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ON THE HOPF ALGEBRAS GENERATED
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1. Following the approaches of refs. [1] and [2] we proposed in [3] a recipe for constructing quantum doubles (quasitriangular Hopf algebras of the special type [4, 5]) associated with invertible solutions of the quantum Yang-Baxter equations (QYBE). Let us briefly recall this procedure.

It is known [1] that any invertible solution R of QYBE

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1}$$

naturally generates a bialgebra $\mathcal{T}$ with generators $\{1, t^i_j\}$ and relations

$$R_{12}T_1T_2 = T_2T_1R_{12}, \quad \Delta(T) = T \otimes T, \quad \varepsilon(T) = 1 \tag{2}$$

(t_{ij} form a matrix T, Δ is a coproduct and ε a counit). We can now define an analogous bialgebra $\mathcal{U} = \{1, u^i_j\}$ by

$$R_{12}U_1U_2 = U_2U_1R_{12}, \quad \Delta(U) = U \otimes U, \quad \varepsilon(U) = 1, \tag{3}$$

and introduce a pairing between these two,

$$< U_1, T_2 > = R_{12}, \tag{4}$$

as a bilinear map $< \cdot, \cdot >: \mathcal{U} \otimes \mathcal{T} \to \mathcal{K}$ into the underlying field $\mathcal{K}$. The pairing (4) proves to be consistent with the bialgebra structure [3] but is, as a rule, degenerate. Removing the degeneracy by factoring out so-called null bi-ideals [2] allows us to introduce antipodes by the relations

$$< S(U_1), T_2 > = < U_1, S^{-1}(T_2) > = R_{12}^{-1}, \tag{5}$$

and then establish the quantum-double structure on $\mathcal{T} \otimes \mathcal{U}$ using the original Drinfeld recipe [4, 5]
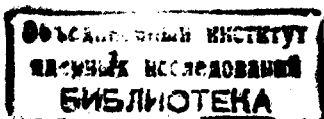
$$\alpha b = \sum \sum < S(\alpha_{(1)}), b_{(1)} > < \alpha_{(3)}, b_{(3)} > b_{(2)}\alpha_{(2)}, \tag{6}$$

where

$$\Delta^2(\alpha) = \sum \alpha_{(1)} \otimes \alpha_{(2)} \otimes \alpha_{(3)}, \quad \Delta^2(b) = \sum b_{(1)} \otimes b_{(2)} \otimes b_{(3)}. \tag{7}$$

In the case (2)-(4) this recipe results in the well known formula

$$R_{12}U_1T_2 = T_2U_1R_{12}. \tag{8}$$

However, it is not very well known that (8) can be interpreted [3] as the quantum-double cross-multiplication condition as well.

In the present paper we develop the method [3] along the following aspects. Firstly, we change the order of certain steps described above: a definition of antipode will now precede the bracketing procedure. This will immediately produce the (R-generated) Hopf algebra, because Reshetikhin's result [6] enables one to introduce invertible antipodes explicitly and so give up implicit definitions (5), where the invertibility of S was not guaranteed. However, after the removal of degeneracy of $< \cdot, \cdot >$ by abovementioned factorization, these two ways lead us to the same quantum double.

Secondly, we now understand why the cross-multiplication relation (8) appears in its final form actually before (and independently of) any factorization. We show that $T \otimes U$ can be provided with the bialgebra (or the Hopf algebra) structure merely due to appropriate features of the pairing, though degenerate.

Therefore, in the present version of the method, the quotienting by null bi-ideals does not look so unpredictably dangerous as it does in [2] and [3]. Now it can at most trivialize the whole output. To show that sometimes it does not, we perform the construction of the quantum double for one of the 4×4 R-matrices listed in [7]. The resulting Hopf algebra is by no means trivial and appears to be quasitriangular. We assume the corresponding universal $\mathcal{R}$-matrix to be a formal power series and evaluate its terms up to the fourth order.

2. Here we are to explain how an antipode can be introduced [6] into the R-generated bialgebra T. For generality, let us consider its inhomogeneous version [3] (cf. [8, 9, 10]) with generators $\{1, t^i_j, E_p\}$ (we prefer to display all the indices):

$$R^{ij}_{mn} t^m_p t^n_q = R^{mn}_{pq} t^j_n t^i_m, \quad E_p t^j_q = R^{mn}_{pq} t^j_n E_m,$$
$$\Delta(t^i_j) = t^i_k \otimes t^k_j, \qquad \varepsilon(t^i_j) = \delta^i_j,$$
$$\Delta(E_j) = E_i \otimes t^i_j + 1 \otimes E_j, \qquad \varepsilon(E_j) = 0. \tag{9}$$

R-matrix is a solution of QYBE (1):

$$R^{ij}_{lm} R^{lk}_{pn} R^{mn}_{qr} = R^{jk}_{lm} R^{im}_{nr} R^{nl}_{pq}. \tag{10}$$

Now let us extend this bialgebra by the inverse elements $\bar{t}^i_j$ (overlining a quantity will always mean its inverse):

$$t^i_k \bar{t}^k_j = \bar{t}^i_k t^k_j = \delta^i_j. \tag{11}$$

As a consequence, the following relations are to be added to (9):

$$R^{ij}_{mn} \bar{t}^n_p \bar{t}^m_q = R^{mn}_{qp} \bar{t}^i_m \bar{t}^j_n, \qquad R^{im}_{nq} \bar{t}^j_m t^n_p = R^{mj}_{pn} t^i_m \bar{t}^n_q, \tag{12}$$

$$\bar{t}^i_q E_p = R^{mj}_{pn} E_m \bar{t}^n_q, \quad \Delta(\bar{t}^i_j) = \bar{t}^k_j \otimes \bar{t}^i_k, \quad \varepsilon(\bar{t}^i_j) = \delta^i_j. \tag{13}$$

Further, assume that R admits not only an inverse matrix $\bar{R}$,

$$R^{ij}_{mn} \bar{R}^{mn}_{pq} = \bar{R}^{ij}_{mn} R^{mn}_{pq} = \delta^i_p \delta^j_q, \tag{14}$$

but also the matrices $\tilde{R}$ and $\bar{\tilde{R}}$ ('twisted inverses' for R and $\bar{R}$, respectively):

$$R^{mj}_{pn} \tilde{R}^{in}_{mq} = \tilde{R}^{mj}_{pn} R^{in}_{mq} = \delta^i_p \delta^j_q, \quad \bar{R}^{mj}_{pn} \bar{\tilde{R}}^{in}_{mq} = \bar{\tilde{R}}^{mj}_{pn} \bar{R}^{in}_{mq} = \delta^i_p \delta^j_q. \tag{15}$$

Let us define the tensors

$$\Omega^i_j \equiv \tilde{R}^{mi}_{jm}, \qquad \bar{\Omega}^i_j \equiv \bar{\tilde{R}}^{mi}_{jm}, \tag{16}$$

which are inverse to each other,

$$\bar{\Omega}^i_k \Omega^k_j = \delta^i_j. \tag{17}$$

This can be easily seen from

$$\tilde{R}^{jk}_{ls} \bar{\tilde{R}}^{il}_{tq} = \bar{R}^{pr}_{ts} R^{ik}_{lm} \bar{\tilde{R}}^{lj}_{pn} \tilde{R}^{nm}_{qr},$$

which, in turn, is a direct consequence of QYBE (10).

In terms of Ω and $\bar{\Omega}$ one can define [6] an antipode

$$S(t^i_j) = \bar{t}^i_j, \quad S(\bar{t}^i_j) = \Omega^i_m t^m_n \bar{\Omega}^n_j, \quad S(E_i) = -E_j \bar{t}^j_i \tag{18}$$

and its inverse

$$\bar{S}(t^i_j) = \bar{\Omega}^i_m \bar{t}^m_n \Omega^n_j, \quad \bar{S}(\bar{t}^i_j) = t^i_j, \quad \bar{S}(E_i) = -\bar{S}(t^j_i)E_j. \tag{19}$$

To confirm the correctness of this definition one can use the following relations:

$$\Omega^n_m t^m_j \bar{t}^i_n = \Omega^i_j, \qquad \bar{\Omega}^n_m \bar{t}^m_j t^i_n = \bar{\Omega}^i_j. \tag{20}$$

For example,

$$S(t^i_k \bar{t}^k_j) = S(\bar{t}^k_j)S(t^i_k) = \Omega^k_m t^m_n \bar{\Omega}^n_j \bar{t}^i_k = \bar{\Omega}^n_j \Omega^i_n = \delta^i_j.$$

Then the cross-multiplication rule (cf. (6))

$$\alpha b = \sum \sum \ll \alpha_{(1)}, b_{(1)} \gg < \alpha_{(3)}, b_{(3)} > b_{(2)}\alpha_{(2)} \tag{34}$$

establishes the bialgebra structure on $\mathcal{A} \otimes \mathcal{B}$.

In (32) $\tilde{1}$ is the unit of $\mathcal{B}$, and $\ll_1 <_2$ in (33) indicates that $\ll \cdot, \cdot \gg$-operation deals with the left multipliers in tensor products; whereas $< \cdot, \cdot >$ with the right ones.

Proof. Fix the bases $\{e_i\}$ in $\mathcal{A}$ and $\{e^j\}$ in $\mathcal{B}$. Denoting the structure constants and the pairing tensors (which are in general degenerate) as follows,

$$e_i\, e_j = c_{ij}^k\, e_k, \quad \Delta(e_i) = f_i^{jk}(e_j \otimes e_k), \quad \varepsilon(e_i) = \varepsilon_i, \quad 1 = E^i e_i,$$

$$e^i e^j = \tilde{f}_k^{ij} e^k, \quad \Delta(e^i) = \tilde{c}_{jk}^i(e^k \otimes e^j), \quad \varepsilon(e^i) = \tilde{E}^i, \quad \tilde{1} = \tilde{\varepsilon}_i\, e^i, \tag{35}$$

$$< e^i, e_j > = \eta_j^i, \qquad \ll e^i, e_j \gg = \chi_j^i,$$

we may list the relations between them which are caused by the bialgebra structure of $\mathcal{A}$ and $\mathcal{B}$,

$$c_{ij}^k\, c_{kn}^m = c_{ik}^m\, c_{jn}^k, \quad c_{ij}^k E^j = c_{ji}^k E^j = \delta_i^k, \quad c_{ij}^k \varepsilon_k = \varepsilon_i\, \varepsilon_j,$$

$$f_n^{ij} f_m^{nk} = f_m^{in} f_n^{jk}, \quad f_i^{jk}\varepsilon_k = f_i^{kj}\varepsilon_k = \delta_i^j, \quad f_i^{jk} E^i = E^j E^k, \tag{36}$$

$$c_{ij}^k\, f_k^{rs} = f_i^{mn} f_j^{pq}\, c_{mp}^r\, c_{nq}^s, \quad E^i \varepsilon_i = 1$$

(the same for quantities with a tilde), by the properties (32),

$$\eta_k^m \tilde{f}_m^{ij} = \eta_m^i \eta_n^j f_k^{mn}, \quad \eta_m^i c_{jk}^m = \eta_j^m \eta_k^n \tilde{c}_{mn}^i, \quad \chi_k^m \tilde{f}_m^{ij} = \chi_m^i \chi_n^j f_k^{nm}, \tag{37}$$

$$\chi_m^i c_{jk}^m = \chi_j^m \chi_k^n \tilde{c}_{nm}^i, \quad \eta_j^i E^j = \chi_j^i E^j = \tilde{E}^i, \quad \eta_j^i \tilde{\varepsilon}_i = \chi_j^i \tilde{\varepsilon}_i = \varepsilon_j, \tag{38}$$

and by the relation (33),

$$\tilde{c}_{mn}^i \eta_q^m \chi_p^n f_j^{pq} = \tilde{c}_{mn}^i \eta_p^n \chi_q^m f_j^{pq} = \tilde{E}^i \varepsilon_j. \tag{39}$$

Now (34) reads

$$e^i e_j = \mathcal{P}_{jq}^{ip}\, e_p e^q, \qquad \mathcal{P}_{jq}^{ip} \equiv \eta_n^m\, \tilde{c}_{mq}^t\, \tilde{c}_{ts}^i\, \chi_r^s\, f_j^{rl} f_l^{pn}. \tag{40}$$

To be convinced that this makes $\mathcal{A} \otimes \mathcal{B}$ a bialgebra, we should verify that the transition from, say, $e^i e_j e_k$ to $e_t e^n$ can be equally well performed in two different ways, which requires

$$\mathcal{P}_{jq}^{ip}\, \mathcal{P}_{kn}^{qm}\, c_{pm}^t = c_{jk}^p\, \mathcal{P}_{pn}^{it}. \tag{41}$$

Analogously, $e^i e^j e_k \to e_t e^n$ implies

$$\mathcal{P}_{kq}^{jp}\, \mathcal{P}_{pm}^{it}\, \tilde{f}_n^{mq} = \tilde{f}_p^{ij}\, \mathcal{P}_{kn}^{pt}, \tag{42}$$

and, at last, $\Delta(e^i e_j) = \Delta(e^i)\Delta(e_j)$ means

$$\tilde{c}_{mn}^i f_j^{pq}\, \mathcal{P}_{pl}^{nk}\, \mathcal{P}_{qs}^{mr} = \mathcal{P}_{jq}^{ip}\, \tilde{c}_{sl}^{q'}\, f_p^{kr}. \tag{43}$$

All the conditions (41)-(43) are verified by direct, though tedious, calculations with repeated use of (36)-(39). For example, when proving (41) or (42), the $cf = ffcc$ relation from (36) is applied twice and $cc = cc$ (or $ff = ff$) many times, whereas in the case of (43) the key property is (39) accompanied by numerous applications of $cc = cc$ and $ff = ff$.

A minor problem is caused by checking the conditions

$$E^j \mathcal{P}_{jq}^{ip} = E^p \delta_q^i, \quad \tilde{\varepsilon}_i \mathcal{P}_{jq}^{ip} = \tilde{\varepsilon}_q \delta_j^p, \quad \varepsilon_p\, \tilde{E}^q\, \mathcal{P}_{jq}^{ip} = \tilde{E}^i \varepsilon_j, \tag{44}$$

which reflect the properties of unit and counit. Proposition 1 is proved.

It is worth noting an alternative form of (40),

$$\mathcal{E}_{in}^{mj}\, e^i e_j = \mathcal{F}_{in}^{mj}\, e_j e^i, \tag{45}$$

where

$$\mathcal{E}_{in}^{mj} = \tilde{c}_{ip}^m\, \eta_q^p\, f_n^{qj}, \qquad \mathcal{F}_{in}^{mj} = \tilde{c}_{pi}^m\, \eta_q^p\, f_n^{jq}. \tag{46}$$

Formula (45) is related to (40) through

$$\mathcal{P}_{jq}^{ip} = \overline{\mathcal{E}}_{mj}^{in}\, \mathcal{F}_{qn}^{mp}, \tag{47}$$

with

$$\overline{\mathcal{E}}_{in}^{mj} = \tilde{c}_{ip}^m\, \chi_q^p\, f_n^{qj}, \quad \overline{\mathcal{E}}_{jn}^{mi}\mathcal{E}_{ri}^{js} = \mathcal{E}_{jn}^{mi}\overline{\mathcal{E}}_{ri}^{js} = \delta_r^m \delta_n^s, \tag{48}$$

and, for completeness,

$$\overline{\mathcal{F}}_{in}^{mj} = \tilde{c}_{pi}^m\, \chi_q^p\, f_n^{jq}, \quad \overline{\mathcal{F}}_{nj}^{im}\mathcal{F}_{ir}^{sj} = \mathcal{F}_{nj}^{im}\overline{\mathcal{F}}_{ir}^{sj} = \delta_r^m \delta_n^s. \tag{49}$$

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The principal goal of the proposition proved was to formulate 'minimal' requirements (33) which yet suffice for the cross-multiplication recipe (34) to be fruitful. Properties of the second pairing, $<< \cdot, \cdot >>$, as well as (33) itself, are motivated by the anticipation of an antipode. The following proposition states this explicitly.

Proposition 2. Let $\mathcal{A}$ and $\mathcal{B}$ be the Hopf algebras and let there exist a pairing $< \cdot, \cdot >: \mathcal{B} \otimes \mathcal{A} \to \mathcal{K}$ with the properties (32) and, in addition,

$$< S(\alpha), a > = < \alpha, \overline{S}(a) >, \quad < \overline{S}(\alpha), a > = < \alpha, S(a) > . \tag{50}$$

Then the rule (6) makes $\mathcal{A} \otimes \mathcal{B}$ a Hopf algebra.

Proof. Using the notation

$$S(e_i) = \xi_i^j e_j, \quad \overline{S}(e_i) = \sigma_i^j e_j, \quad S(e^j) = \tilde{\sigma}_i^j e^i, \quad \overline{S}(e^j) = \tilde{\xi}_i^j e^i, \tag{51}$$

we write down the Hopf-algebra properties of $\mathcal{A}$ and $\mathcal{B}$ as

$$\sigma_i^k \xi_k^j = \xi_i^k \sigma_k^j = \delta_i^j, \quad E^j \xi_j^i = E^j \sigma_j^i = E^i, \quad \varepsilon_i \xi_j^i = \varepsilon_i \sigma_j^i = \varepsilon_j, \tag{52}$$

$$c_{ij}^k \xi_k^m = c_{pq}^m \xi_i^q \xi_j^p, \quad f_k^{ij} \sigma_m^k = f_m^{pq} \sigma_q^i \sigma_p^j, \tag{53}$$

$$c_{ij}^k \sigma_k^m = c_{pq}^m \sigma_i^q \sigma_j^p, \quad f_k^{ij} \xi_m^k = f_m^{pq} \xi_q^i \xi_p^j, \tag{54}$$

$$c_{nr}^j \xi_s^r f_i^{ns} = c_{rn}^j \xi_s^r f_i^{sn} = c_{nr}^j \sigma_s^r f_i^{sn} = c_{rn}^j \sigma_s^r f_i^{ns} = E^j \varepsilon_i, \tag{55}$$

(the same for quantities with a tilde), and the conditions (50) as

$$\eta_i^k \tilde{\sigma}_k^j = \eta_k^j \sigma_i^k, \quad \eta_i^k \tilde{\xi}_k^j = \eta_k^j \xi_i^k . \tag{56}$$

The bialgebra part of the proof is already done in the Proposition 1 because of the following identification:

$$<< \alpha, a >> \equiv < S(\alpha), a >, \quad \text{i.e.} \quad \chi_i^j = \eta_k^j \sigma_i^k = \eta_i^k \tilde{\sigma}_k^j .$$

The conditions (33) are readily checked,

$$<<_1 <_2 \Delta(\alpha), \Delta(a) >>_1>_2 \equiv < (S \otimes id) \circ \Delta(\alpha), \Delta(a) >$$

$$= < m \circ (S \otimes id) \circ \Delta(\alpha), a > = \varepsilon(\alpha) < \tilde{1}, a > = \varepsilon(\alpha) \varepsilon(a) .$$

So it remains to prove that $S(e^i e_j) = S(e_j) S(e^i)$, i.e.

$$\mathcal{P}_{jq}^{ip} \tilde{\sigma}_r^q \xi_p^s \mathcal{P}_{sm}^{rk} = \xi_j^k \tilde{\sigma}_m^i . \tag{57}$$

It can be done applying the relations from (54) four times and then, twice, (39).

One easily observes that our bialgebras (Hopf algebras) $\mathcal{T}$ and $\mathcal{U}$ in Sect.2 fit the above Propositions. This explains the appearance of the cross-multiplication relations (8),(31) prior to factorization. The role of the latter is to produce ortonormalized bases,

$$< e^i, e_j > \equiv \eta_j^i = \delta_j^i,$$

that enables one to rewrite (45) in the form of quasicocommutativity condition [4]

$$\mathcal{R}\Delta(x) = \Delta'(x)\mathcal{R}, \quad \Delta' \equiv P \circ \Delta, \quad P(a \otimes b) = b \otimes a \tag{58}$$

with the universal $\mathcal{R}$-matrix

$$\mathcal{R} = e_i \otimes e^i. \tag{59}$$

4. The method described in the present paper creates quantum doubles out of arbitrary invertible Yang-Baxter R-matrices taken as an input. However, an output (quasitriangular Hopf algebras) might sometimes appear almost trivial if the factorization involved were 'rude' enough to crash down interesting features of original bialgebras. Fortunately, this does not necessarily take place. In [3] (cf. [2]) it is shown how $sl_q(2)$ is recovered by this method. Another illustration is given below.

Let us take as an input the R-matrix [11, 12, 7]

$$R = \begin{pmatrix} 1 & q & -q & q^2 \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{60}$$

and consider the homogeneous case of the R-generated algebras (without E- and F-generators), assuming the notation

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad U = \begin{pmatrix} w & x \\ y & z \end{pmatrix} .$$

To remove the degeneracy of the pairing (4), we should require

$$c = y = 0, \quad ad = da = wz = zw = 1. \tag{61}$$

The procedure described in Sect.2 results in the Hopf algebra with generators $\{1, a, \bar{a}, b, w, \bar{w}, x\}$ whose multiplicative relations are

$$[a, b] = q(a^2 - 1), \quad [w, x] = q(w^2 - 1),$$

$$[a, x] = qa(w - \bar{w}), \quad [w, b] = q(a - \bar{a})w, \tag{62}$$

$$[b, x] = q(a + \bar{a})x - q(w + \bar{w})b, \quad aw = wa,$$

and the corresponding ones for inverse generators. We see that q may be absorbed into b and x (so we actually use the $\mathcal{R}$-matrix given by (60) with $q = 1$ [13]). Denoting also

$$a = e^g, \quad w = e^h, \quad x = -v, \tag{63}$$

we eventually come to

$$T = \begin{pmatrix} e^g & b \\ 0 & e^{-g} \end{pmatrix}, \quad U = \begin{pmatrix} e^h & -v \\ 0 & e^{-h} \end{pmatrix}.$$

The elements of these matrices form the Hopf algebra

$$[g, b] = [h, b] = e^g - e^{-g}, \quad [g, v] = [h, v] = e^{-h} - e^h,$$

$$[b, v] = (e^g + e^{-g})v + (e^h + e^{-h})b, \quad [g, h] = 0,$$

$$\Delta(b) = e^g \otimes b + b \otimes e^{-g}, \quad \Delta(v) = e^h \otimes v + v \otimes e^{-h}, \tag{64}$$

$$\Delta(g) = g \otimes 1 + 1 \otimes g, \quad \Delta(h) = h \otimes 1 + 1 \otimes h, \quad S^{\pm 1}(g) = -g,$$

$$S^{\pm 1}(h) = -h, \quad S^{\pm 1}(b) = -b \pm e^g \mp e^{-g}, \quad S^{\pm 1}(v) = -v \mp e^h \pm e^{-h}.$$

The pairing relations are the following:

$$< 1, 1 > = < h, b > = < v, g > = 1, \quad < v, b > = -1,$$

$$< 1, b > = < 1, g > = < h, 1 > = < v, 1 > = < h, g > = 0. \tag{65}$$

By construction, the Hopf algebra (64) has to be a quantum double. So, it should possess a universal $\mathcal{R}$-matrix. Assuming exponential Ansatz, we can write down several terms of its formal power expansion in g and h:

$$\mathcal{R} = \exp\Big\{g \otimes v + b \otimes h$$

$$-\frac{1}{6}\Big(g \otimes hvh + gbg \otimes h + g^2 \otimes (hv + vh) + (gb + bg) \otimes h^2\Big) + \dots\Big\}, \tag{66}$$

where discarded terms are of the fifth order in g and h. To check (58) and the quasitriangularity conditions

$$(\Delta \otimes id)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (id \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{67}$$

for the $\mathcal{R}$-matrix (66), the program FORM [14] has been essentially used.

A detailed study of this and other R-generated quasitriangular Hopf algebras is a subject of further investigations.

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