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ON SQUARING THE PRIMARY CONSTRAINTS
IN A GENERALIZED HAMILTONIAN DYNAMICS

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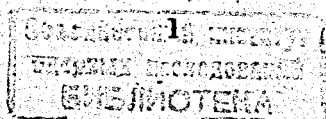
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1. The generalized Hamiltonian dynamics describing the systems with constraints is widely used now in investigating the theoretical models in a contemporary elementary particle physics. For example, the gauge symmetries of various types, without which every model does not practically works, inevitably entail the constraints in the phase space. Despite quite a large attention paid to the Hamiltonian systems with constraints (see, for example, papers [1-3] and references there in) some topics here still require a careful consideration. The present note is dealing with one of these problems, namely, the procedure of squaring of the primary constraints widely used in practical calculations will be investigated. By making use of a concrete example, a model of the so-called relativistic particle with curvature and torsion in the three-dimensional space-time [4, 5], we will show that this procedure can result finally in an erroneous answer.

The layout of the paper is the following. In the second section highlights about the primary constraints are given and the procedure of squaring of these constraints is explained. In the third section a generalized Hamiltonian description of a relativistic particle with curvature and torsion is developed by making use of the primary constraints in their original form, i.e. in the form that follows directly from the definition of the canonical momenta. In the third section the Hamiltonian description of this model is given by employing the squared primary constraints. It is shown that in this case the final result is erroneous. In section 5 it is argued that the Hamiltonian formalism with the use of squared primary constraints describes in the case under consideration another model with an action analogous but not identical with the initial action.

2. The primary constraints are a starting point in generating a complete set of constraints in a generalized Hamiltonian formalism



[1-3]. The requirement of preserving the primary constraints under time evolution entails the secondary constraints that in their turn should be preserved in time too. This results in the tertiary constraints and so on.¹

The primary constraints follow directly from the definition of the canonical momenta

$$p_i(q, \dot{q}) = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i}, \quad i = 1, \dots, n. \quad (2.1)$$

Given a degenerated or a singular Lagrangian $L(q, \dot{q})$, the functions $p_i(q, \dot{q})$ obey $m = n - r$ constraints

$$\varphi_s(q, p) = 0, \quad s = 1, \dots, m, \quad (2.2)$$

where n is the number of degrees of freedom and r is the rank of the Hessian

$$\frac{\partial^2 L(q, \dot{q})}{\partial \dot{q}_i \partial \dot{q}_j}, \quad 1 \leq i, j \leq n. \quad (2.3)$$

Upon substituting the functions $p_i(q, \dot{q})$ in (2.2) by (2.1) all the primary constraints (2.2) vanish identically with respect to q and \dot{q} .

In the case of the Lagrangian linear in velocities $r = 0$ and the definitions (2.1) are the primary constraints themselves

$$p_i = f_i(q), \quad i = 1, \dots, n. \quad (2.4)$$

Often it turns out to be convenient to deal with primary constraints preliminary transformed instead of using them in their original form (2.1) or (2.4). Squaring the left- and the right-hand side of (2.1) and projecting this equation onto suitable linearly independent vectors $\hat{n}_i^{(\alpha)}(q)$, $\alpha = 1, \dots, m - 1$ one obtains

$$\sum_{i=1}^n p_i^2 = \sum_{i=1}^n \left(\frac{\partial L(q, \dot{q})}{\partial \dot{q}_i} \right)^2,$$

¹According to the Dirac terminology [6] all the constraints except the primary ones are called the secondary constraints.

$$\sum_{i=1}^n p_i \hat{n}_i^{(\alpha)}(q) = \sum_{i=1}^n \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i} \hat{n}_i^{(\alpha)}(q), \quad \alpha = 1, 2, \dots, m - 1. \quad (2.5)$$

In the theory of the relativistic strings and membranes [7], for example, this procedure enables one to get immediately relations like (2.2) independent of the velocities, i.e., the primary constraints in the Hamiltonian form. However, squaring primary constraints does not prove to be always correct, this will be illustrated further by of a concrete example.

3. Let us consider the model of the so-called relativistic particle with curvature and torsion in the three-dimensional space-time. This model is defined by the action [4, 5]

$$S = -m \int ds - \alpha \int k(s) ds - \beta \int \kappa(s) ds, \quad (3.1)$$

where α and β are dimensionless parameters, m is a parameter with the dimension of mass, ds is a differential of the length of the world curve $x^\mu(s)$, $\mu = 0, 1, 2$, $k(s)$ is the curvature of this curve

$$k^2 = - \frac{d^2 x_\mu}{ds^2} \frac{d^2 x^\mu}{ds^2} \quad (3.2)$$

and $\kappa(s)$ is its torsion

$$\kappa(s) = k^{-2} \varepsilon_{\mu\nu\rho} x'^{\mu} x''^{\nu} x'''^{\rho}, \quad (3.3)$$

where $\varepsilon_{\mu\nu\rho}$ is a completely antisymmetric unit tensor of the third rank, $\varepsilon_{012} = +1$, the prime denotes the differentiation with respect to s . The Lorentz metric with signature $(+, -, -)$ is used.

The models of this kind have been considered recently in investigating the boson-fermion transformations in external Chern-Simons fields [8-10], as the one dimensional version of the rigid string [11, 12] and in polymer physics [13].

Given an arbitrary parametrization of the world curve $x^\mu(\tau)$, $\mu = 0, 1, 2$, the action (3.1) can be rewritten as

$$S = -m \int d\tau \sqrt{\dot{x}^2} - \alpha \int d\tau \frac{\sqrt{(\dot{x}\ddot{x})^2 - \dot{x}^2 \ddot{x}^2}}{\dot{x}^2}$$

$$- \beta \int d\tau \sqrt{\dot{x}^2} \frac{\varepsilon_{\mu\nu\rho} \dot{x}^\mu \ddot{x}^\nu \ddot{x}^\rho}{(\dot{x}\dot{x})^2 - \dot{x}^2 \ddot{x}^2}, \quad (3.4)$$

$$\dot{x} \equiv dx(\tau)/d\tau, \quad \dot{x}^2 > 0, \quad D = 3.$$

It depends on the particle velocity, its acceleration, and on the third derivatives of the particle coordinates with respect to τ . Therefore, the canonical variables are to be introduced according to Ostrogradsky [14, 15]

$$\begin{aligned} q_1 &= x, & q_2 &= \dot{x}, & q_3 &= \ddot{x} \\ p_1 &= -\frac{\partial L}{\partial \dot{x}} - \dot{p}_2, & p_2 &= -\frac{\partial L}{\partial \ddot{x}} - \dot{p}_3, & p_3 &= -\frac{\partial L}{\partial \ddot{x}}, \end{aligned} \quad (3.5)$$

where L is the Lagrangian function in (3.4).

The action (3.4) is invariant under reparametrization $\tau \rightarrow f(\tau)$. Hence, the Lagrangian in (3.4) is singular or degenerated and, as a consequence, the phase space should be restricted by constraints.

In paper [3] it has been shown that on introducing the canonical variables (3.5) the Hamiltonian formalism for theories with higher derivatives is constructed completely analogous to the Dirac generalized Hamiltonian dynamics dealing with singular Lagrangians depending only on the coordinates and velocities.²

The lagrangian in (3.4) is linear in \ddot{x}^μ therefore the definition of the canonical momenta p_3^μ is a constraint itself

$$\varphi_\mu^{(1)} = p_{3\mu} - \beta \frac{\sqrt{q_2^2}}{g} \varepsilon_{\mu\nu\lambda} q_2^\nu q_3^\lambda \approx 0, \quad (3.6)$$

$$\mu, \nu, \lambda = 0, 1, 2,$$

where $g = (q_2 q_3)^2 - q_2^2 q_3^2$ and sign \approx means weak equality [6].

According to Ostrogradsky, the canonical Hamiltonian is

$$H = -p_1 \dot{x} - p_2 \ddot{x} - p_3 \ddot{x} - L = -p_1 q_2 - p_2 q_3 + m \sqrt{q_2^2} + \alpha \frac{\sqrt{g}}{q_2^2}. \quad (3.7)$$

²In paper [5] the action (3.9) has been cast at the beginning into an equivalent form without higher derivatives and then the Hamiltonian formalism has been developed.

The Poisson brackets are defined in a standard way

$$(f, g) = \sum_{a=1}^3 \left(\frac{\partial f}{\partial p_a^\mu} \frac{\partial g}{\partial q_{a\mu}} - \frac{\partial f}{\partial q_a^\mu} \frac{\partial g}{\partial p_{a\mu}} \right). \quad (3.8)$$

The evolution of the model under consideration is determined by a total Hamiltonian

$$H_T = H + \sum_{\mu=0}^2 \lambda^\mu \varphi_\mu^{(1)}, \quad (3.9)$$

where λ^μ , $\mu = 0, 1, 2$ are the Lagrange multipliers.

The primary constrains are mutually in involution in a strong sense

$$\left(\varphi_\mu^{(1)}, \varphi_\nu^{(1)} \right) = 0 \quad (3.10)$$

The requirement of preserving the primary constraints under time evolution

$$\frac{d \varphi_\mu^{(1)}}{d\tau} = \left(\varphi_\mu^{(1)}, H_T \right) \approx 0, \quad \mu = 0, 1, 2 \quad (3.11)$$

results in the three secondary constraints

$$\begin{aligned} \varphi_\mu^{(2)} &= p_{2\mu} - \frac{\alpha}{q_2^2 \sqrt{g}} [(q_2 q_3) q_{2\mu} - q_2^2 q_{3\mu}] + \\ &+ \beta \varepsilon_{\mu\nu\lambda} q_2^\nu q_3^\lambda \frac{(q_2 q_3)}{g \sqrt{q_2^2}} \approx 0, \quad \mu = 0, 1, 2. \end{aligned} \quad (3.12)$$

Imposing the stationarity condition on constraints (3.12) one derives

$$\begin{aligned} \frac{d \varphi_\mu^{(2)}}{d\tau} &= \left(\varphi_\mu^{(2)}, H_T \right) = p_{1\mu} + m \frac{q_{2\mu}}{\sqrt{q_2^2}} - \beta \frac{\varepsilon_{\mu\nu\lambda} q_2^\nu q_3^\lambda}{q_2^2 \sqrt{q_2^2}} + \\ &+ \sum_{\nu=0}^2 \left(\varphi_\mu^{(2)}, \varphi_\nu^{(1)} \right) \lambda^\nu \approx 0, \quad \mu = 0, 1, 2. \end{aligned} \quad (3.13)$$

By rather involved calculations it can be shown that

$$\begin{pmatrix} (2) \\ \varphi_\mu, \varphi_\nu \end{pmatrix} = \frac{\alpha}{\sqrt{g}} b_\mu b_\nu, \quad (3.14)$$

where b_μ is a unit space-like vector directed along the binormal of the world curve

$$b_\mu = \frac{\varepsilon_{\mu\nu\rho} q_2^\nu q_3^\rho}{\sqrt{g}}, \quad b_\mu b^\mu = -1. \quad (3.15)$$

Projecting (3.13) onto q_2^μ , q_3^μ and taking into account (3.15) we obtain two constraints of the third generation

$$\begin{aligned} (3) \varphi_1 &= p_1 q_2 - m \sqrt{q_2^2} \approx 0, \\ (3) \varphi_2 &= p_1 q_3 - m \frac{q_2 q_3}{\sqrt{q_2^2}} \approx 0. \end{aligned} \quad (3.16)$$

Projection of (3.13) onto b^μ gives the relationship between the Lagrange multipliers

$$p_1 b - \frac{\beta}{q_2^2} \left(\frac{g}{q_2^2} \right)^{1/2} + \frac{\alpha}{\sqrt{g}} b^\lambda \approx 0. \quad (3.17)$$

Differentiating the constraints (3.16) with respect to τ one obtains

$$\frac{d}{d\tau} (3) \varphi_1 = \left((3) \varphi_1, H_T \right) = (3) \varphi_2 \approx 0, \quad (3.18)$$

$$\frac{d}{d\tau} (3) \varphi_2 = \left((3) \varphi_2, H_T \right) = m \frac{g}{(q_2^2)^{3/2}} + (\lambda b) (p_1 b) \approx 0. \quad (3.19)$$

Thus we have two equations (3.17) and (3.19) for two unknown quantities $(p_1 b)$ and (λb) . The exact solutions to these equations will not be required further because we concentrate now upon the relation in the model under consideration between the mass of the

particle $M^2 = p^2$ and its spin. When (3.16) is taken into account, the energy-momentum vector p_1^μ assumes the form

$$p_1^\mu = m \frac{q_2^\mu}{\sqrt{q_2^2}} - (p_1 b) b^\mu. \quad (3.20)$$

This vector is conserved under the time evolution as $(p_1^\mu, H_T) = 0$. On squaring (3.20), we have

$$M^2 = p_1^2 = m^2 - (p_1 b)^2. \quad (3.21)$$

In case of the three-dimensional space-time the spin of the particle is defined by

$$S = \frac{1}{2\sqrt{|p_1^2|}} \varepsilon_{\mu\nu\lambda} p_1^\mu M^{\nu\lambda}, \quad (3.22)$$

where $M_{\mu\nu}$ are the Lorentz generators

$$M_{\mu\nu} = \sum_{a=1}^3 (q_{a\mu} p_{a\nu} - q_{a\nu} p_{a\mu}). \quad (3.23)$$

When substituting (3.23) into (3.22) the spin of the particle becomes

$$S = \frac{1}{\sqrt{|p_1^2|}} \varepsilon_{\mu\nu\lambda} p_1^\mu (q_2^\nu p_2^\lambda + q_3^\nu p_3^\lambda). \quad (3.24)$$

Now let us calculate S on the submanifold of the phase space defined by the constraint equations (3.6), (3.12) and by expansion (3.20). As a result we derive

$$S = \pm \alpha \sqrt{\mu^2 - \varepsilon} - \beta \mu, \quad (3.25)$$

where $\mu = m/\sqrt{|p_1^2|} \geq 1$, $\varepsilon = \text{sign } p_1^2$. Thus, the Regge trajectory is split into two branches, i.e., the mass being fixed, there are two states with different spin values. As it will be shown further, it is just this peculiarity of the spectrum that will be lost in dealing with squared primary constraints.

4. In this section we construct the Hamiltonian formalism in the model under consideration starting, instead of (3.6), with squared primary constraints

$$\phi_1^{(1)} = p_3^2 + \beta^2 \frac{q_2^2}{g} \approx 0, \quad (4.1)$$

$$\phi_2^{(1)} = p_3 q_2 \approx 0, \quad (4.2)$$

$$\phi_3^{(1)} = p_3 q_3 \approx 0. \quad (4.3)$$

The constraint $\phi_1^{(1)}$ is obtained by moving the second term in (3.6) into the right-hand side and by squaring this equation. The constraints (4.2) and (4.3) are the projections of (3.6) onto q_2 and q_3 , respectively.

The canonical Hamiltonian (3.7) remains, obviously, the same but the total Hamiltonian \bar{H}_T is constructed now with primary constraints (4.1) - (4.3)

$$\bar{H}_T = H + \sum_{a=1}^3 \mu^a \phi_a^{(1)}. \quad (4.4)$$

Here μ^a , $a = 1, 2, 3$ are new Lagrange multipliers.

The primary constraint (4.1) - (4.3) are mutually in involution in a weak sense

$$\begin{aligned} \left(\phi_1^{(1)}, \phi_2^{(1)} \right) &= 0, \quad \left(\phi_1^{(1)}, \phi_3^{(1)} \right) = 2 \phi_3^{(1)} \approx 0, \\ \left(\phi_2^{(1)}, \phi_3^{(1)} \right) &= \phi_2^{(1)} \approx 0. \end{aligned} \quad (4.5)$$

The requirement of preserving the primary constraints (4.1) - (4.3) under time evolution results in the three secondary constraints

$$\phi_1^{(2)} = p_2 p_3 - \beta^2 \frac{q_2 q_3}{g} \approx 0, \quad (4.6)$$

$$\phi_2^{(2)} = p_2 q_2 \approx 0, \quad (4.7)$$

$$\phi_3^{(2)} = p_2 q_3 - \alpha \frac{\sqrt{g}}{q_2^2}. \quad (4.8)$$

The constraints (4.6) - (4.8) are in a complete agreement with constraints (3.12). Really, projecting (3.12) onto (3.6), q_2^μ and q_3^μ we arrive at the constraints (4.6) - (4.8).

Disagreement appears in the following. The constraints (4.6) - (4.8) turn out to be in involution in a weak sense with primary constraints (4.1) - (4.3)

$$\left(\phi_a^{(2)}, \phi_b^{(1)} \right) \approx 0, \quad a, b = 1, 2, 3, \quad (4.9)$$

while the constraints (3.12) and (3.6) do not commute (see eq. (3.14)).

Differentiating constraints (4.6) - (4.8) with respect to τ ,

$$\frac{d \phi_a^{(2)}}{d\tau} = \left(\phi_a^{(2)}, \bar{H}_T \right) \approx \left(\phi_a^{(2)}, H \right) \approx 0, \quad a = 1, 2, 3 \quad (4.10)$$

three new constraints are derived

$$\phi_1^{(3)} = p_1 p_3 + p_2^2 + \alpha \frac{p_2 q_3}{\sqrt{g}} + \beta^2 \frac{q_3^2}{g} \approx 0, \quad (4.11)$$

$$\phi_2^{(3)} = p_1 q_2 - m \sqrt{q_2^2} \approx 0, \quad (4.12)$$

$$\phi_3^{(3)} = \dot{p}_1 q_3 - m \frac{q_2 q_3}{\sqrt{q_2^2}} \approx 0. \quad (4.13)$$

Constraints (4.12) and (4.13) are completely equivalent to (3.16) but the constraint (4.11) has no counterpart between the constraints derived in the preceding section.

The requirement of the stationarity of the constraints (4.11) - (4.13) enables one to fix two Lagrangian multipliers μ_1 and μ_3 while μ_2 remains arbitrary [4].

It is convenient to use further the proper time gauge

$$q_2^2 = \text{const}, \quad q_2 q_3 = 0. \quad (4.14)$$

In this case, three vectors q_2 , q_3 , and p_3 form, owing to (4.1) – (4.3) and (4.14) a complete orthogonal basis. From (4.6) – (4.8) we deduce

$$p_2^\mu = -\alpha \frac{q_3^\mu}{\sqrt{-q_2^2 q_3^2}}. \quad (4.15)$$

Given (4.11) – (4.13), we can derive in the same way

$$p_1^\mu = m \frac{q_2^\mu}{\sqrt{q_2^2}} + p_3^\mu \frac{q_3^2}{q_2^2}. \quad (4.16)$$

For the mass squared this yields

$$M^2 = p_1^2 = m^2 + \beta^2 \frac{q_3^2}{(q_2^2)^2} = m^2 - \beta^2 k^2(s) \quad (4.17)$$

instead of (3.21) This expression, as well as (3.21), is not positive definite because of $q_3^2 < 0$ and $k^2(s) > 0$.

Let us calculate the spin of the particle according to (3.24). We should here evaluate a quantity $V = \varepsilon_{\mu\nu\lambda} q_2^\mu q_3^\nu p_3^\lambda$ on the physical submanifold of the phase space. By making use of the primary constraints in the form (4.1) – (4.3) we can find V up to sign and, as a consequence, the spin of the particle will be determined up to sign. To remove this ambiguity we fix the sign of V using the calculations of the preceding section, which gives

$$V = -\beta \sqrt{q_2^2}. \quad (4.18)$$

Finally, the particle spin is given by

$$S = \alpha \sqrt{\mu^2 - \varepsilon} - \beta \mu, \quad (4.19)$$

where μ and ε are the same parameters as in eq. (3.25).

Thus, dealing with squared primary constraints (4.1) – (4.3) we have lost the two-valuedness of the Regge trajectory.

5. In conclusion it should be noted the following. The squared primary constraints (4.1) – (4.3) appear inevitably when treating the action (3.1) in the space-time with dimension $D > 3$. In this case the torsion of the world curve is determined not by eq. (3.3), linear in \ddot{x} , but by the nonlinear expression

$$\kappa(s) = \frac{\sqrt{\det(d_{\alpha\beta})}}{k^2(s)}, \quad (5.1)$$

$$d_{\alpha\beta} = x^{(\alpha)} x_{\mu}^{(\beta)}, \quad x^{(\alpha)} \equiv d^\alpha x / ds^\alpha, \quad \alpha, \beta = 1, 2, 3.$$

The definition (5.1) makes sense for $D = 3$ too. In this case it gives an absolute value of the torsion defined by (3.3). The action (3.1) with torsion given by (5.1) has been considered in [4] and the mass spectrum (4.19) squared has been derived there.

Thus, the use of primary constraints in the squared form (4.1) – (4.3) results really in replacing the model (3.1), (3.3) by (3.1), (5.1). It has been shown recently in non-manifest way in paper [16] where the model (3.1), (3.3) was treated by making use of the squared primary constraints (4.1)–(4.3) in the total Hamiltonian.

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