1 Introduction

Recently, in a number of papers (cf. [1]–[5]) the exact evaluation of the phase space path integrals was studied. The (formal) path integral generalization [1] of the Duistermaat-Heckman localization formula (DH-formula) [6], [7] was used for this purpose . Due to this formula if M is the compact $2N$-dimensional symplectic manifold with the symplectic structure $\omega = \frac{1}{2\pi}\omega_{ij}dx^i \wedge dx^j$ and Hamiltonian H define the circle action on M then

$$\int_M e^H(\omega)^N = \sum_{dH=0} \frac{e^H\sqrt{det\omega_{ij}}}{\sqrt{det\frac{\partial^2 H}{\partial x^i \partial x^j}}}. \qquad (1.1)$$

This formula was applied at first to the evaluation of path integral in Ref.[8] where Dirac's operator index was calculated.

This approach turns out to be convenient for a large class of problems [2]-[5] including conceptually new, geometric interpretation of supersymmetric theories [4] and revision of two-dimensional Yang-Mills theory [5].

The derivation of (1.1) can be formulated in terms of supersymmetry which corresponds to the *equivariant transformations* [7]. Namely, the r. h. s. of (1.1) can be presented in the form

$$Z_0 = \frac{1}{(2\pi)^N}\int_M e^{H(x)}\sqrt{det\omega_{ij}}d^{2N}x = \frac{1}{(\pi)^N}\int_{\mathcal{M}} \exp(H(x) + \frac{1}{2}\omega_{ij}\theta^i\theta^j)d^{2N}xd^{2N}\theta, \qquad (1.2)$$

where θ^i are auxiliary Grassmannian fields corresponded to the basic 1-forms dx^i, $\mathcal{M}$ is the supermanifold associated with the tangent bundle of M. With respect to $\theta^i \leftrightarrow dx^i$, to any differential form $\omega^k = \omega^k_{i_1 \dots i_k}dx^{i_1}\wedge \dots \wedge dx^{i_k}$ on M the monomial $\tilde{\omega}^k = \omega^k_{i_1\dots i_k}\theta^{i_1}\dots\theta^{i_k}$ on $\mathcal{M}$ is corresponded.

Since H defines on M a circle action one can construct the metric g_{ij} and the odd function ("gauge fermion")

$$\tilde{Q}_H = \xi^i_H g_{ij}\theta^j, \qquad (1.3)$$

which are both Lie-invariant with respect to ξ^i_H, where

$$\xi^i_H \equiv \frac{\partial H}{\partial x^j}\omega^{ji}. \qquad (1.4)$$

Then, both integrals: (1.2) and

$$Z_\lambda = \int_{\mathcal{M}} \exp(H + \frac{1}{2}\omega_{ij}\theta^i\theta^j - \lambda\hat{d}_H\tilde{Q}_H)d^{2N}d^{2N}\theta, \qquad (1.5)$$

(where $\hat{d}_H = \hat{d} + \hat{\imath}_H$ is equivariant differential along ξ^i_H, and λ is a numerical parameter) are invariant under supersymmetry transformations generated by the equivariant differential $\hat{d}_H = \hat{d} + \hat{\imath}_H$:

$$\delta x^i = \theta^i, \quad \delta\theta^i = \xi^i_H. \qquad (1.6)$$

Moreover, the integral (1.5) is λ –independent.

Taking the limits $\lambda \to 0$ and $\lambda \to \infty$ we obtain the standard DH-formula (1.1).

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EQUIVARIANT LOCALIZATION: BV-GEOMETRY AND SUPERSYMMETRIC DYNAMICS

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where θ^i are auxiliary Grassmannian fields corresponded to the basic 1-forms dx^i, $\mathcal{M}$ is the supermanifold associated with the tangent bundle of M. With respect to $\theta^i \leftrightarrow dx^i$, to any differential form $\omega^k = \omega^k_{i_1 \ldots i_k}dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ on M the monomial $\tilde{\omega}^k = \omega^k_{i_1 \ldots i_k}\theta^{i_1}\ldots\theta^{i_k}$ on $\mathcal{M}$ is corresponded.

Since H defines on M a circle action one can construct the metric g_{ij} and the odd function ("gauge fermion")

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(where $\hat{d}_H = \hat{d} + \hat{\imath}_H$ is equivariant differential along ξ^i_H, and λ is a numerical parameter) are invariant under supersymmetry transformations generated by the equivariant differential $\hat{d}_H = \hat{d} + \hat{\imath}_H$:

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Moreover, the integral (1.5) is λ-independent.

Taking the limits $\lambda \to 0$ and $\lambda \to \infty$ we obtain the standard DH-formula (1.1).

By this reason, further, we will call by DH-supermanifold the one associated with tangent bundle of a compact manifold which is provided by Hamiltonian dynamics and by Riemannian structure, Lie-invariant with respect to this dynamics.

We will denote by $\mathcal{M}$ a supermanifold associated with the tangent bundle of symplectic manifold M.

Since any function on $\mathcal{M}$ can be interpreted in terms of differential forms on M one can formulate the exterior differential calculus on M in terms of functions on $\mathcal{M}$.

The derivation of DH-formula (1.1) presented above was used in Ref.[1]–[3] for the path integral generalization of DH-formula. For this purpose, the formalism developed by E. Gozzi et al. for the path integral formulation of classical mechanics [9] is used. In this formalism the canonical symplectic structure constructed on the cotangent bundle of $\mathcal{M}$ is used.

However, there is the alternative possibility of Hamiltonian description of DH-localization which allows to avoid the introduction of the structure of cotangent bundle of $\mathcal{M}$ by using the *odd symplectic structure* constructed on $\mathcal{M}$ [10].

In this work we will present this approach more completely.

We will show that $\mathcal{M}$ can be provided with the basic objects of BV-formalism [11], [12]: the odd symplectic structure Ω_1 and the nilpotent operator Δ.

Using the symplectic structure ω on M we define the natural lift of an arbitrary Hamiltonian mechanics (H,ω,M) on the certain odd one $(Q_H,\Omega_1,\mathcal{M})$ which is defined by the odd function $\dot{Q}_H$ and the odd symplectic structure Ω_1 on $\mathcal{M}$. This odd mechanics defines the Lie derivatives of differential forms along ξ^i_H. Moreover, it is invariant under supersymmetry transformations (1.6) which are generated on the odd symplectic structure by the initial Hamiltonian H and by the symplectic structure ω. If $\mathcal{M}$ is the DH-supermanifold then the "gauge fermion" (1.3) plays the role of the additional (odd) motion integral of $(Q_H,\Omega_1,\mathcal{M})$.

Hence this odd Hamiltonian mechanics gives the natural description of the symmetries of the integrals (1.2), (1.5) and, therefore, of the DH-localization.

The existence of Riemannian metric on M allow us to construct on $\mathcal{M}$ the *even symplectic structure*. We show that if $\mathcal{M}$ is DH-supermanifold then one can construct the second, *even Hamiltonian structure* for $(Q_H,\Omega_1,\mathcal{M})$.

It is interesting to point out that such bi-Hamiltonian systems (with even and odd symplectic structures) were studied earlier without any connection with DH-formula. The example of such a system (one-dimensional supersymmetric Witten's mechanics) was present at first by D. V. Volkov et al. [13]. Later such systems were studied in more details in Ref.[14]–[18]. Particularly, in Ref. [14] such structure was studied for the supersymmetric integrable mechanics in terms of "action-angle" variables. In Ref.[15] there is the proof, that only a finite number of such bi-Hamiltonian systems for fixed even and odd symplectic structures exist. The bi-Hamiltonian systems with even and odd symplectic structures on the superspaces were studied in Ref.[16], and on Kählerian supermanifolds in Ref.[17]. In Ref.[18] the procedure of Hamiltonian reduction of such systems to complex projective superspaces was considered.

Let us cite also the Ref.[19] where it is shown that the odd symplectic structure has nontrivial geometrical properties with no analogues in the even symplectic structure.

But the odd symplectic structure (with the connected object– operator Δ) is well-

known in physics as the basic object of the covariant quantization formalism for the theories with arbitrary constraints– Batalin-Vilkovisky formalism (BV-formalism) [11], [12]. BV-formalism is considered now as the background in the construction of invariant string field theory (cf.[20]). This is the stimulative factor in the investigation of the geometry of BV-formalism and the odd symplectic structure [21].

However, the odd symplectic structure is used in BV–formalism for the formulation of so-called "master equation" which is *kinematic* one.

As we noted above, the equivariant localization gives the example of applying of the odd symplectic structure for the formulation of *dynamical* (Hamiltonian) equations (the existence of Hamiltonian systems with odd symplectic structure was first pointed out in [22]).

The paper is organized as follows.

In *Section 2* we present the basic properties of the odd symplectic structure and operator Δ and construct theirs on the supermanifold $\mathcal{M}$ associated with the tangent bundle of the symplectic manifold M.

In *Section 3* we map (lift) the arbitrary Hamiltonian mechanics (H,ω,M) on the odd supersymmetric Hamiltonian mechanics $(Q_H,\Omega_1,\mathcal{M})$ and interpret its supersymmetry in terms of equivariant transformations. We demonstrate that if $\mathcal{M}$ is DH-supermanifold then one can define the even Hamiltonian structure on it determining the same dynamics as $(Q_H,\Omega_1,\mathcal{M})$.

In *Section 4* using the results of the previous sections we repeat the derivation of DH-localization formula.

For rigorous definitions and conventions in on Hamiltonian mechanics and symplectic geometry used here we refer to Ref.[23], and Ref.[24] on supergeometry and superanalyses

2 BV-Geometry

Odd Poisson bracket (antibracket) of functions $f(z)$ and $g(z)$ on the supermanifold is the bi-linear differential operation

$$\{f,g\}_1 = \frac{\partial_r f}{\partial z^A}\Omega_1^{AB}(x)\frac{\partial_l g}{\partial z^B}\quad, \tag{2.1}$$

which satisfy the conditions

$$p(\{f,g\}_1) = p(f) + p(g) + 1 \quad \text{(grading condition)},$$

$$\{f,g\}_1 = -(-1)^{(p(f)+1)(p(g)+1)}\{g,f\}_1 \quad \text{("antisymmetricity")}, \tag{2.2}$$

$$(-1)^{(p(f)+1)(p(h)+1)}\{f,\{g,h\}_1\}_1 + \text{cycl.perm.(f, g, h)} = 0 \quad \text{(Jacobi id.)}, \tag{2.3}$$

where z^A are the local coordinates, and $\frac{\partial_r}{\partial z^A}$ and $\frac{\partial_l}{\partial z^A}$ denotes correspondingly right and left derivatives.

On the supermanifold with an equal number of even and odd coordinates the odd bracket can be nondegenerate one. Then one can corresponds to it the odd symplectic structure

$$\Omega_1 = dz^A\Omega_{(1)AB}dz^B \tag{2.4}$$

where $\Omega_{(1)AB}\Omega_1^{BC} = \delta_A^C$. Locally, the nondegenerate odd Poisson bracket can be transformed to the canonical form [22]:

$$\{f,g\}_1^{\rm can} = \sum_{i=1}^{N}\left(\frac{\partial_r f}{\partial x^i}\frac{\partial_l g}{\partial \theta_i} - \frac{\partial_r f}{\partial \theta_i}\frac{\partial_l g}{\partial x^i}\right), \tag{2.5}$$

where $p(\theta_i) = p(x^i) + 1$.

It was shown in Ref.[11], [19] that the odd Poisson bracket has not invariant volume forms. Hence, if the supermanifold is provided with the odd bracket (2.1) and with the volume form

$$dv = e^{\rho(z)} d^{2N} z \quad , \tag{2.6}$$

where e^{ρ}– some integral density, then one can define on it the odd differential operator of the second order, so called "operator Δ" which is invariant under the transformations conserving the symplectic structure and the volume form [15]. Its action on the function $f(x,\theta)$ is the divergency of the Hamiltonian vector field $\mathrm{D}_f = \{\ \ ,f\}_1$ with the volume form (2.6):

$$\Delta_\rho f = \frac{1}{2} div_\rho \mathrm{D}_f \equiv \frac{1}{2}\frac{\mathcal{L}_{\mathrm{D}_f} dv}{dv}, \tag{2.7}$$

where $\mathcal{L}_{\mathrm{D}_f}$ — Lie derivative along D_f , or in the coordinate form

$$\Delta_\rho f = \frac{1}{2}\frac{\partial^R}{\partial z^A}\left(\{z^A, f\}_1\right) + \frac{1}{2}\{\rho, f\}_1 \equiv \Delta_0 + \frac{1}{2}\{\rho, f\}_1. \tag{2.8}$$

The operator Δ which is used in Batalin-Vilkovisky formalism obeys to the nilpotency condition

$$\Delta_\rho^2 = 0, \tag{2.9}$$

which holds if the density $\rho(z)$ satisfy some conditions. In more details the properties of Δ are considered in Ref. [21].

It is known that any supermanifold one can associate with some vector bundle [24]. On the supermanifold associated with the cotangent bundle of any manifold it can be constructed the odd symplectic structure [22].

Indeed, let T^*M be the cotangent bundle to the manifold M, x^i are the local coordinates on M and (x^i, θ_i) are the corresponding local coordinates on T^*M with the transition functions

$$\tilde{x}^i = \tilde{x}^i(x), \quad \tilde{\theta}_i = \sum_{i=1}^{N}\frac{\partial x^j}{\partial \tilde{x}^i}\theta_j. \tag{2.10}$$

Considering for every map the superalgebra generated by (x^i, θ_i) where x^i are even and θ_i are odd, transforming from map to map like (x^i, θ_i) in the (2.10) we go to supermanifold $\mathcal{M}$ which is associated to T^*M in the coordinates (x^i, θ_i) . Obviously, on this supermanifold in coordinates (x^i, θ_i) one can globally define the canonical odd symplectic structure with the canonical odd bracket (2.5) . (By the same way one can define an odd Poisson bracket on the supermanifold associated with cotangent bundle of a *supermanifold.*)

Let us construct an odd symplectic structure and the operator Δ on the supermanifold $\mathcal{M}$ associated with the tangent bundle of the symplectic manifold M.

Let

$$\{f(x), g(x)\} = \frac{\partial f}{\partial x^i}\omega^{ij}\frac{\partial g}{\partial x^j} \tag{2.11}$$

is the nondegenerate Poisson bracket on M.

Then the function

$$F(z) = \frac{1}{2}\theta_i\omega^{ij}\theta_j = -\frac{1}{2}\theta^i\omega_{ij}\theta^j \tag{2.12}$$

is globally defined on $\mathcal{M}$ and satisfy the condition

$$\{F,F\}_1 = 0. \tag{2.13}$$

Using this function one can map any function $f(x)$ on M to $\mathcal{M}$:

$$f(x) \to Q_f(z) = \{f(x), F(x,\theta)\}_1 \quad . \tag{2.14}$$

This map has the following important property

$$\{f(x), g(x)\} = \{f(x), Q_g(x,\theta)\}_1 \quad \text{for any} \quad f(x), g(x). \tag{2.15}$$

Then doing the coordinate transformation

$$(x^i, \theta_i) \to (x^i, \theta^i = \{x^i, F(z)\}_1) \ , \tag{2.16}$$

From (2.16) we see that θ^i transforms like dx^i, then it can be interpreted as the basis of 1-form on M. Hence, $z^A = (x^i, \theta^i)$ plays the role of the local coordinates of the supermanifold associated with tangent bundle of the symplectic manifold M.

An odd symplectic structure in these coordinates takes the form

$$\Omega_1 = \omega_{ij} dx^i \wedge d\theta^j + \omega_{ij,k}\theta^j dx^k \wedge dx^j. \tag{2.17}$$

The corresponding odd Poisson bracket is defined by following basic relations:

$$\{x^i, x^j\}_1 = 0, \quad \{x^i, \theta^j\}_1 = \omega^{ij}, \quad \{\theta^i, \theta^j\}_1 = \frac{\partial \omega^{ij}}{\partial x^k}\theta^k, \tag{2.18}$$

where ω^{ij} is the matrix of the even Poisson bracket (2.11) on M. The operator Δ connected with it takes the form

$$\Delta_\rho = \omega^{ij}\frac{\partial^2}{\partial x^i \partial \theta^j} + \frac{1}{2}\omega^{ij}_{,k}\theta^k\frac{\partial^2}{\partial \theta^i \partial \theta^j} + \frac{1}{2}\{\rho, \ \ \}_1. \tag{2.19}$$

It is easy to check that if

$$\rho = kF \tag{2.20}$$

(k is an arbitrary numerical constant) then this operator is nilpotent one:

$$\Delta_{kF}^2 = 0. \tag{2.21}$$

Thus, we provide the supermanifold $\mathcal{M}$ associated with tangent bundle of symplectic manifold M with the geometrical structures of Batalin-Vilkovisky formalism. As we noticed in Introduction, such supermanifold presents in the Duistermaat-Heckman localization procedure.

In the next Section we will study the lift (2.14) of the Hamiltonian mechanics on the symplectic manifold.

3 Bi-Hamiltonian Dynamics

Let the symplectic manifold M is provided with the Hamiltonian mechanics $(H(x), \omega, M)$, where the symplectic structure ω defines the nondegenerate Poisson bracket (2.11) on M and $H(x)$ is the Hamiltonian on it.

Using (2.14), (2.15) let us map this mechanics on $(Q_H, \Omega_1, \mathcal{M})$, where

$$Q_H = \{H, F\}_1 \tag{3.1}$$

is the odd Hamiltonian on $\mathcal{M}$ (the mapping of Hamiltonian H on $\mathcal{M}$), and Ω_1 is the odd symplectic structure (2.17) on it which defines the antibracket (2.18).

It is easy to see that the equations of motion of $(Q_H, \Omega_1, \mathcal{M})$ in terms of (x^i, θ^i) takes the form:

$$\frac{dx^i}{dt} = \{x^i, Q_H\}_1 = \xi^i_H, \qquad \frac{d\theta^i}{dt} = \{\theta^i, Q_H\}_1 = \frac{\partial \xi^i_H}{\partial x^j}\theta^j. \tag{3.2}$$

This mechanics is supersymmetric one . Indeed, using (2.13) (3.1) and definition of H we see that H, F, Q_H forms the simplest superalgebra:

$$\begin{aligned} &\{H \pm F, H \pm F\}_1 = \pm 2Q_H, \\ &\{H + F, H - F\}_1 = \{H \pm F, Q_H\}_1 = \{Q_H, Q_H\}_1 = 0 \end{aligned} \tag{3.3}$$

Then, let us interpret it in terms of differential forms on M.

The following correspondence is obvious one:

$$\begin{aligned} \{H, \ \}_1 = \xi^i_H \frac{\partial}{\partial \theta^i} \ &\to \ \hat{\imath}_H - \text{operator of inner product on the vector field } \xi_H, \\ \{F, \ \}_1 = \theta^i \frac{\partial}{\partial x^i} \ &\to \ \hat{d} - \text{operator of exterior differentiation}, \\ \{Q, \ \}_1 = \xi^i_H \frac{\partial}{\partial x^i} + \xi^i_{H,k}\theta^k \frac{\partial}{\partial \theta^i} \ &\to \ \hat{\mathcal{L}}_H - \text{Lie derivative along } \xi_H. \end{aligned} \tag{3.4}$$

Using Jacobi identities (2.3) we obtain

$$\{H, F\}_1 = Q_H \to \hat{d}\hat{\imath}_H + \hat{\imath}_H \hat{d} = \hat{\mathcal{L}}_H - \text{homotopy formula}. \tag{3.5}$$

As we see, the supercharge $H + F$, which defines the supersymmetry transformation (1.6), corresponds to the operator of equivariant differentiation.

Now let us allow that some Riemannian metric g_{ij} is defined on M and ξ^i_H is its Killing vector.

Then the "gauge fermion" (1.3) is the motion integral of the odd mechanics $(Q_H, \Omega_1, \mathcal{M})$:

$$\mathcal{L}_H g = 0 - \text{Killing equation} \to \{Q_H, \tilde{Q}_H\}_1 = 0. \tag{3.6}$$

The functions F and H commute with $\tilde{Q}$ by the following way:

$$\{F, \tilde{Q}_H\}_1 = -F_2, \quad \{H, Q_H\}_1 = H_2 \ , \tag{3.7}$$

where

$$H_2 = \xi^i_H g_{ij} \xi^j_H, \quad F_2 = \frac{1}{2}\theta^i \omega_{(2)ij}\theta^j, \quad \omega_{(2)ij} = \frac{\partial (g_{ik}\xi^k_H)}{\partial x^j} - \frac{\partial (g_{jk}\xi^k_H)}{\partial x^i}. \tag{3.8}$$

It is easy to check up that the mechanics (H, ω, M) and (H_2, ω_2, M) define the same Hamiltonian vector field on M (it was shown at first in Ref. [3]):

$$\frac{\partial H}{\partial x^i}\omega^{ij} = \frac{\partial H_2}{\partial x^i}\omega^{ij}_2. \tag{3.9}$$

Obviously, one can separate two different cases:

i) The case, when these Hamiltonian systems coincides with each other (up to a constant multiplier):

$$H = H_2, \quad \omega = \omega_2. \tag{3.10}$$

Then $H, F, Q_H, \tilde{Q}_H$ form the closed superalgebra

$$\begin{aligned} &\{H \pm F, H \pm F\}_1 = \pm 2Q_H, \{H \pm F, \tilde{Q}_H\}_1 = H \mp F, \\ &\{H + F, H - F\}_1 = \{H \pm F, Q_H\}_1 = \{Q_H, \tilde{Q}_H\}_1 = 0. \end{aligned} \tag{3.11}$$

It coincides with the superalgebra of 1D Witten's mechanics.

In the case of path integral generalization (see below) this corresponds to the topological field theory [2].

ii) These Hamiltonian systems are different ones:

$$H \neq H_2, \quad \omega \neq \omega_2. \tag{3.12}$$

Then (H_2, ω_2, M) defines the second Hamiltonian structure.

If the Poisson brackets corresponded to ω and ω_2 satisfy the compatibility condition (e. i. if any linear combination of that satisfy the Jacobi identity) then the initial mechanics (H, ω, M) is integrable one.

Due to M provided with both symplectic and Riemannian structures we can define on the supermanifold $\mathcal{M}$ the *even symplectic structure.* For this purpose let us consider on $\mathcal{M}$ the following local 1-form:

$$\mathcal{A}_\alpha = A_{(\alpha)i}dx^i + \theta^i g_{ij} D\theta^j. \tag{3.13}$$

Here $A_\alpha = A_{(\alpha)i}dx^i$ is local (pre)symplectic 1-form on M:

$$dA_\alpha = \omega_\alpha,$$

and

$$D\theta^i = d\theta^i + \Gamma^i_{kl}\theta^k dx^l,$$

where Γ^i_{kl} are Cristoffel symbols for metrics g_{ij} on M).

It is easy to see that the exterior differential of this 1-form is *globally* defined on $\mathcal{M}$ and represents the following symplectic structure:

$$\tilde{\Omega}_\alpha = d\mathcal{A}_\alpha = \frac{1}{2}(\omega_{(\alpha)ij} + R_{ijkl}\theta^k\theta^l)dx^i \wedge dx^j + g_{ij}D\theta^i \wedge D\theta^j, \tag{3.14}$$

where R_{ijkl}–curvature tensor on M (of course, this is not the unique form of an even symplectic structure on $\mathcal{M}$).

The Poisson bracket corresponded to this structure is the following one:

$$\{f(z),g(z)\}_\alpha = \nabla_i f(z)(\omega_{(\alpha)ij} + R_{ijkl}\theta^k\theta^l)^{-1}\nabla_j g(z) + \frac{1}{2}\frac{\partial_r f(z)}{\partial\theta^i}g^{ij}\frac{\partial_l g(z)}{\partial\theta^j}, \qquad (3.15)$$

where $g^{ik}g_{kj} = \delta^i_j$,

$$\nabla_i = \frac{\partial}{\partial x^i} - \Gamma^k_{ij}\theta^j\frac{\partial_l}{\partial\theta^k}.$$

Let us assume in (3.13)-(3.15) : $\alpha = 0,2,\ \omega_0 \equiv \omega,\quad H_0 \equiv H$.

Then it is easy to see that $(\mathcal{H}_\alpha, \Omega_\alpha, \mathcal{M})$ and $(Q_H, \Omega_1, \mathcal{M})$ are defining the same Hamiltonian dynamics on $\mathcal{M}$:

$$\{z^A, \mathcal{H}_\alpha\}_\alpha = \{z^A, Q_H\}_1, \quad \text{where} \quad \mathcal{H}_\alpha = H_\alpha + F_2, \qquad \alpha = 0,2\,. \qquad (3.16)$$

Thus, we have shown that on the DH-supermanifold there exists the exotic structure of bi-Hamiltonian dynamics with even and odd symplectic structures. Such dynamics has been considered in Refs.[13]-[18].

Without loss of generality, we can choose in presented constructions the metric g_{ij} by such a way that $\omega^{ij}g_{jk} = I^i_k$ defines an almost complex structure ($I^i_l I^l_k = -\delta^i_k$) which is Lie-invariant with respect to ξ^i_H . In the case, if $\hat{I}$ is the complex structure, the (super)manifolds M and $\mathcal{M}$ are Kählerian ones. Moreover, $\mathcal{M}$ is provided by both even and odd Kählerian structures. Bi-Hamiltonian dynamics with even and odd symplectic structures and BV-structures on such supermanifolds have been considered in [17],[18].

It is obvious that such dynamics includes the integrable systems on the orbits of coadjoint representation of semisimple Lie groups (which are Kählerian manifolds).

4 Equivariant Localization

Now we demonstrate the derivation of DH-formula (1.1) using the constructions presented above.

We can present the integral (1.5) in the form

$$Z_\lambda = \frac{1}{(\pi)^N}\int_{\mathcal{M}} \exp\,(H - F - \lambda\{H + F, \tilde{Q}\}_1)d^{4N}z, \qquad (4.1)$$

where F and $\tilde{Q}$ are defined by the expressions (2.12) , (1.3) and $\{\ ,\ \}_1$ by (2.18).

The vector fields (3.4) conserve the "volume form" $d^{4N}z = d^{2N}x d^{2N}\theta$ (or, equivalently, $\Delta_0 H = \Delta_0 F = \Delta_0 Q_H = 0$). From (3.4), (3.6) we deduce

$$\{H + F, e^{(H-F-\lambda\{H+F,\tilde{Q}\})_1}\}_1 = 0, \quad \{Q, e^{(H-F-\lambda\{H+F,\tilde{Q}\}_1)}\}_1 = 0.$$

Therefore, the integral (4.1) is invariant under equivariant and Lie transformations along ξ^i_H. Using the definition of operator Δ (2.7) we can present this fact in the form:

$$\Delta_{H-F}(H + F) = \Delta_{H-F}Q = 0. \qquad (4.2)$$

We have also

$$\{Q, \tilde{Q}e^{(H-F-\lambda\{H+F,\tilde{Q}\}_1)}\}_1 = 0.$$

Using these expressions and the fact that the integral of an equivariantly exact form vanishes on the compact manifold we show :

$$\begin{aligned}\frac{dZ_\lambda}{d\lambda} &= -\frac{\lambda}{\pi^N}\int_{\mathcal{M}}\{H + F, \tilde{Q}\}_1 e^{(H-F-\lambda\{H+F,\tilde{Q}\}_1)}d^{4N}z = \\ &= -\frac{\lambda}{\pi^N}\int_{\mathcal{M}}\{H + F, \tilde{Q}e^{(H-F-\lambda\{H+F,\tilde{Q}\}_1)}\}_1 d^{4N}z + \\ &+ \frac{\lambda}{\pi^N}\int_{\mathcal{M}}\tilde{Q}\{H + F, e^{(H-F-\lambda\{H+F,\tilde{Q}\}_1)}\}_1 d^{4N}z = 0.\end{aligned}$$

Thus,taking the limits $\lambda \to 0$, $\lambda \to \infty$ and due to

$$\delta(\xi^i_H) = \frac{1}{\pi^N}\lim_{\lambda\to\infty}\sqrt{\lambda^{2N}det g_{ij}}\,e^{-\lambda\xi^i_H g_{ij}\xi^j_H},$$

we obtain the DH-localization formula:

$$\begin{aligned}Z_0 &= \frac{1}{(2\pi)^N}\int_M e^H\sqrt{det\omega_{ij}}d^{2N}x = \lim_{\alpha\to\infty}\frac{1}{\pi^N}\int_{\mathcal{M}} e^{(H-F-\lambda(H_2-F_2))}d^{4N}z = \\ &= \int_M e^H\delta(\xi_H)\sqrt{\det\omega_{ij}}\sqrt{\det\frac{\partial\xi^i_H}{\partial x^j}}d^{2N}x. \qquad (4.3)\end{aligned}$$

The path integral generalization of the presented constructions one can accomplish by the lifting it on the loop space, similarly to [1]-[3] .For this, we have to consider the integral in (1.2) and (4.1) as the path integral over loop space $L\mathcal{M}$ with the boundary conditions on z^A: $z^A(0) = z^A(T)$ and replace the Hamiltonian H by action S and lift on the loop space the constructions obtained in previous Sections:

$$\begin{aligned}H &\to iS = i\int_0^T A_i dx^i - H dt, \text{where} \quad dA = \omega, \\ F(z) &\to F^L(z(t)) = -\int_0^T dt\omega_{ij}(x(t))\theta^i(t)\theta^j(t), \\ g_{ij}(x)dx^i dx^j &\to \int_0^T dt g_{ij}(x(t))dx^i(t)dx^j(t), \\ \{f(z), g(z)\}_1 &\to \{f(z(t)), g(z(t))\}^L_1 = \int_0^T dt\frac{\delta_r f(z(t))}{\delta z^A}\Omega_{(1)AB}(z(t))\frac{\delta_l g(z(t))}{\delta z^A},\end{aligned}$$

where $\Omega_{(1)AB}(z(t))$ is defined by (2.18).

Then, we get

$$\begin{aligned}\xi^i_H &\to \xi^i_S = \{x^i(t), S\}^L_1 = \dot{x}^i - \xi^i_H, \\ Q &\to Q_S = \{S, F^L\}^L_1 = \int_0^T dt\xi^i_S\omega_{ij}(x(t))\theta^j(t), \\ \tilde{Q} &\to \tilde{Q}_S = \int_0^T dt\xi^i_S(x(t))g_{ij}(x(t))\theta^i(t),\end{aligned}$$

and the argument of δ-function in the last integral in (4.3) changes from ξ^i_H to ξ^i_S, i. e. the path integral localizes to the ordinary integral over the classical phase space.

5 Conclusion

We have shown that the supermanifold of Duistermaat-Heckman localization can be provided with :

1. the structures of Batalin-Vilkovisky formalism, namely, with odd symplectic structure and nilpotent operator Δ;

2. the structure of supersymmetric bi-Hamiltonian dynamics with even and odd symplectic structures.

These structures allow one to describe the equvariant localization by Hamiltonian way without introducing the additional structure of cotangent bundle of supermanifold (associated with tangent bundle of the initial manifold), and thus - without introducing the additional $4N$ variables.

Using the first structure one can consider the derivation of the degenerate DH-formula (including path integral generalized case) via BV-formalism .

The second structure gives the example of dynamical application of the odd symplectic structure. It establishes the correspondence between initial dynamics and its supersymmetrization and gives also the supersymmetrization way for the wide class of integrable systems. One can consider this correspondence in a quantum level too, taking into account that replacing in path integral generalization of (4.1)

$$\{S - F^L, \tilde{Q}_S\}_1^L \rightarrow \{H - F, \tilde{Q}_S\}_1^L$$

one obtains the partition function of the supersymmetric dynamics, considered above (the case (3.10), if $\lambda \neq 0, \infty$.

Let us notice that the representation of the initial integral in the form (1.2) formally coincides to the form of the integral from differential forms on a *supermanifold* [24] and the presented description is a symmetrical one according to the initial and auxiliary coordinates. Thus, it can be generalized for the super-Hamiltonian systems.

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