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A NOTE ON (APPLIED) QUANTUM GROUPS

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The past few years have witnessed a great deal of interest in the study of the quantum groups [1-10]. Such an upsurge of interest has primarily been motivated by the fact that the usual symmetry groups and symmetry algebras have played pivotal role in providing deeper understanding of the basic laws of nature. It is, therefore, expected that the quantum groups, which are generalization of the usual groups in some sense, might shed more light on the working of many diverse and intricate physical phenomena of nature.

Quantum groups present examples of Hopf algebra (see, for instance ref.[5] for review). In this note, a nut-and-bolt approach is made to introduce quantum groups without going into details of (anti)homeomorphism, coproduct, counit, antipode etc. Furthermore, the definition of the deformed traces has been exploited to derive Hamiltonians for the quantum groups $GL(2)$ and $GL(1|1)$ in terms of the q-oscillators. In addition, the non-relativistic free particle, harmonic oscillator and relativistic free particle are discussed in the q-deformed Lagrangian formulation where ideas of differential calculus on quantum plane and q-deformed Poisson-brackets are summoned for the definition of the q-deformed Legendre transformations, derivation of the q-deformed Hamilton's equations and Euler-Lagrange equations of motion.

Reduced to bare essentials, quantum groups are a deformation of the usual groups in which a(multi) smooth c-number parameter(s) is(are) introduced without violating the essential ingredients of group properties. This non-zero parameter(q) is popularly known as "deformation-parameter". A particular limit of the deformation parameter ($q \rightarrow 1$) leads to the emergence of the usual groups. It is a very tempting question as to what extent the concept of quantum groups can be applicable to the realm of physics. In fact, the "smoothness" of the deformation parameter guarantees that the predictions of the deformed symmetries would be arbitrarily close to that of the usual symmetries [6]. These deformed groups have found applications in many diverse areas of research in mathematics and physics such as: knot theory, non- commutative geometry, integrable models, statistical mechanics, conformal field theories, solution of the Yang-Baxter equations etc. (see, for instance, refs. 1-10 and references therein).

Deformation of a symmetry group can be achieved either by introducing a deformation parameter in its algebraic structure or in its group structure. For instance, it was firstly demonstrated by Reshetikhin, Sklyanin [7] and Kulish [8] that following SU(2) algebra constituted by raising and lowering operators ($J_\pm$) and third component of the angular momentum (J_3):

$$\Big[J_3, J_\pm\Big] = \pm J_\pm \tag{1}$$

$$\Big[J_+, J_-\Big] = J_3 \tag{2}$$

can be deformed without spoiling the basic feature of SU(2). In this endeavour, equation (1) remains intact but equation (2) undergoes following modification:

$$\Big[J_+, J_-\Big] = \frac{q^{J_3} - q^{-J_3}}{q - q^{-1}} \tag{3}$$

where non-zero smooth c-number parameter "q" is the deformation parameter. It is straightforward to see that $q = 1$ leads to the rederivation of the SU(2) algebra. Furthermore, the famous Jacobi-identities are also satisfied for the commutators (1) and (3). However, it should be added that for the validity of these identities in the general case, one has to define deformed commutators in a special way (see, for instance, ref.[9] for the Virasoro algebra).

In the framework of the q-deformation of a given symmetry algebra, the usual quantization by introduction of the Planck's constant h can also be discussed [10]. For instance, in equation (3), if one takes $q = e^h$, then, the limit $q \rightarrow 1$ corresponds to the limit $h \rightarrow 0$. As is well-known, the latter limit leads to the famous correspondence - principle due to which, the quantum mechanical equations reduce to their classical counterpart. Recently there has been an attempt to discuss these deformations on completely independent grounds in the hope to obtain possible inter-relatioships amongst quantum groups, non-commutative geometry and space-time quantization [11]. In this work, a framework has been developed by which one can discuss in detail the q-deformation, $\hbar$-deformation and their various combinations in different orders for a given classical system. In fact, as a starting point, the simple cases of non-relativistic free particle and harmonic oscillator are extensively discussed by exploiting techniques of the various deformations [11].

Now we shall discuss about the introduction of deformation parameter in the "group-structure" of a given undeformed group. For simplicity and clarity, let us start with the general linear group of 2×2 non-singular matrices; namely, GL(2). An arbitrary non-singular element T ($detT \neq 0$) of this group can be represented as:

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2) \tag{4}$$

where a,b,c and d are c-number elements of the matrix T. It is straightforward to see that under matrix product (closure relation $(\cdot)$), matrices (4) form a group because identity, inverse and associativity can be defined explicitly. The other trivial properties of matrices (4), which would play significant role in the context of deformed groups are as follows:

(i) All the c-number elements a,b,c and d commute amongst themselves;namely:

$$\begin{aligned} ab &= ba, \quad ac = ca, \quad ad = da \\ bc &= cb, \quad bd = db, \quad cd = dc. \end{aligned} \tag{5}$$

(ii) The determinant $\mathcal{D} = ad - bc$ is the central and commutes with all the elements

$$\mathcal{D}\Big(\{a,b,c,d\}\Big) = \Big(\{a,b,c,d\}\Big)\mathcal{D}. \tag{6}$$

With this simple background, let us discuss the deformation of GL(2) by introducing a deformation parameter q. The elements a,b,c,and d of the 2×2 quantum matrix T, belonging to the quantum group $GL_q(2)$, exihibit different braiding relations in rows and columns as given below:

$$\begin{aligned} ab &= qba, \qquad ac = qca, \qquad cd = qdc \\ bd &= qdb, \quad bc = cb, \quad [a,d] = (q - q^{-1})bc \end{aligned} \tag{7}$$

where $q \in \mathcal{C}/\{0\}$. It is easy to note that, in contrast to (5), the elements of q-matrix obey non-commuting relations (7). The particular value of $q = 1$ leads to the rederivation of (5). Furthermore, once again under matrix multiplication, the q-matrices T form a group. To see it clearly, we define the non-zero q-determinant $\mathcal{D}$ as follows:

$$\mathcal{D} \equiv det_q T = ad - qbc. \tag{8}$$

Now it is obvious that the identity, inverse and associativity can be defined explicitly for the q-matrices T. In fact, the matrix multiplication $T = T'T''$ preserves relations (7) if we assume the commutativity of the elements of T' and T'' (i.e. $\left[T'_{ij}, T''_{kl}\right] = 0$). The preservation of the multiplication law can be explained through Yang-Baxter equations. Actually, the elements of $GL_q(2)$ act on a two-dimensional Manin's plane [4], therefore, the tensor products:

$$\begin{aligned} T_1 &= T \otimes 1 \\ T_2 &= 1 \otimes T \end{aligned} \tag{9}$$

act on the product space of the vector spaces. It is interesting to note that the R-matrix:

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & s & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{10}$$

with $s = q - q^{-1}$, satisfies following relations with T_1 and T_2:

$$R\, T_1\, T_2 = T_2\, T_1\, R \tag{11}$$

as a consequence of the braiding relations (7), obeyed by the elements of T. In anology with T_1 and T_2, if we define now another set of matrices T_1' and T_2' from a different matrix T' of $GL_q(2)$ with elements that commute with those of T, then, following equation:

$$R\, T_1\, T_1'\, T_2\, T_2' = T_2\, T_2'\, T_1\, T_1'\, R \tag{12}$$

is logically satisfied due to (11) and its anologue for the primed matrices. This equation, which is nothing but the Yang-Baxter equation, establishes the group properties under the matrix multiplication and thereby it also provides the reasoning for the preservation of the multiplication law.(see, e.g. ref.[12] for details)

It is interesting to check that, once again, the q-determinant $\mathcal{D}$ is central under braiding relations (7). Furthermore, it is straightforward to see that the group property under multiplication is also satisfied if q is changed to q^{-1} in equation (7).(we shall dwell a bit more on this symmetry later on, in the context of $GL_{qp}(2)$)

How to obtain relations (7) ?? Historically, it was the definition of the q-determinant and the requirement of the associativity which led to the derivation of relations (7) in the context of inverse scattering method for the quantum Liouville model described on a lattice [3]. However, later on, Manin [4] proposed a simpler method to obtain equation

(7) by introducing the concept of quantum hyperplane. In this technique, one introduces "q-coordinates" x and y which transform under $GL_q(2)$ as follows:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{13}$$

where $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_q(2)$. Furthermore, there exist another set of q-coordinates $\tilde{x}$ and $\tilde{y}$ which transform under $GL_q(2)$ as given below:

$$(\tilde{x}', \tilde{y}') = (\tilde{x}, \tilde{y}) \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{14}$$

Both these sets of q-coordinates describe the quantum hyperplane and satisfy non-commutative relations as follows:

$$xy = qyx \qquad \text{and} \qquad \tilde{x}\,\tilde{y} = q\,\tilde{y}\,\tilde{x}\,. \tag{15}$$

If we require the validity of relations (15) in the primed quantum hyperplanes also ,then, we derive all the relations of equation (7) for the elements of the q-matrix T.

The braiding relations for the elements of a q-matrix belonging to $SL_q(2)$ can be derived from (13), (14) and (15) with an additional restriction that $\mathcal{D} = det_q T = ad - qbc = 1$. Further appropriate restrictions on the elements of $SL_q(2)$ lead to the emergence of the quantum groups $U_q(2)$ and $SU_q(2)$. (see, for instance ref.[13], for details).

Manin's hyperplane approach to the definitions of the quantum group is very general and can be exploited for the multi-parameter deformation of $GL(N)$. For instance, the most general deformation of the group $GL(N)$ contains $[\frac{N(N-1)}{2} + 1]$ parameters. To illustrate this point succinctly, we discuss here two-parameter (p and q) deformation of $GL(2)$. For such a two-parameter deformation we require following conditions:

$$\begin{aligned} xy &= qyx, \qquad x'y' = qy'x' \\ \tilde{x}\,\tilde{y} &= p\,\tilde{y}\,\tilde{x}, \qquad \tilde{x}'\,\tilde{y}' = p\,\tilde{y}'\,\tilde{x}' \end{aligned} \tag{16}$$

where $p, q \in C/\{0\}$. It will be noticed that the above equation (16) is just the generalization of relations (15) to the two-parameter case which subsequently leads to the following braiding relations for the elements of the quantum matrix:

$$\begin{aligned} ab &= pba, \quad cd = pdc, \quad ac = qca, \quad bd = qdb \\ bc &= \frac{q}{p}cb, \quad [a, d] = (p - q^{-1})bc = (q - p^{-1})cb. \end{aligned} \tag{17}$$

In this case the q-determinant is defined as follows:

$$\mathcal{D} = ad - qcb = ad - pbc = da - q^{-1}bc = da - p^{-1}cb. \tag{18}$$

It is worth mentioning that here, the determinant $\mathcal{D}$ is not the central because it satisfies following relations:

$$\begin{aligned} a(\mathcal{D}, \mathcal{D}^{-1}) &= (\mathcal{D}, \mathcal{D}^{-1})a, \quad d(\mathcal{D}, \mathcal{D}^{-1}) = (\mathcal{D}, \mathcal{D}^{-1})d \\ b(\mathcal{D}, \mathcal{D}^{-1}) &= (\frac{q}{p}\mathcal{D}, \frac{p}{q}\mathcal{D}^{-1})b, \quad c(\mathcal{D}, \mathcal{D}^{-1}) = (\frac{p}{q}\mathcal{D}, \frac{q}{p}\mathcal{D}^{-1})c \end{aligned} \tag{19}$$

where $\mathcal{D}^{-1}$ is defined through the inverse of the matrix T as given below:

$$T^{-1}_{ij} = \mathcal{D}^{-1} \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix} = \begin{pmatrix} d & -p^{-1}b \\ -pc & a \end{pmatrix} \mathcal{D}^{-1}. \tag{20}$$

It will be noticed that the substitution $p = q$ in equations (16-19), leads to the rederivation of the corresponding equations for $GL_q(2)$ and, once again, the determinant $\mathcal{D}$ becomes the central. Moreover, the limit $p = q = 1$ corresponds to the undeformed group $GL(2)$ and, all above equations ultimately reduce to equations (5) and (6). The generalization of two-parameters to multi-parameters is straightforward. However, we shall not venture into the derivations of the corresponding relations in this short note.

We shall discuss now the deformed traces for the one- and two-parameter quantum group $GL(2)$. In the realm of physics, the importance of the notion of traces, ranging from Yang-Mills theory to the partition functions for conformal field theories, is rather well-known. It is, therefore, of immense value to obtain an appropriate definition for the quantum trace in the context of the deformed groups and apply them to some physical problems. One of the simplest way to define such "deformed traces" is to take the help of the representation theory. For instance, it is well known that, for a particular element of a given group, two matrix representations are equivalent if they are related to each-other by similarity transformations. The key feature of this equivalency is the equality of the trace for both the matrix representations. The well-known gauge invariance of the kinetic energy term of the Yang-Mills theory centres around this basic property. Thus, it would be fruitful to define deformed trace by exploiting the idea of its invariance under similarity transformations.

Let us take the example of the non-Abelian gauge theory and pin-point few technical terms which would be useful in the context of the quantum group. As is well established, under local gauge transformations, the gauge fields (A_μ), the covariant derivatives (D_μ), and the field strength tensors $(F_{\mu\nu})$ transform as follows:

$$\begin{aligned} A_\mu &\rightarrow UA_\mu U^{-1} + (\partial_\mu U)U^{-1} \\ D_\mu &\rightarrow UD_\mu U^{-1} \\ F_{\mu\nu} &\rightarrow UF_{\mu\nu}U^{-1} \end{aligned} \tag{21}$$

where unitary matrices U are group valued and carry in their womb transformation generators and transformation parameters for the group under consideration. In mathematical language, the transformations (21) form the "gauge orbits" for a given transformation group. For the case of the quantum group $GL_{qp}(2)$, we define the quantum orbit in the space of 2×2 quantum matrices E with following transformation properties:

$$E_{ij} \rightarrow T_{ik}\, E_{kl}\, T^{-1}_{lj} \tag{22}$$

where T_{ij} are elements of $T \in GL_{qp}(2)$ and commute with the elements of 2×2 q-matrix E (i.e.$[T_{ij}, E_{kl}] = 0$). For such a situation, the q-orbit invariant quantum trace for the matrix $E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$ is as follows[14]:

$$Tr_r(E) = r^{-1}E_{11} + rE_{22} \tag{23}$$

where $r = (qp)^{\frac{1}{2}}$. For the case $q = p$, equation (23) reduces to the q-trace of the quantum group GL_q. As is trivially evident, the condition $q = p = 1$ leads to the ordinary definition of the trace. Furthermore, equation (23) agrees with the assertion [6,16] that two-(or multi) parameter deformed quantities can always be recast into a form which would depend only on a single parameter. For instance, in (23) only $r = (qp)^{\frac{1}{2}}$ appears as a single parameter eventhough we took q and p as independent parameters. (The independency of these parameters turns up in the comultiplication, which we donot intend to discuss here.)

Now we shall concentrate on the qp-oscillator realisation of the quantum group $GL_{qp}(2)$ and derive Hamiltonians as bilinears of these oscillators which remain invariant under the coaction of this group. For this purpose, we inrtroduce two sets of oscillators A_i and $\tilde{A}_i$ (i=1,2) which are, in the language of differential geometry on q-plane, precisely the coordinates and derivatives on the quantum hyperplane. Let us demand following $GL_{qp}(2)$ transformations:

$$\begin{aligned} A_i &\rightarrow T_{ij}\, A_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \\ \tilde{A}_i &\rightarrow \tilde{A}_j\, T_{ij}^{-1} = (\tilde{A}_1, \tilde{A}_2)\mathcal{D}^{-1} \begin{pmatrix} d, & -q^{-1}b \\ -qc, & a \end{pmatrix}. \end{aligned} \tag{24}$$

It is straightforward now to see that following q-matrix:

$$E_{ij} = A_i\, \tilde{A}_j = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \otimes (\tilde{A}_1, \tilde{A}_2) \tag{25}$$

obeys the transformation properties of the q-orbit (22) as a consequence of its construction in equation (25) and transformations (24). At this juncture, the q-invariant trace defined by equation (23) can be exploited to derive the $GL_{qp}(2)$ invaraint Hamiltonian given by:

$$H_{qp} = r^{-1} A_1\, \tilde{A}_1 + r A_2\, \tilde{A}_2\,. \tag{26}$$

Furthermore, in the space of 2×2 q-matrices $\tilde{E}$, the other oscillator realisation which would trivially remain invariant under transformations (24) is as follows:

$$E_{ij} = \tilde{A}_i\, A_j = (\tilde{A}_1, \tilde{A}_2) \otimes \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}. \tag{27}$$

Thus, another $GL_{qp}(2)$ invariant hamiltonian $\tilde{H}$ in terms of the bilinears of the q-oscillators is as follows:

$$\tilde{H}_{qp} = \sum_{i=1}^{2} \tilde{A}_i\, A_i \tag{28}$$

which is nothing other than the usual trace of the matrix $\tilde{E}$ in (27).

It is rather instructive to note that the notion of the q-orbit invariant deformed trace provides a kind of generality for the derivation of the Hamiltonians because, in ref.[13], only the analogue of (28) is mentioned. However, it is evident from the above discussion that there exists another representation of the Hamiltonian in terms of the bilinears of the q-oscillators given by equation (26). It has been demonstrated in ref.[14] that these Hamiltonians are related to each-other due to the presence of q-oscillator algebras that are covariant under the coaction of $GL_{qp}(2)$.

Why are equations (26) and (28) called Hamiltonians?? The answer to this question comes from the algebraic relations that are satisfied by these and other combinations of the q-oscillators. To make this point transparent, it is instructive to take the supersymmetric quantum group $GL_{qp}(1|1)$ characterized by 2×2 matrices T^s with two odd (β, γ) and two even (a,d) elements that satisfy following non-commutative braiding relations for the non-zero c-number deformation parameters q and p :

$$\begin{aligned} a\beta &= p\beta a, \quad d\beta = p\beta d, \quad a\gamma = q\gamma a, \quad d\gamma = q\gamma d \\ \beta^2 &= \gamma^2 = 0, \quad \beta\gamma = -\frac{q}{p}\gamma\beta, \quad [a,d] = (q - p^{-1})\gamma\beta = (q^{-1} - p)\beta\gamma. \end{aligned} \tag{29}$$

In analogy with the case of $GL_{qp}(2)$, the set of bosonic oscillators $(A, \tilde{A})$ and the set of fermionic oscillators $(B, \tilde{B})$ transform as follows under $GL_{qp}(1|1)$:

$$\begin{aligned} \begin{pmatrix} A \\ B \end{pmatrix} &\rightarrow \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \equiv T^s \begin{pmatrix} A \\ B \end{pmatrix} \\ (\tilde{A}, \tilde{B}) &\rightarrow (\tilde{A}, \tilde{B}) \begin{pmatrix} a^{-1} + a^{-1}\beta d^{-1}\gamma a^{-1}, & -a^{-1}\beta d^{-1} \\ -d^{-1}\gamma a^{-1}, & d^{-1} - d^{-1}\beta a^{-1}\gamma d^{-1} \end{pmatrix} \equiv (\tilde{A}, \tilde{B})(T^s)^{-1}. \end{aligned} \tag{30}$$

The super q-matrix E^s constructed from these oscillators:

$$E^s_{ij} = \begin{pmatrix} A\,\tilde{A}, & A\,\tilde{B} \\ B\,\tilde{A}, & B\,\tilde{B} \end{pmatrix} \tag{31}$$

obeys the transformation laws of the super q-orbit. The invariance of the q-supertrace results in following expression for the super q-orbit invariant Hamiltonian:

$$H^s_{qp} = A\,\tilde{A} - B\,\tilde{B} = \lambda(\tilde{A}\,A + \tilde{B}\,B) \tag{32}$$

where λ is a constant which turns out to be equal to 1 or r^{-2} due to the requirement of the graded associativity for the q- super oscillators. In fact, this requirement is equivalent to the derivation and solution of the graded Yang-Baxter equations. The latter value of λ leads to an oscillator algebra which, in turn, yields following covariant super algebraic relations that include $N = 2$ supersymmetric quantum mechanical algebra:

$$\begin{aligned} &\{Q, \bar{Q}\} = H^s_{qp}, \; Q^2 = \bar{Q}^2 = 0, \\ &[Q, Y] = Q, \; [\bar{Q}, Y] = -\bar{Q}, \\ &[H^s_{qp}, Y] = [H^s_{qp}, Q] = [H^s_{qp}, \bar{Q}] = 0, \end{aligned} \tag{33}$$

where $Q = E^s_{12} = A\,\tilde{B}$, $\bar{Q} = E^s_{21} = B\,\tilde{A}$ and $Y = \frac{1}{2}(A\,\tilde{A} + B\,\tilde{B})$. As a consequence of the SUSY quantum mechanics, it is clear that the notion of the q-orbit invariant quantum trace provides the expression of the Hamiltonian $H^s_{qp} = \{Q, \bar{Q}\}$. It is precisely this reason that the oscillator bilinears (26) and (28) are called Hamiltonians as they satisfy Witten-type algebra along with other oscillators[14]. The other reason for such a nomenclature is the appearance of these expressions as the sum of products of "creation" and "annihilation" operators which resembles the "usual" Hamiltonian for the harmonic oscillator. The name "oscillator" is also cited because of this close kinship. The notion of the deformed traces has been exploited by Isaev and Popowicz (in ref.[15]) to discuss Yang-Mills theory. Other attempts have also been made in this direction [15].

In addition to providing the derivation of the Hamiltonians, the transformations (24) lead to the existence of two-types of qp-oscillator algebras in terms of the q-deformed commutators as discussed in ref.[14]. One of the interesting feature of these algebras is the existence of following symmetry:

$$i = 1 \leftrightarrow i = 2 \qquad \text{and} \qquad p, q \leftrightarrow p^{-1}, q^{-1}. \tag{34}$$

In the case of one-parameter deformed group $SU_q(2)$, analogue of this symmetry (i.e.$q \to q^{-1}$) is mentioned [13] which is the trivial symmetry present in the braiding relations (7). However, it is illuminating to note that the generalization of this symmetry, in the case of two-parameter deformed group $GL(2)$, emerges very naturally in the discussion of ref.[14] as the requirement of the associativity condition on all the q-oscillators. It is worth emphasising here that the requirement of associativity is equivalent to the derivation and solution of the Yang-Baxter equations.

As an illustartion of the application of the quantum group to physical problems, we consider now the motion of a q-deformed free non-relativistic particle on a quantum-line. This line is characterized by phase variables $x(t)$ and $p(t)$ on the q-deformed manifold which satisfy following relationship:

$$x(t)\; p(t) = q\; p(t)\; x(t) \tag{35}$$

where t is a real evolution parameter. The free motion is endowed with translational invariance on the q-deformed manifold. As a result, equation of motion is nothing but the conservation of momentum given by:

$$\dot{p}(t) = 0. \tag{36}$$

For the hermitian operators $x(t)$ and $p(t)$ (with $|q| = 1$), the q-deformed Hamiltonian describing the free motion is as follows[11]:

$$H = \frac{q}{1+q^2}\; p\; m^{-1}\; p. \tag{37}$$

The q-deformed manifold, with above hermitian phase variables, is characterized by an $SL_q(2)$ invariant q-deformed symplectic form B_q defined in terms of the anti-symmetric matrix $(\Omega_{AB}(q))$ as follows:

$$B_q(z, z') = z^A\; \Omega_{AB}(q)\; z^{B'} = -q^{-\frac{1}{2}}\; x\; p' + q^{\frac{1}{2}}\; p\; x' \tag{38}$$

where $z^A = (x, p)$ and $\Omega_{AB} = \begin{pmatrix} 0, & -q^{\frac{-1}{2}} \\ q^{\frac{1}{2}}, & 0 \end{pmatrix}$. Thus, the q-Poisson bracket between two-variables f and g is defined as:

$$\{f, g\}_q = \frac{\partial f}{\partial z^A}\, \Omega^{AB} \frac{\partial g}{\partial z^B} \tag{39}$$

where $\Omega^{AB} = \begin{pmatrix} 0, & q^{\frac{-1}{2}} \\ -q^{\frac{1}{2}}, & 0 \end{pmatrix}$ is the inverse of the matrix Ω_{AB} defined on the quantum manifold. Due to the presence of the equation (39), the Hamilton's equations of motion are as follows:

$$\dot{x} = \{x, H\}_q = q^{\frac{1}{2}}\, m^{-1} p \quad \text{and} \quad \dot{p} = \{p, H\}_q = 0. \tag{40}$$

It is worth mentioning that the differentials present in equation (39) and defined on the quantum plane (characterized by equation (35)) obey following rule[17]:

$$\begin{aligned} \frac{\partial\, (p^n x^m)}{\partial p} &= p^{n-1}\, x^m\, \frac{(1 - q^{2n})}{(1 - q^2)} \\ \frac{\partial (p^n x^m)}{\partial x} &= p^n\, x^{m-1}\, q^n\, \frac{(1 - q^{2m})}{(1 - q^2)}. \end{aligned} \tag{41}$$

While computing the Poisson- brackets, it is essential to bring the Hamiltonian in a form $p^r m^s$ (where r and s are real numbers) by using following non-commutative relation:

$$m^{-1}\, p = q\, p\, m^{-1} \tag{42}$$

resulting from (35) due to equations of motion (40). This demonstrates that the mass is a non-commutative number on a quantum line. Consistent with equations (35), (36), (40) and (41), following q-deformed Legendre transformations yields the first order Lagrangian (L_F) as :

$$L_F = q^{\frac{1}{2}}\, p\, \dot{x} - \frac{q}{1 + q^2}\, p\, m^{-1}\, p \tag{43}$$

which, in turn, leads to the derivation of (40) due to the following equation of motion :

$$\frac{\partial L_F}{\partial p} = 0, \tag{44}$$

and the q-deformed Euler-Lagrange equations of motion. (We shall discuss it in detail for the q-deformed harmonic oscillator.) The equivalent second order Lagrangian is given by:

$$L_S = \frac{q^2}{1 + q^2}\, m\, (\dot{x})^2 \tag{45}$$

which emerges due to the application of equations (40) and (43). It is now straightforward to check that the following Legendre transformations:

$$H = q^{\frac{1}{2}}\, p\, \dot{x} - L_S \tag{46}$$

reproduces Hamiltonian (37), we started with. The Hamilton's equation of motion , consistent with Poisson-bracket (39), can be derived by requiring the invariance of the action ($S = \int L_F dt$) defined in terms of the first order Lagrangian (43). This is given by:

$$\delta S = 0 = \int (q^{\frac{1}{2}}\ \delta p\dot{x} + q^{\frac{1}{2}}\ p\delta\dot{x} - \delta x\frac{\partial H}{\partial x} - \delta p\frac{\partial H}{\partial p})dt. \tag{47}$$

Now, taking all the variations to the left side by applying following non-commutative relations resulting from equations (35), (36):

$$\delta\dot{x}\ p = q\ p\ \delta\dot{x} \quad \text{and} \quad \dot{x}\ \delta p = q\ \delta p\dot{x}, \tag{48}$$

and dropping off the total drivative term by choosing appropriate boundary conditions on the transformation parameter(s) of the variations, we obtain following equations of motion:

$$\dot{x} = q^{-1/2}\frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = q^{1/2}\frac{\partial H}{\partial x} \tag{49}$$

which are in total agreement with the definition of the Poisson bracket (39). (We follow here the convention of taking all the symmetry variations to the left). Furthermore, it can be noticed that the basic Poisson-brackets for the "canonical" variables :

$$\{x, p\}_q = q^{-1/2} \quad \text{and} \quad \{p, x\}_q = -q^{1/2} \tag{50}$$

can be generalized to q-commutators in the canonical quantization procedure as given below:

$$\begin{aligned} [x, p]_q &\equiv x\ p - q\ p\ x = i\ \hbar\ q^{-1/2} \\ [p, x]_q &\equiv p\ x - q\ x\ p = -i\ \hbar\ q^{1/2}. \end{aligned} \tag{51}$$

To derive the q-deformed Euler-Lagrange equations, it is instructive to consider one dimensional q-deformed harmonic oscillator with following q-deformed hamiltonian:

$$H^{osc} = \frac{q}{1+q^2}\ p\ m^{-1}\ p + \frac{(q^{-3}\omega^2)}{1+q^2} x\ m\ x. \tag{52}$$

Exploiting q-deformed Poisson-brackets defined in (39), we obtain following Hamilton's equations of motion:

$$\dot{x} = \{x, H^{osc}\}_q = q^{\frac{1}{2}}\ m^{-1} p \quad \text{and} \quad \dot{p} = \{p, H^{osc}\}_q = -q^{-1/2}\omega^2\ m\ x, \tag{53}$$

which result in, if we assume following non-commutative relations between $x(t)$ and m:

$$x(t)\ m = q\ m\ x(t). \tag{54}$$

It is worth pointing out that all the relations (35), (42), (53) and (54) are consistent with one-another on the q-deformed manifold. It is obvious from (42) and (54) that the mass parameter is, once again, non-commutative number with respect to phase variables $x(t)$ and $p(t)$. The first order Lagrangian (L_f^{osc}) is as follows:

$$L_f^{osc} = q^{\frac{1}{2}}\, p\, \dot{x} - \frac{q}{1+q^2}\, p\, m^{-1}\, p - \frac{(q^{-3}\omega^2)}{1+q^2} x\, m\, x. \tag{55}$$

One of the ineteresting but unusual features of above Lagrangian is the definition of the q-deformed "canonical" momentum that emerges from it. For instance, it can be clearly seen, using the rules of the differential calculus on the q-plane and $\dot{x}p = qp\dot{x}$ (emrging from (35)and (53)), that following equations are derived from L_f^{osc}:

$$\frac{\partial L_f^{osc}}{\partial \dot{x}} = q^{3/2}\, p \quad \text{and} \quad \frac{\partial L_f^{osc}}{\partial x} = -\, q^{-1}\omega^2\, m\, x. \tag{56}$$

It is evident now that, to attain conformity with Hamilton's equations of motion (53), following q-deformed Euler-Lagrange equation of motion very naturally emerges:

$$q^{-3/2} \frac{d}{dt}\Big(\frac{\partial L_f^{osc}}{\partial \dot{x}}\Big) = q^{1/2}\, \frac{\partial L_f^{osc}}{\partial x}. \tag{57}$$

In the $q \to 1$ limit, above equation reduces to the usual Euler- Lagrange equations.

The second-order Lagrangian, derived from the first order Lagrangian (55), is as follows:

$$L_S^{osc} = \frac{q^2}{1+q^2}\, m\, (\dot{x})^2 - \frac{(q^{-3}\omega^2)}{1+q^2} x\, m\, x. \tag{58}$$

Consistent with the Hamilton's equations of motion (53), the Euler-Lagrange equations of motion derived from the second-order Lagrangian is as follows:

$$q^{-7/2} \frac{d}{dt}\Big(\frac{\partial L_S^{osc}}{\partial \dot{x}}\Big) = q^{1/2}\, \frac{\partial L_S^{osc}}{\partial x}. \tag{59}$$

The difference in the powers of " q " in equations (57) and (59), is not surprising. This discrepancy emrges due to the presence of different powers of " q" and velocities "$\dot{x}$" in the first- and second order Lagrangians. It is now straightforward to see that the Legendre transformation (46), along with the equations of motion (53), lead us back to the oscillator Hamiltonian (52), we started with.

It is very tempting venture to discuss Lagrangian formulation of the q- deformed free relativistic particle on the same footing as that of the q-deformed free non-relativistic particle. In fact, this attempt is in progress [18], and the preliminary results demonstrate that the mass-shell condition:

$$p_\mu\, p^\mu - m^2 = 0, \tag{60}$$

remains the same for the q-deformed free relativistic particle as well. This result is in conformity with the derivation of the q-deformed Klein-Gordon equation and Dirac equation in ref.[19] by exploiting the definition of the q-deformed Dirac matrices. The other interesting feature of the q-deformed relativistic free particle is the emergence of mass and einbein field (metric) $e(\tau)$ as non-commutative objects. The three q-deformed equivalent Lagrangians for the q-relativistic particle are as given below:

$$L_o = \frac{q(1+q)}{1+q^2}\, m\, (\dot{x}_\mu \dot{x}^\mu)^{1/2}$$

$$L_f = q^{1/2}\, p_\mu \dot{x}^\mu - \frac{q}{1+q^2}\, [p_\mu\, e\, p^\mu - m\, e\, m]$$
$$L_S = \frac{q^2}{1+q^2}\, e^{-1}\, (\dot{x}_\mu)^2 + \frac{q}{1+q^2}\, m\, e\, m \tag{61}$$

where $e\ m = q\ m\ e$ is assumed and other non-commutative relations emerge from the equations of motion and the basic non-commutative relation describing a quantum world-line on a q-deformed D-dimensional Minkowski manifold as given below:

$$x_\mu(\tau)\, p^\mu(\tau) = q\, p^\mu(\tau)\, x_\mu(\tau) \tag{62}$$

where $\mu = 0, 1, 2 .... D-1$ are Minkowski indices and τ is the proper-time evolution parameter.

It is rather well-known that the understanding of the free relativistic particle is key to the modern development of the string theories. The string and free relativistic particle actions are endowed with many symmetries e.g. reparametrization, gauge and Weyl symmetries — to name a few. It would be, therefore, interesting to perform BRST quantization of the q-deformed relativistic particle by exploiting the gauge symmetry present in the corresponding action. The q-constraint analysis, q-reparametrization symmetries and q-BRST quantization etc. for this system are under consideration and would be reported elsewhere [18].

It is worthwhile to stress here that the thorough understanding of q-deformed relativistic particle might turn out to be the corner stone for the development of the q-deformed field theory.

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