_1\theta_2 + (q^2\rho-1)\theta_2\theta_4\}\,,\\
\theta_4\theta_2 &= \frac{1}{q^{-2}+q^2\rho}\{-(1+q^2)\theta_2\theta_4 + (q^{-2}-\rho)\theta_1\theta_2\}\,;
\end{aligned} \qquad (4.21)$$



$$\begin{aligned}
\textit{type II}:\quad \theta_2^2 &= \theta_3^2 = 0\,, \qquad \theta_3\theta_2 = -\theta_2\theta_3\,,\\
\theta_1^2 &= \frac{\mu}{1-\mu}\theta_2\theta_3\,, \qquad \theta_4^2 = \frac{1-q^{-2}-\mu}{1-\mu}\theta_2\theta_3\,,\\
\theta_4\theta_1 &= -\theta_1\theta_4 + \lambda\frac{(q+q^{-1})\mu-q}{1-\mu}\theta_2\theta_3\,,\\
\theta_3\theta_1 &= -\frac{1}{1-\mu(1+q^{-2})}\left\{(1-\mu)\theta_1\theta_3 + \frac{\mu}{q^2}\theta_3\theta_4\right\}\,,\\
\theta_4\theta_3 &= -\frac{1}{1-\mu(1+q^{-2})}\left\{\frac{1-\mu}{q^2}\theta_3\theta_4 + (\mu-1+q^{-2})\theta_1\theta_3\right\}\,,\\
\theta_2\theta_1 &= -(1-\mu)\theta_1\theta_2 + \mu\theta_2\theta_4\,,\\
\theta_4\theta_2 &= -q^2(1-\mu)\theta_2\theta_4 + q^2(\mu-1+q^{-2})\theta_1\theta_2\,.
\end{aligned} \qquad (4.22)$$

Here we use the parameter $\mu = \frac{\sigma+[N]_q\kappa_q}{\sigma+[N]_q}\,|_{N=2}$ instead of $\sigma$ for convenience. In this notation the case (4.11) corresponds to $\mu = 0$. Obvious restrictions $\mu \neq \{1, (1+q^{-2})^{-1}\}$ and $\rho \neq \{-1, -q^{-2}, -q^{-4}\}$ arise when passing from covariant relations to formulation in components (see remark 2 to Theorem 1).

Let us compare these results with those presented for the $GL_{p,q}(2)$ case in Ref. [16]. First, we note that by assumption the left-invariant 1-forms in [16] admit the decomposition $\Omega = T^{-1}\cdot dT$, and the external derivative satisfies the undeformed version of Leibnitz rules. Hence, the formula $d\cdot\Omega = -\Omega^2 - \Omega\cdot d$ is postulated. Moreover, the relation $d\Omega \sim [\omega,\Omega]_+$ is also implied. Hence, the differential calculi obtained in [16] must be of the second type. Indeed the relations (8.5) of [16] can be transformed into the form (4.22) if we note that due to the conditions (6.25) [16] the parameter N (see (8.1-3) [16] ) obeys the following quadratic relation:

$$(1+r^2)\,(2+\mathrm{N}\,(1+r-(1+r^2)s)) = (1+r+r\mathrm{N})^2\,. \qquad (4.23)$$

Here $r = pq$ (see Eq. (8.5) [16]) is the only combination of deformation parameters that enters into the external algebra of invariant forms. This is not surprising since the $GL_{p,q}(2)$ $\mathbf{R}$-matrix, when suitably normalized, satisfies the Hecke relation

$$\mathbf{R}^2 = \mathbf{1} + (r^{\frac{1}{2}} - r^{-\frac{1}{2}})\mathbf{R}\,. \qquad (4.24)$$

Hence, we expect that the parameter $r^{\frac{1}{2}}$ of [16] corresponds to our $q^{-1}$ ( the inverse power here is due to the substitution $q \leftrightarrow q^{-1}$ that should be done to pass from the right-invariant forms of Eq. (4.22) to the left-invariant ones).

The variable $s$ of (4.23) parametrizes different external algebras in [16] and it should correspond to our $\mu$. Actually, using (4.23) it is strightforward to check that Eqs. (8.5) of [16] are equivalent to (4.22) with the following substitutions to be made:

$$r \leftrightarrow q^{-2} \quad , \quad \frac{r^{-1} - 1 - rN}{r + r^{-1}} \leftrightarrow \mu \, .$$

Summarizing all the above, we conclude that our type II differential complexes, on the one hand, generalize formulae given in [16] to arbitrary $N$ and, on the other hand, include the bicovariant calculi considered in [10]-[15].

# 5 Conclusion

Here we make some comments on constructing the differential calculi for type II complexes (3.34),(3.36),(4.4), and briefly discuss the problem of $SL_q(N)$-reduction of the $GL_q(N)$- differential calculi.

Since all the type II differential complexes are isomorphic (see the first remark to Theorem 1), we expect that $\sigma = -\kappa_q[N]_q$ differential calculus (4.7),(4.8) can be transformed to the case of any $\sigma$. To realize this transformation, we consider the new set of generators $\{T_{ij}^{(g)}\}$ of the algebra $Fun(GL_q(N))$

$$T_{ij}^{(g)} = g(z) T_{ij} \, , \tag{5.1}$$

where $z = det_q(T)$ and $g(z)$ is an arbitrary function of $z$. It is clear that $det_q(T^{(g)}) = z g(z)^N$, and therefore the choice $g(z) = z^{-1/N}$ leads to the $SL_q(N)$-case of [13, 15]. Using commutation relations (4.7) and (4.10) one can deduce

$$z\, dT = q^2 dT\, z \, , \quad \lambda q^N dT = [T, \, \omega] \Rightarrow \lambda q^N dg(z) = \omega(g(q^2 z) - g(z)) \, . \tag{5.2}$$

Now we introduce new Cartan 1-forms $\Omega^g = dT^{(g)}(T^{(g)})^{-1}$ related to the old ones via the following formulae

$$\Omega^g = g dT T^{-1} g^{-1} + dg \cdot g^{-1} = \Omega G(z) + \omega \frac{G(z) - 1}{\lambda q^N} \, . \tag{5.3}$$

Here $G(z) = g(q^2 z) g^{-1}(z)$. Note that eqs.(4.7) and (4.11) give the following formulae

$$\lambda q^N d\Omega = -\{\Omega, \omega\} = \lambda q^N \Omega^2 \tag{5.4}$$

Using these equations and relations (4.7) and (4.11) we obtain the set of $GL_q(N)$-differential calculi parametrized by the function $G(z)$:

$$\begin{gathered}
\mathbf{R}\mathbf{T}^{(g)}\mathbf{T}^{(g)\prime} = \mathbf{T}^{(g)}\mathbf{T}^{(g)\prime}\mathbf{R} \, , \\
\mathbf{R}\Omega^g\mathbf{R}\Omega^g + \Omega^g\mathbf{R}\Omega^g\mathbf{R}^{-1} = \mathbf{R}U^g\mathbf{R} + U^g \, , \\
\mathbf{T}^{(g)}\Omega^{g\prime} = \mathbf{R}\Omega^g\mathbf{R}\mathbf{T}^{(g)}G(z) + \Omega^g\mathbf{T}^{(g)}(G(z) - 1) + \\
\omega^{(g)}(1 - \mathbf{R}^2 G(z))\mathbf{T}^{(g)} \left( [N]_q + \tfrac{\lambda q^N G(z)}{G(z) - 1} \right)^{-1} ,
\end{gathered} \tag{5.5}$$



where

$$U^g = (\Omega^g)^2 (1 - G(z)) + \omega \Omega^g \frac{G(q^2 z) - G(z)}{\lambda q^N} \, , \tag{5.6}$$

$$\omega^{(g)} = Tr_q \Omega^g = \omega \left( G(z) + \frac{[N]_q (G(z) - 1)}{\lambda q^N} \right) \tag{5.7}$$

Let us consider the case when the function $G(z)$ is a constant. For example for $g(z) = z^\alpha$ we have $G(z) = q^{2\alpha}$ and Eqs.(5.5) give the one-parametric set of differential calculi which can be naturally related to the set of type II differential complexes (3.49):

$$\begin{gathered}
\mathbf{R}\mathbf{T}^{(\alpha)}\mathbf{T}^{(\alpha)\prime} = \mathbf{T}^{(\alpha)}\mathbf{T}^{(\alpha)\prime}\mathbf{R} \, , \\
\mathbf{R}\Omega^{(\alpha)}\mathbf{R}\Omega^{(\alpha)} + \Omega^{(\alpha)}\mathbf{R}\Omega^{(\alpha)}\mathbf{R}^{-1} = \mu_\alpha(\mathbf{R}(\Omega^{(\alpha)})^2\mathbf{R} + (\Omega^{(\alpha)})^2) \, , \\
\mathbf{T}^{(\alpha)}\Omega^{(\alpha)\prime} = \mathbf{R}\Omega^{(\alpha)}\mathbf{R}\mathbf{T}^{(\alpha)}(1 - \mu_\alpha) - \Omega^{(\alpha)}\mathbf{T}^{(\alpha)}\mu_\alpha - \\
\left( \tfrac{\mu_\alpha}{\xi(\mu_\alpha)} \right) \mathbf{R}(1 + \tfrac{\mu_\alpha}{\lambda}\mathbf{R})\omega^{(\alpha)}\mathbf{T}^{(\alpha)} ,
\end{gathered} \tag{5.8}$$

where $\mu_\alpha = 1 - q^{2\alpha}$, $\xi(\mu_\alpha) = q^N(1 - \mu_\alpha) - \frac{\mu_\alpha}{\lambda}[N]_q$ and

$$\omega^{(\alpha)} = Tr_q \Omega^{(\alpha)} = q^{-N} \xi(\mu_\alpha) \omega \, . \tag{5.9}$$

Now let us explore the possibilities of $SL_q(N)$-reduction of these calculi. First, if we put $\alpha = -1/N$ (as it was done in Refs.[13, 15]), then in the commutation relations (5.8) we have the unavoidable additional 1-form generator $\omega^{(-1/N)}$ and, thus, the number of Cartan's 1-forms is $N^2$ but not $N^2 - 1$ as in the undeformed case of $SL(N)$. Second, one could try to put $\omega^{(\alpha)}$ to zero choosing parameters $\alpha$ and $q$ as

$$q^{-\alpha} \xi(\mu_\alpha) = q^{\alpha + N} + [\alpha]_q [N]_q = 0 \, . \tag{5.10}$$

In particular, this equation is fulfilled for $q$ being a root of unity: $q^{-2N} = q^{\pm 2} = q^{2\alpha}$, which doesn't contradict the condition $\alpha = -1/N$. However, for the case of (5.10), in the third equation of (5.8) we have $\frac{0}{0}$-indefiniteness, that is solved as

$$q^{N - \alpha} \omega^{(\alpha)} \left( q^{\alpha + N} + [\alpha]_q [N]_q \right)^{-1} = \omega \, , \tag{5.11}$$

and we cannot put it to zero having in mind that $\lambda q^N dT^{(\alpha)} = [T^{(\alpha)}, \, \omega]$ and $[det_q T^{(\alpha)}, \, \omega] \neq 0$. Therefore, the differential calculi (5.8) do not admit the correct $SL_q(N)$-reduction even for special values of the quantization parameter $q$.

Now, how one may hope to construct the consistent bicovariant differential calculus on $SL_q(N)$? The nice way of making the reduction from the $GL_q(N)$-case doesn't work for type II differential calculi (5.8). May be, the cross-multiplication presented in (5.8) is not the unique possibility of construsting the differential calculi starting with the type II complexes. May be, the difficulties will be overcome if we use the type IA complexes instead of the type II ones. But here for $\rho \neq 1$ we meet serious problems when constructing the local coordinate representation of the type $\Omega = dT \cdot T^{-1}$. So, only the type IA differential complex with $\rho = 1$ seems to be a good candidate for the construction of consistent differential calculus on $GL_q(N)$ with its possible reduction to $SL_q(N)$. We hope to revert to these problems in further publications.


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# A Appendix

Here we present the 'ordered' expressions for quadratic combinations $(\tilde{\Omega} \circ \tilde{\Omega}^2)_{21}$, $(\tilde{\Omega}^2 \circ \tilde{\Omega})_{21}$ (see (3.42)), and collect some formulae used in the derivation of these expressions.

Consider the sequence $x_i$ defined iteratively

$$x_0 = q^N + q^{-N}, \quad x_1 = [N]_q + \lambda q^N ; \quad x_{i+2} = x_i + \lambda x_{i+1} . \tag{A.1}$$

Define

$$y_i = (-1)^i \frac{x_{1-i}x_1 - x_{-i}x_2}{|x|} , \quad y_{i,k} = (-1)^{i+k} \frac{x_{1-i-k}x_{1-k} - x_{-i-k}x_{2-k}}{|x|} , \tag{A.2}$$

where $|x| \equiv [N+2]_q[N-2]_q$. It is strightforward to show that $y_{i,k} = y_i$ for any $i$ and $k$, and $y_i$ are calculated by the following simple iteration:

$$y_0 = 1 , \quad y_1 = \lambda , \quad y_{i+2} = y_i + \lambda y_{i+1} . \tag{A.3}$$

When simplfying the final expressions for $(\tilde{\Omega} \circ \tilde{\Omega}^2)_{21}$ and $(\tilde{\Omega}^2 \circ \tilde{\Omega})_{21}$ we use the following properties of the matrix functions $x(\mathbf{R})$, $\bar{x}(\mathbf{R})$:

$$\mathbf{R}^k x(\mathbf{R}) = x_k \mathbf{1} + x_{k+1}\mathbf{R} , \quad \mathbf{R}^k \bar{x}(\mathbf{R}) = \frac{(-1)^k}{|x|}(-x_{2-k}\mathbf{1} + x_{1-k}\mathbf{R}) ,$$
$$\mathbf{R}^k x_i \bar{x}(\mathbf{R}) = (-1)^{i+k}\left(y_{-i-k}\mathbf{1} - x_{1-k}\mathbf{R}^{1-i}\bar{x}(\mathbf{R})\right) ,$$



together with (A.1)-(A.3). The result is

$$\begin{aligned}
(\tilde{\Omega}^2 \circ \tilde{\Omega})_{21} = E^{-1}(\mathbf{R}) \Big[ & \epsilon(\mathbf{R}) \left\{ 1 + \tfrac{x(\mathbf{R})}{|x|}\delta(\mathbf{R}) \right\} (\tilde{\Omega} \circ \tilde{\Omega}^2)_{12} \\
& + \mathbf{R} \left\{ \epsilon(\mathbf{R}) \left( \lambda - \tfrac{\delta x_{-1} + q^N x_{-3}}{|x|} \right) + \tfrac{\tau\sigma}{|x|} \left( [N]_q + \mathbf{R}\delta(\mathbf{R}) \right) \right\} (\tilde{\Omega}^2 \circ \tilde{\Omega})_{12} \\
& - \tau \mathbf{R}^2 \epsilon(\mathbf{R}) \left\{ \bar{x}(\mathbf{R}) + \rho \tfrac{x(\mathbf{R})}{|x|} \mathbf{R}^{-2} \right\} \omega (\tilde{\Omega} \circ \tilde{\Omega})_{12} \, . \\
& + \left\{ \tfrac{\delta x_1 + q^N x_{-1}}{|x|} \mathbf{R}^2 (1 + \mathbf{R}^2 \bar{x}(\mathbf{R})) + \tfrac{\tau\sigma}{|x|} \mathbf{R} \left( [N]_q + \lambda q^N (1 + \mathbf{R}^2) + (1 + \lambda^2) \mathbf{R}\delta(\mathbf{R}) \right) \right\} (\tilde{\Omega}^3)_1 \\
& - \left\{ \tfrac{\delta x_1 + q^N x_{-1}}{|x|} \mathbf{R} (1 + \mathbf{R}^2 \bar{x}(\mathbf{R})) + \tfrac{\tau\sigma}{|x|} \left( q^N (1 + \mathbf{R}^2) + \lambda \mathbf{R}\delta(\mathbf{R}) \right) \right\} (\tilde{\Omega}^3)_2 \\
& + \left\{ \tau \bar{x}(\mathbf{R}) \epsilon(\mathbf{R}) \left( \lambda \mathbf{R} - \tfrac{\delta x_0 + q^N x_{-2}}{|x|} \right) - \tfrac{\tau\rho x_3}{|x|} \mathbf{R} \epsilon(\mathbf{R}) + \lambda q^N \tfrac{\tau\rho}{|x|} \mathbf{R}^{-1} F(\mathbf{R}) \right\} \omega(\tilde{\Omega}^2)_1 \\
& + \left\{ -\tau \bar{x}(\mathbf{R}) \epsilon(\mathbf{R}) \left( 1 + \tfrac{\delta x_0 + q^N x_{-2}}{|x|} \right) + \tfrac{\tau\rho x_2}{|x|} \epsilon(\mathbf{R}) + \lambda q^N \tfrac{\tau\rho}{|x|} \mathbf{R}^{-1} F(\mathbf{R}) \right\} \omega(\tilde{\Omega}^2)_2 \Big] ,
\end{aligned} \tag{A.4}$$

$$\begin{aligned}
(\tilde{\Omega} \circ \tilde{\Omega}^2)_{21} = E^{-1}(\mathbf{R}) \Big[ & \left\{ \mathbf{R} \epsilon(\mathbf{R}) \tfrac{\delta x_1 + q^N x_{-1}}{|x|} - \tfrac{\tau\sigma}{|x|} \left( \lambda q^N \mathbf{R}^3 - x_{-1}\mathbf{R} + \mathbf{R}^2 \delta(\mathbf{R}) \right) \right\} (\tilde{\Omega} \circ \tilde{\Omega}^2)_{12} \\
& + \Big\{ \mathbf{R}^2 \epsilon(\mathbf{R}) \left( \mathbf{1} - \bar{x}(\mathbf{R})\delta(\mathbf{R}) + \tau\sigma \tfrac{x_{-2} + x_0}{|x|} \right) + \tau\sigma (\mathbf{R}^4 + \mathbf{R}^6) \delta(\mathbf{R}) \bar{x}^2(\mathbf{R}) \\
& - \tfrac{(\tau\sigma)^2}{|x|} \mathbf{R}^2 \Big\} (\tilde{\Omega}^2 \circ \tilde{\Omega})_{12} - \tau \mathbf{R}^2 \left( \bar{x}(\mathbf{R}) + \rho \tfrac{x(\mathbf{R})}{|x|} \mathbf{R}^{-2} \right) \Big( \epsilon(\mathbf{R}) + \tau\sigma \mathbf{R}^2 \bar{x}(\mathbf{R}) \Big) \omega (\tilde{\Omega} \circ \tilde{\Omega})_{12} \\
& + \left\{ \tfrac{\delta x_1 + q^N x_{-1}}{|x|} \mathbf{R}^3 G(\mathbf{R}) + \tau\sigma \mathbf{R}^2 \left( \tfrac{x_2}{|x|} + \tfrac{x_3}{|x|} \mathbf{R} \bar{x}(\mathbf{R}) \delta(\mathbf{R}) \right) - \tfrac{(\tau\sigma)^2}{|x|} (1 + \lambda^2) \mathbf{R}^2 \right\} (\tilde{\Omega}^3)_1 \\
& - \left\{ \tfrac{\delta x_1 + q^N x_{-1}}{|x|} \mathbf{R} G(\mathbf{R}) + \tau\sigma \mathbf{R} \left( \tfrac{x_1}{|x|} + \tfrac{x_2}{|x|} \mathbf{R} \bar{x}(\mathbf{R}) \delta(\mathbf{R}) \right) - \lambda \mathbf{R} \tfrac{(\tau\sigma)^2}{|x|} \right\} (\tilde{\Omega}^3)_2 \\
& + \Big\{ \tfrac{\tau\rho}{|x|} \mathbf{R}^2 \Big( \epsilon(\mathbf{R}) (-x_2 + (1 + \lambda^2 - x_3 \bar{x}(\mathbf{R})\mathbf{R}) \delta(\mathbf{R})) + \tau\sigma (1 + \lambda^2) \Big) \\
& + \tau \bar{x}(\mathbf{R}) \left( \epsilon(\mathbf{R}) \left( \tau\sigma \tfrac{x_0}{|x|} + \mathbf{R}^2 \right) - \lambda\tau\sigma \mathbf{R}^3 \bar{x}(\mathbf{R}) \right) \Big\} \omega(\tilde{\Omega}^2)_1 \\
& + \Big\{ \tfrac{\tau\rho}{|x|} \Big( \mathbf{R} \epsilon(\mathbf{R}) (x_1 + (x_2 \mathbf{R} \bar{x}(\mathbf{R}) - \lambda) \delta(\mathbf{R})) - \lambda\tau\sigma \mathbf{R} \Big) \\
& + \tau^2 \sigma \bar{x}(\mathbf{R}) \left( \tfrac{x_0}{|x|} \epsilon(\mathbf{R}) + \mathbf{R}^2 \bar{x}(\mathbf{R}) \right) \Big\} \omega(\tilde{\Omega}^2)_2 \Big] .
\end{aligned} \tag{A.5}$$

where

$$\delta(\mathbf{R}) = \delta + q^N \mathbf{R}^2 , \quad \epsilon(\mathbf{R}) = 1 + \mathbf{R}^2 \bar{x}(\mathbf{R}) \delta(\mathbf{R}) ,$$

$$F(\mathbf{R}) = 1 + \mathbf{R}^{-4} \frac{x(\mathbf{R})}{|x|} \delta(\mathbf{R}) , \quad G(\mathbf{R}) = \epsilon(\mathbf{R}) + \tau\sigma \mathbf{R}^2 \bar{x}(\mathbf{R}) ,$$

$$E(\mathbf{R}) = \left(1 + \frac{\delta x_0 + q^N x_{-2}}{|x|}\right) \epsilon(\mathbf{R}) + \frac{\tau\sigma}{|x|} (q^{-N} + q^N \mathbf{R}^2) . \qquad (A.6)$$

Note that these expressions are significantly simplified under the restriction $\tau\sigma = 0$.

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Ковариантные дифференциальные комплексы на квантовых линейных группах

Рассматриваются возможные конструкции внешних алгебр для картановских форм на $GL_q(N)$ и $SL_q(N)$. Исходными пунктами анализа являются положения о том, что

1) 1-формы реализуют присоединенное представление квантовой группы;

2) составленные из 1-форм мономы допускают единственное упорядочение.

Для полученных внешних алгебр определено дифференциальное отображение, характеризуемое стандартным свойством нильпотентности, и, вообще говоря, деформированным правилом Лейбница. Обсуждается место известных примеров дифференциальных исчислений на $GL_q(N)$ в предлагаемой классификационной схеме, а также проблемы $SL_q(N)$-редукции.

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Covariant Differential Complexes on Quantum Linear Groups

We consider the possible covariant external algebra structures for Cartan's 1-forms ($\Omega$) on $GL_q(N)$ and $SL_q(N)$. Our starting point is that $\Omega$ s realize an adjoint representation of quantum group and all monomials of $\Omega$ s possess the unique ordering. For the obtained external algebras we define the differential mapping $d$ possessing the usual nilpotence condition, and the generally deformed version of Leibnitz rules. The status of the known examples of $GL_q(N)$-differential calculi in the proposed classification scheme and the problems of $SL_q(N)$-reduction are discussed.

The investigation has been performed at the Bogoliubov Laboratory of Theoretical Physics, JINR.

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