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THE LATTICE SPINOR QED HAMILTONIAN. CRITIQUE OF THE CONTINUOUS SPACE APPROACH

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INTRODUCTION

Wilson formulated the lattice approach to the QED [1]. In the present work, we derive the lattice spinor QED Hamiltonian, starting with the continuous time modification of the Wilson's approach that was given by Kogut and Susskind [2] (**Sec. I**).

Almost all textbooks (see, for instance, [3-6]) and many works use the continuous space approach to the QED, so it seems to be quite natural and necessary to supplement our derivation by at least a short descripion of the continuous space QED Hamiltonian (see **Sec. II** until item **2.3.**).

In items **2.3.-2.3.6.1.** we compare the lattice and continuous space approaches to the QED and try to substantiate our opinion that *the lattice approach is obviously preferable.* **In particular, it is impossible to construct a continuous space QED** ***Hamiltonian*** **which is compatible with the condition of gauge invariance** (items 2.3.1-2).

We state that however one chooses the lattice *Lagrange function* so that to ensure its correspondence with the continuous space approach *Lagrangian* , such a correspondence, in fact, is rather problematic (item 2.3.3.0.).

Papers [7-8] prove unboundedness from below of the spinor QED *Hamiltonian* . But these papers use the continuous space approach to QED. Thus, the consideration of these papers is not perfect from the point of view of gauge invariance (see items 2.3.2. and 2.3.3.1). Lattice consideration of QED, which is irreproachably gauge invariant, does not confirm the result of works [7-8].

In Summary we list the virtues of our derivation of the lattice QED Hamiltonian as compared with the continuous space QED Hamiltonian derivations we know [3-6].

In Appendix we give a new derivation of the lattice QED Hamiltonian, different from that given in Sec.I.

Concerning our notation. In **Sec. II** we consider the continuous space QED in a periodicity cube $V = L^3$ so that the field variables depend on the continuous coordinate $\mathbf{x} \equiv (x_1, x_2, x_3), 0 < x_s \le L, s = 1,2,3.$

In **Sec.I**, we consider the lattice QED: $\mathbf{x} = a\mathbf{j}, x_s = aj_s, j_s = 1,2,...,M, s = 1,2,3, L = aM$, here a is the lattice length. We denote our lattice by R.

Three-component vectors are denoted by boldface letters, for instance, in **Sec.II** the spatial parts of the basic vector potentials are denoted by $\mathbf{A} = \mathbf{A}(\mathbf{x}) \equiv (A_1, A_2, A_3)$ and $\mathbf{B} = \mathbf{B}(\mathbf{x}) \equiv (B_1, B_2, B_3)$. Note that in **Sec.I** the spatial components of the vector potentials are denoted by [1-2] $A_s(\mathbf{j}, \mathbf{j} + \hat{s})$ and $B_s(\mathbf{j}, \mathbf{j} + \hat{s})$, $s = 1,2,3$, here $\hat{s}$ is the unit vector in the direction of the s-th axis, and the scalar potential is denoted by $A_0(\mathbf{j})$.

We do omit the time argument t of the potentials $A_\mu, \mu = 1,2,3,4$ and spinor function ψ. Analogously, we omit almost everywhere the time arguments t and t_0 of the functions $B_s, B_s^{tr}, B_s^{long}, s = 1,2,3, \psi_1, \psi_2$, and the time arguments of continuous space and lattice Fourier components $q(\mathbf{k}, \lambda, t, t_0), \mathbf{k} \neq 0$, $q(\mathbf{n}, \lambda, t, t_0), \mathbf{n} \neq 0$ and $\mathbf{q}(0, t, t_0)$ (the last quantity is the zero momentum mode [9]) of the function B.

Dot denotes the differentiation with respect to the time, $x_4 = it, A_4 = iA_0$. The derivatives $\frac{\partial}{\partial q(\mathbf{n}, \lambda)}_\psi$ are usual partial derivatives, defined by the condition

$$[\frac{\partial}{\partial q(\mathbf{n}, \lambda)}_\psi, \psi] = [\frac{\partial}{\partial q(\mathbf{n}, \lambda)}_\psi, \psi^*] = 0,$$

see eqs. (1.14), (1.16) and (A2-A4).

Sec.I. DERIVATION OF THE LATTICE QED HAMILTONIAN

We start with the Lagrange function (1.1)-(1.2d) where the term $U(\mathbf{A})$, eq. (1.2b), is chosen simpler than the corresponding term of the Lagrange function in papers [1-2], [10-12]. The Lagrange function (1.1)-(1.2d), unlike the Lagrange function [1-2], [10-12], is not periodical in variables $B_s(\mathbf{j}, \mathbf{j} + \hat{s})$); instead, it is more easy to handle. In order to get the *Hamiltonian* that gives the Lagrange function [1-2], [10-12] one has only to calculate anew in our *Hamiltonian* , eqs. (13-16), the term $H_{0ph,l}(pot)$, eq. (1.14).

Introducing new variables via A_0 dependent gauge transformation (the De-Witt gauge transform), eq. (1.4), we express the Lagrange function in terms of these variables, eq. (1.6). This way represented, the Lagrange function depends not only on the time derivative of the longitudinal potential, but also on this potential itself. A new change of the spinor field variable, eq. (1.9), cf. [4], Chapter 12, eq. (110), enables us to get the representation (1.10) of the Lagrange function. This representation does not contain explicit dependence on the longitudinal potential itself, it depends only on the time derivative of this potential. Then, the straightforward standard procedure gives at once the expression (1.13) of the lattice QED Hamiltonian $H_{QED,l}$. Note that this expression depends only on the derivatives with respect to the longitudinal potential variables, but not on these variables themselves.

In Appendix we give, starting with the representation (1.6) of the Lagrange function, another derivation of the same Hamiltonian.

The lattice QED is defined on the spatial lattice R, see **Introduction.**

The Lagrange function of the spinor lattice QED is [1], [2]

$$L(\mathbf{A}, A_0, \psi) = T(\dot{\mathbf{A}} + \mathbf{g}) - U(\mathbf{A}) - D(\psi, \mathbf{A}) - D_4(\psi, A_4), \quad (1.1)$$

$$\mathbf{g} \equiv (g_1, g_2, g_3), \quad g_s(\mathbf{j}, \mathbf{j}+\hat{s}) = [A_0(\mathbf{j}+\hat{s}) - A_0(\mathbf{j})]/a \equiv \frac{\Delta(s)}{a} A_0(\mathbf{j}). \quad (1.1a)$$

Here, $\Delta(s)$ is the finite difference operator: for any function $\Phi(\mathbf{j})$ one has $\Delta(s)\Phi(\mathbf{j}) = \Phi(\mathbf{j} + \hat{s}) - \Phi(\mathbf{j})$ and $\hat{s}$ is the unit vector in the direction of the s-th axis. Formally, $\frac{\Delta(s)}{a} \to \frac{\partial}{\partial x_s}$ as $a \to 0$.

Now we define the terms in the r.h.s. of eq. (1.1):

$$T(\mathbf{Z}) = \frac{1}{2} a^3 \sum_{\mathbf{j}, s; \mathbf{j} \in R, s=1,2,3} Z_s(\mathbf{j}, \mathbf{j} + \hat{s})^2, \quad (1.2a)$$

$$U(\mathbf{A}) = \frac{a^3}{4} \sum_{\mathbf{j}, s, r} [\frac{\Delta(r)}{a} A_s(\mathbf{j}, \mathbf{j} + \hat{s}) - \frac{\Delta(s)}{a} A_r(\mathbf{j}, \mathbf{j} + \hat{r})]^2, \quad (1.2b)$$

$$D_4(\psi, A_4) = a^3 \sum_{\mathbf{j}} \psi(\mathbf{j})^* [\partial/\partial x_4 - ieA_4(\mathbf{j})] \psi(\mathbf{j}), \quad (1.2c)$$

$$D(\psi, \mathbf{A}) = a^3 \sum_{\mathbf{j}} \{ \sum_{s=1,2,3} [\psi(\mathbf{j})^* \gamma_4 \gamma_s \psi(\mathbf{j} + \hat{s}) e^{-ieaA_s(\mathbf{j}, \mathbf{j}+\hat{s})}$$

$$-\psi(\mathbf{j} + \hat{s})^* \gamma_4 \gamma_s \psi(\mathbf{j}) e^{ieaA_s(\mathbf{j}, \mathbf{j}+\hat{s})}]/(2a) + m\psi(\mathbf{j})^* \gamma_4 \psi(\mathbf{j})\}. \quad (1.2d)$$

1. The Lagrange function (1.1), (1.1a), (1.2a-d) is invariant under gauge transformations

$$A_0(\mathbf{j}) \to A_0(\mathbf{j}) - \partial\lambda(\mathbf{j}, t)/\partial t, \quad \psi(\mathbf{j}) \to \psi(\mathbf{j}) e^{ie\lambda(\mathbf{j}, t)},$$

$$A_s(\mathbf{j}, \mathbf{j} + \hat{\mathbf{s}}) \to A_s(\mathbf{j}, \mathbf{j} + \hat{\mathbf{s}}) + \frac{\Delta(s)}{a} \lambda(\mathbf{j}, t), \quad (1.3)$$

here $\lambda(\mathbf{j}, t), \mathbf{j} \in R$ are arbitrary *continuous* (cf. item **2.3.1**) real functions of the time.

Let us take any value of t_0 and introduce the functions $\mathbf{B} \equiv (B_1, B_2, B_3), \psi_1$ and Λ via the formula

$$\Lambda(\mathbf{j}) \equiv \Lambda(\mathbf{j}, t, t_0) = \int_{t_0}^{t} A_0(\mathbf{j}, \tau) d\tau,$$

$$\psi(\mathbf{j}, t) \equiv \psi(\mathbf{j}) = \psi_1(\mathbf{j}) e^{-ie\Lambda(\mathbf{j})} \equiv \psi_1(\mathbf{j}, t, t_0) e^{-ie\Lambda(\mathbf{j}, t, t_0)},$$

$$B_s(\mathbf{j}, \mathbf{j} + \hat{s}) \equiv B_s(\mathbf{j}, \mathbf{j} + \hat{s}, t, t_0) = A_s(\mathbf{j}, \mathbf{j} + \hat{s}) + \frac{\Delta(s)}{a} \Lambda(\mathbf{j}). \quad (1.4)$$

1.1. One can check that if the potentials $A_\mu, \mu = 1, 2, 3, 4$ and function ψ undergo the transformation (1.3), then the quantities $B_s, s = 1, 2, 3$ and ψ_1 undergo the transformation

$$B_s(\mathbf{j}, \mathbf{j} + \hat{s}) \to B_s(\mathbf{j}, \mathbf{j} + \hat{s}) + \frac{\Delta(s)}{a} \lambda(\mathbf{j}, t_0),$$

$$\psi_1(\mathbf{j}) \to \psi_1(\mathbf{j}) e^{ie\lambda(\mathbf{j}, t_0)}. \quad (1.5)$$

1.2. Equation (1.4) gives the transformation of the variables $\psi, \mathbf{A}, A_0 \to \psi_1, \mathbf{B}, 0$. This transformation belongs to the class of transformations (1.3) with $\lambda(\mathbf{j}, t) = \Lambda(\mathbf{j}, t, t_0)$. This fact enables one to easily express the Lagrange function (1.1)-(1.2d) in terms of the quantities $B_s, s = 1, 2, 3$, and ψ_1:

$$L(\mathbf{A}, A_0, \psi) = L(\mathbf{B}, 0, \psi_1) \equiv L_1$$

$$= T(\dot{\mathbf{B}}) - U(\mathbf{B}) - D(\psi_1, \mathbf{B}) - D_4(\psi_1, 0). \quad (1.6)$$

As far as we know, this change of the variables descends from ref. [13]; it reflects the well-known fact that among the four components $A_\mu, \mu = 1, 2, 3, 4$, of the vector potential there are only three essentially independent functions. In order to overcome this point, one usually imposes some restriction on the functions A_μ, for instance, $\partial_\mu A_\mu = 0$ and, thus, fixes the gauge, cf. [3-4]. *This way one destroys the manifest gauge invariance of the consideration. Our consideration, on the contrary, is gauge invariant throughout.*

Now we introduce the lattice Fourier representation of the vector function $\mathbf{B}$ and function $\lambda(\mathbf{j}, t_0)$:

$$\mathbf{B} = \mathbf{B}^{tr} + \mathbf{B}^{long} + \mathbf{q}(0)/\sqrt{V},$$

$$B_s^{tr}(\mathbf{j}, \mathbf{j}+\hat{s}) = \sum_{\mathbf{n}\neq 0, \mathbf{n}\in R, \lambda=1,2} e(\mathbf{n}, \lambda)_s q(\mathbf{n}, \lambda) F^{(\mathbf{n}\cdot\mathbf{j})}/\sqrt{V}, F = e^{2\pi i/M},$$

$$B_s^{long}(\mathbf{j}, \mathbf{j}+\hat{s}) = \sum_{\mathbf{n}\neq 0, \mathbf{n}\in R} e(\mathbf{n}, 3)_s q(\mathbf{n}, 3) F^{(\mathbf{n}\cdot\mathbf{j})}/\sqrt{V},$$

$$\mathbf{e}(\mathbf{n}, \lambda_1)^* \cdot \mathbf{e}(\mathbf{n}, \lambda_2) = \delta_{\lambda_1, \lambda_2}, \quad \mathbf{e}(\mathbf{n}, \lambda)^* = \mathbf{e}(-\mathbf{n}, \lambda),$$

$$e(\mathbf{n}, 3)_s = (F^{n_s} - 1)/N(\mathbf{n}), \quad N^2(\mathbf{n}) = \sum_{s=1,2,3} 2[1 - \cos(2\pi n_s/M)], \tag{1.7}$$

$$\lambda(\mathbf{j}, t_0) = \sum_{\mathbf{n}\in R} l(\mathbf{n}, t_0) F^{(\mathbf{n}\cdot\mathbf{j})}/\sqrt{V}. \tag{1.7a}$$

Note the orthogonality relation

$$S \equiv \sum_{\mathbf{j}\in R} F^{(\mathbf{j}\cdot\mathbf{n})} = M^3 \text{ if } \mathbf{n} = 0 \bmod M \quad \text{and} \quad S = 0 \text{ otherwise}. \tag{1.7b}$$

One has

$$B_s^{long}(\mathbf{j}, \mathbf{j}+\hat{s}) = \frac{\Delta(s)}{a} f(\mathbf{j}),$$

$$f(\mathbf{j}) = a \sum_{\mathbf{n}\neq 0, \mathbf{n}\in R} \frac{q(\mathbf{n}, 3) F^{(\mathbf{n}\cdot\mathbf{j})}}{N(\mathbf{n})\sqrt{V}}. \tag{1.8}$$

1.2.1. Equations (1.2a) and (1.7) give

$$T(\dot{\mathbf{B}}) = \frac{1}{2}\{\dot{\mathbf{q}}(0)^2 + \sum_{\mathbf{n}\in R, \lambda=1,2,3; \mathbf{n}\neq 0} \dot{q}(\mathbf{n}, \lambda)\dot{q}(-\mathbf{n}, \lambda)\}. \tag{1.8a}$$

1.3. Introducing a new spinor variable ψ_2,

$$\psi_1(\mathbf{j}) = \psi_2(\mathbf{j}) e^{ief(\mathbf{j})}, \tag{1.9}$$

one can express the Lagrange function in terms of the variables $q(\mathbf{n}, \lambda), \mathbf{n} \neq 0, \mathbf{q}(0)$ and ψ_2:

$$L(\mathbf{B}, 0, \psi_1) = L_2 = T_1 - U(\mathbf{B}^{tr}) - D(\psi_2, \mathbf{B}^{tr} + \mathbf{q}(0)/\sqrt{V}) - D_4(\psi_2, 0),$$

$$T_1 = \sum_{\mathbf{n}\neq 0, \mathbf{n}\in R} [\frac{1}{2} \sum_{\lambda=1,2,3} \dot{q}(\mathbf{n}, \lambda)\dot{q}(-\mathbf{n}, \lambda) - \dot{q}(\mathbf{n}, 3)\frac{ae\rho(\mathbf{n})}{N(\mathbf{n})\sqrt{V}}]$$

$$+ \frac{1}{2}\dot{\mathbf{q}}(0)^2, \quad \rho(\mathbf{n}) = a^3 \sum_{\mathbf{j}\in R} \psi_2(\mathbf{j})^* \psi_2(\mathbf{j}) F^{(\mathbf{n}\cdot\mathbf{j})}. \tag{1.10}$$

Here, we have used the formulas

$$U(\mathbf{B}) = U(\mathbf{B}^{tr}) \text{ and } D(\psi_1, \mathbf{B}) = D(\psi_2, \mathbf{B}^{tr} + \mathbf{q}(0)/\sqrt{V}),$$

$$T(\mathbf{B}) + D_4(\psi_1, 0) = T_1 + D_4(\psi_2, 0). \tag{1.10a}$$

1.4. Now the standard recipe [3]

$$p_r(\mathbf{n}, \lambda) = \frac{\partial L_r}{\partial \dot{q}(\mathbf{n}, \lambda)_{\psi_r}}, \quad \pi_r(\mathbf{j}) = \frac{\partial L_r}{\partial \dot{\psi}_r(\mathbf{j})}, \quad r = 1, 2,$$

$$H_r = \sum_{\mathbf{n}, \lambda} p_r(\mathbf{n}, \lambda)\dot{q}(\mathbf{n}, \lambda) + \sum_{\mathbf{j}} \pi_r(\mathbf{j})\dot{\psi}_r(\mathbf{j}) - L_r, \tag{1.11}$$

enables one to compare the classical lattice spinor QED Hamiltonian to each of the Lagrange functions (1.6) and (1.10).

Putting here $r=2$, $p_2(\mathbf{n},\lambda)=-i\frac{\partial}{\partial q(\mathbf{n},\lambda)}_{\psi_2}$, expressing $q(\mathbf{n},\lambda)$ in terms of $p_2(\mathbf{n},\lambda)$ and ψ_2 and imposing the commutation relation

$$\psi_2(\mathbf{j}_1)\psi_2^*(\mathbf{j}_2)+\psi_2^*(\mathbf{j}_2)\psi_2(\mathbf{j}_1)=\delta_{\mathbf{j}_1,\mathbf{j}_2}/a^3. \qquad (1.12)$$

one gets the quantum lattice QED Hamiltonian

$$H_{QED,l}=H_{0ph,l}+D(\psi_2,\mathbf{B}^{tr}+\mathbf{q}(0)/\sqrt{V})+H_{c,l}-$$

$$-\frac{1}{2}(\frac{\partial}{\partial\mathbf{q}(0)})^2+H_{long,l}. \qquad (1.13)$$

Here, the Hamiltonian $H_{0ph,l}$ describes free transversal photons on the lattice,

$$H_{0ph,l}=\frac{1}{2}\sum_{\mathbf{n}\neq 0,\mathbf{n}\in R,\lambda=1,2}[-\frac{\partial}{\partial q(\mathbf{n},\lambda)}_{\psi_2}\frac{\partial}{\partial q(-\mathbf{n},\lambda)}_{\psi_2}$$

$$+\frac{N^2(\mathbf{n})}{a^2}q(\mathbf{n},\lambda)q(-\mathbf{n},\lambda)]\equiv H_{0ph,l}(kin)+H_{0ph,l}(pot), \quad (1.14)$$

the term $H_{c,l}$ is the lattice Coulomb Hamiltonian,

$$H_{c,l}=\frac{e^2a^2}{2V}\sum_{\mathbf{n}\neq 0,\mathbf{n}\in R}\frac{\rho(\mathbf{n})\rho(-\mathbf{n})}{N^2(\mathbf{n})}, \qquad (1.15)$$

the Hamiltonian $D(\psi_2,\mathbf{B}^{tr}+\mathbf{q}(0)/\sqrt{V})$ corresponds to the part $H_{0f}+H_1+H_{p1}$ of the Hamiltonian (2.2).

The longitudinal Hamiltonian $H_{long,l}$ is

$$H_{long,l}=\sum_{\mathbf{n}\neq 0,\mathbf{n}\in R}[-\frac{1}{2}\frac{\partial}{\partial q(\mathbf{n},3)}_{\psi_2}\frac{\partial}{\partial q(-\mathbf{n},3)}_{\psi_2}-\frac{iae\rho(-\mathbf{n})}{N(\mathbf{n})\sqrt{V}}\frac{\partial}{\partial q(\mathbf{n},3)}_{\psi_2}]. \qquad (1.16)$$

1.5. *We state that the Hamiltonian (1.13)-(1.16) is gauge invariant.* Actually, it follows from eqs. (1.3), (1.5) and (1.7-9) that if the potentials $A_\mu, \mu=1,2,3,4$, and the function ψ undergo gauge transformation (1.3), then the variables $q(\mathbf{n},\lambda)$, $\mathbf{q}(0)$ and ψ_2 undergo the transformation

$$q(\mathbf{n},\lambda)\to q(\mathbf{n},\lambda), \quad \lambda=1,2, \quad q(\mathbf{n},3)\to q(\mathbf{n},3)+$$

$$+l(\mathbf{n},t_0)N(\mathbf{n})/a, \quad \mathbf{q}(0)\to\mathbf{q}(0), \quad \psi_2\to\psi_2 e^{iel(0,t_0)/\sqrt{V}}. \qquad (1.17)$$

Equations (1.10)-(1.17) prove the statement of this item.

1.6. Note also that the Euler-Lagrange equations that are given by the Lagrange function (1.1) do not coincide with those given by the Lagrange function (1.6). Let us define the quantity $X(\mathbf{j},\mathbf{A},A_0,\psi)$ as

$$X\equiv X(\mathbf{j},\mathbf{A},A_0,\psi)=\sum_{s=1,2,3}\frac{\Delta(-s)}{a}[\frac{\partial}{\partial t}A_s(\mathbf{j},\mathbf{j}+\hat{s})+\frac{\Delta(s)}{a}A_0(\mathbf{j})]$$

$$-e\psi^*(\mathbf{j})\psi(\mathbf{j}).$$

The (only) difference is that while the Lagrange function (1.1) gives equation $X=0$, the Lagrange function (1.6) gives equation $Y\equiv\partial X(\mathbf{j},\mathbf{B},0,\psi_1)/\partial t=0$. First of these two equations (the well-known QED Gauss law) entails the equality

$$\sum_{\mathbf{j}\in R}\psi^*(\mathbf{j})\psi(\mathbf{j})=0 \text{ if } e\neq 0:$$

the total charge of the Universe equals zero. But the sum here is positive. Thus, we feel that this statement looks strange and dubious. The second equation, $Y=0$, is free from this weak, in our opinion, point (it entails only the conservation of the total charge of the Universe).

Sec. II. CONTINUOUS SPACE QED IN A PERIODICITY CUBE. MAIN FORMULAE AND CRITICISM

The continuous space spinor QED is determined by the Lagrangian

$$\mathcal{L} = -(\partial_\mu A_\nu - \partial_\nu A_\mu)^2/4 - \psi^* \gamma_4[\gamma_\mu(\partial_\mu - ieA_\mu) + m]\psi. \quad (2.1)$$

Here, e is the electron charge and m, the electron mass parameter. Starting with the Lagrangian (2.1) one can derive the QED Hamiltonian. We give here the following continuous space QED Hamiltonian

$$H_{QED} = H_{0ph} + H_{0f} + H_1 + H_c + H_{p0} + H_{p1} + H_{long}. \quad (2.2)$$

2. The seven terms in the r.h.s. of eq. (2.2) are defined by eqs. (1.14), (2.3), (2.4), (2.5), (2.7), and (1.16).

The Hamiltonian (2.2) depends on the electromagnetic field variables $q(\mathbf{k},\lambda), \mathbf{k} \neq 0, \lambda = 1,2,3$ and $\mathbf{q}(0)$ (these variables are defined by eq.(2.6)) and on the derivatives with respect to these variables. It depends also on the fermion operators $\psi_2 \equiv \psi_2(\mathbf{x})$ and $\psi_2^* \equiv \psi_2^*(\mathbf{x})$ which obey the standard anticommutation rules. The functions $\mathbf{B} \equiv \mathbf{B}(\mathbf{x})$ (see eq. (2.6)) and function (operator) ψ_2 are connected with the potentials $A_\mu, \mu = 1,2,3,4$ and the function $\psi \equiv \psi(\mathbf{x}, t)$ of the Lagrangian (2.1) by the formulas which are continuous space analogues of the lattice eqs. (1.4), (1.8) and (1.9).

2.1. Formally, one can get the Lagrangian (2.1) considering the limit $a \to 0, L = const$ of the lattice Lagrange function (1.1), (1.2), In order to formally derive the Hamiltonian (2.2) one also takes the limit $a \to 0$ of the lattice Hamiltonian of **Sec. I**, see item 2.3.3.

2.2. In eq. (2.2.) the Hamiltonian H_{0ph} describes free transversal photons, its expression formally coincides with that given by eq. (1.14) where, however, one has to substitute $\mathbf{k}$ and $|\mathbf{k}|$ for $\mathbf{n}$ and $N(\mathbf{n})/a$ and change the summation region. The summation variable $\mathbf{k}$ accepts values $\mathbf{k} = 2\pi\mathbf{n}/L$, the components of the vector $\mathbf{n}$ are integer. *This time there is no restriction* $\mathbf{n} \in R$.

The Hamiltonian H_{0f} describes free fermions (electrons and positrons),

$$H_{0f} = \int_V \psi_2^* \gamma_4[\sum_{s=1,2,3} \gamma_s \partial/\partial x_s + m]\psi_2 d^3x, \quad (2.3)$$

the Hamiltonian H_1 describes the interaction of the fermions and transversal photons,

$$H_1 = -ie \int_V \psi_2^* \gamma_4 \sum_{s=1,2,3} \gamma_s B_s^{tr} \psi_2 d^3x, \quad (2.4)$$

and the Hamiltonian H_c describes the Coulomb interaction of fermions,

$$H_c = \frac{e^2}{2V} \sum_{\mathbf{k}\neq 0} \frac{\hat{\rho}(\mathbf{k})\hat{\rho}(-\mathbf{k})}{\mathbf{k}^2}, \quad \hat{\rho}(\mathbf{k}) = \int_V \psi_2^* \psi_2 e^{i(\mathbf{k}\cdot\mathbf{x})} d^3x. \quad (2.5)$$

These first four terms of the Hamiltonian (2.2) are known since long ago, [3-6].

2.2.1. In the last time it became clear that when considering QED in a periodicity cube V it is necessary to take into account the zero momentum mode of the vector potential: the vector potential consists not only of transversal and longitudinal parts, it contains also the spatially independent zero momentum part [9]:

$$\mathbf{B}(\mathbf{x}) = \mathbf{B}(\mathbf{x})^{tr} + \mathbf{B}(\mathbf{x})^{long} + \mathbf{q}(0)/\sqrt{V}.$$

$$\mathbf{B}(\mathbf{x})^{tr} = \sum_{\mathbf{k}\neq 0, \lambda=1,2} \mathbf{e}(\mathbf{k},\lambda) q(\mathbf{k},\lambda) e^{i(\mathbf{k}\cdot\mathbf{x})}/\sqrt{V},$$

$$\mathbf{B}(\mathbf{x})^{long} = \sum_{\mathbf{k}\neq 0} \mathbf{e}(\mathbf{k},3) \dot{q}(\mathbf{k},3) e^{i(\mathbf{k}\cdot\mathbf{x})}/\sqrt{V},$$

$$\mathbf{e}(\mathbf{k},\lambda_1)^*\mathbf{e}(\mathbf{k},\lambda_2) = \delta_{\lambda_1,\lambda_2}, \quad \mathbf{e}(\mathbf{k},\lambda)^* = \mathbf{e}(-\mathbf{k},\lambda),$$

$$\mathbf{e}(\mathbf{k},3) = i\mathbf{k}/|\mathbf{k}|. \qquad (2.6)$$

The zero momentum degree of freedom (that was disregarded in old QED considerations) contributes new terms H_{p0} and H_{p1} to the QED Hamiltonian [7],

$$H_{p0} = -\frac{1}{2}(\frac{\partial}{\partial \mathbf{q}(0)})^2, \quad H_{p1} = -\sum_{s=1,2,3} \frac{ieq(0)_s}{\sqrt{V}} \int_V \psi_2^* \gamma_4 \gamma_s \psi_2 d^3x. \qquad (2.7)$$

If one needs to take into account in the Hamiltonian also the longitudinal degrees of freedom, one has to add to the QED Hamiltonian the term H_{long}, [16]. One can get this term, substituting $\mathbf{k}, |\mathbf{k}|$ and $\hat{\rho}(-\mathbf{k})$ for $\mathbf{n}, N(\mathbf{n})/a$ and $\rho(-\mathbf{n})$ in eq. (1.16).

2.3. Now we shall criticise the continuos space approach to the QED. We are going to substantiate the statement that the *lattice approach to the QED is obviously preferable to the continuous space approach.* This our statement is not new [14], but some of our arguments, maybe, are: see items **2.3.1.-2.3.3.**, **2.3.5.** and **2.3.6.1.**.

2.3.1. An important moment of the derivation of the lattice Hamiltonian, eq. (1.13), from the Lagrange function, eq. (1.1), are two sequential changes of the field variables, eqs. (1.4) and (1.9), which enable one to get new and more convenient representations, eqs. (1.6) and (1.10), of the Lagrange function.

In order to produce analogous changes of the variables in the continuous space approach, one has to use the differentiation formula of the type

$$\frac{\partial}{\partial x_\mu} e^{F(x)} = e^{F(x)} \frac{\partial F(x)}{\partial x_\mu}$$

. This formula, however, is correct only if the function $F(x)$ is continuous in the variables $x_\mu, \mu = 1, 2, 3, 4$.

2.3.2. Thus, the function

$$\Lambda(\mathbf{x}, t, t_0) = \int_{t_0}^{t} A_0(\mathbf{x}, s) ds,$$

(eq. (1.4)), has to be continuous in $\mathbf{x}$ and t and the function

$$f(\mathbf{x}) = \int_V d^3y \sum_{\mathbf{k}\neq 0} \frac{e^{i(\mathbf{k}\cdot[\mathbf{x}-\mathbf{y}])}(\mathbf{k}\cdot\mathbf{A}(\mathbf{y}))}{V\mathbf{k}^2};$$

(eqs. (1.8-9)), has to be continuous in $\mathbf{x}$.

In order to ensure, e.g., the continuity of the function $f(\mathbf{x})$ one has to impose some restrictions on the variables $q(\mathbf{k}, 3)$ while the Hamiltonian (2.2) assumes no such restrictions. This defect is inherent to all the continuous space derivations of the QED Hamiltonian.

2.3.2.1. *By the way, the remark of item 2.3.1. narrows the class of gauge transformations, which leave the Lagrangian (2.1) invariant: the function $\lambda(\mathbf{x}, t)$ should be continuous.*

2.3.3. The papers [7] and [8] pretend to have proven that the continuous space spinor QED Hamiltonian is unbounded from below. There exists no such claims for the lattice QED. Moreover, it is evident that the the consideration of paper [7] does not work in lattice QED: in order to get the result [7] one has to substitute the exponential $exp \equiv \exp(-ieaq(0)_s/\sqrt{V})$

(see eqs. (1.13) and (1.2d)) by two first terms of its decomposition ($exp = 1 - ieaq(0)_s/\sqrt{V}$). Actually, let us consider the quantity

$$Q \equiv \{\psi_2(\mathbf{j})^* \gamma_4 \gamma_s \psi_2(\mathbf{j}+\hat{s}) e^{-iea[B_s^{tr}(\mathbf{j},\, \mathbf{j}+\hat{s}) + q_s(0)/\sqrt{V}]}$$

$$- \psi_2(\mathbf{j})^* \gamma_4 \gamma_s \psi_2(\mathbf{j}-\hat{s}) e^{iea[B_s^{tr}(\mathbf{j}-\hat{s},\, \mathbf{j}) + q_s(0)/\sqrt{V}]}\}/(2a), \qquad (2.8)$$

see eq. (1.2d). Formally, one has

$$Q \to \psi_2(\mathbf{x})^* \gamma_4 \gamma_s \{\frac{\partial}{\partial x_s} - ie[B_s^{tr}(\mathbf{x}) + q_s(0)/\sqrt{V}]\}\psi_2(\mathbf{x}) \quad \text{as} \quad a \to 0. \qquad (2.9)$$

However, there is no reason to believe that the zero momentum mode variable $\mathbf{q}(0) \equiv [q_1(0), q_2(0), q_3(0)]$ *is restricted by the condition* $|eaq_s(0)/\sqrt{V}| \ll 1$ as $a \to 0$ which is necessary to derive eq. (2.9). *Thus, the limit (2.9) looks like to be wrong.* Paper [16] illustrates this statement by the example of the massive QED in two space time dimensions.

As for the more sophisticated consideration of paper [8], it also does not work in the lattice QED (see our forthcoming paper [15]). *The above remark shows that the $a \to 0$ procedure of item 2.1. of the derivation of the continuous space QED Hamiltonian from the lattice QED Hamiltonian is unlawful.*

2.3.3.0. Let us represent the free photon lattice *Hamiltonian* , eq. (1.14), as

$$H_{0ph,l} = \sum_{\mathbf{n} \in R, \lambda = 1,2} \frac{N(\mathbf{n})}{a} c(\mathbf{n},\lambda)^* c(\mathbf{n},\lambda) + const$$

where $c(\mathbf{n},\lambda)^*$ and $c(\mathbf{n},\lambda)$ are the photon creation and annihilation operators.

Let us also consider the quantity $E \equiv e^{ieaB_s^{tr}(\mathbf{j},\mathbf{j}+\hat{s})} \equiv e^W$.

It is possible to represent E in the normal form:

$$E = \sum_{n=0}^{\infty} E_n,$$

here, E_0 does not depend on the operators $c(\mathbf{n},\lambda)^*$ and $c(\mathbf{n},\lambda)$, E_2 is the second order normal form in these operators, etc. Using the definitions of this item, the lattice Fourier representation of the function $\mathbf{B}^{tr}$, eq. (1.7), and the formula

$$q(\mathbf{n},\lambda) = \sqrt{a/[2N(\mathbf{n})]}[c(\mathbf{n},\lambda) + c(-\mathbf{n},\lambda)^*],$$

one can show (see also [15]) that, for instance,

$$E_0 = 1 - \sum_{\mathbf{n} \in R, \mathbf{n} \neq 0} \frac{e^2 |e_s(\mathbf{n},\lambda)|^2}{4M^3 \{\sum_{r=1,2,3}[1 - \cos(2\pi n_r/M)]\}^{1/2}} + O(e^4) etc.$$

Here, all the coefficients by the powers of e^2 have finite limits as $a \to 0$.

We feel that this result contradicts the statement by Wilson (see [1], in between eqs. (3.11) and (3.12)).

So, one chooses the lattice *Lagrange function* so as to ensure the correspondence with the continuous approach *Lagrangian* as $a \to 0$.

But consideration of this item shows that, however small be the lattice length a, all the terms of the expansion $E = \sum_{n=0}^{\infty} W^n/(n!)$ have to be retained - one cannot neglect here the terms$\sim W^n, n > 1$.

Thus, the mentioned above correspondence does not hold.

This our statement, essentially, is not new: Wilson [17] has pointed out the possibility to introduce into a lattice Action terms which formally tend to zero as $a \to 0$ but which, in fact, influence the properties of a quantum system.

(Taking this Wilson's terms into account in our consideration there changes only the operator $D(\psi_2, \mathbf{B}^{tr} + \mathbf{q}(0)/\sqrt{V})$ in eq. (1.13)).

2.3.3.0.1. Maybe, one should use the freedom of choice of such terms to ensure the Lorentz invariance of the theory (see also items 2.3.4.-5.).

2.3.3.1. One should note also that the consideration of paper [7] is obviously non-gauge invariant. The consideration of paper [8] pretended to be faultless from the point of view of gauge invariance. In fact, however, this claim is wrong: it is based on work [13] which missed to notice the point which is discussed in items **2.3.1-2.**

2.3.4. And our last remark concerns the cut-off problem. The lattice approach to a QFT model is equivalent to imposing some definite method of the cut-off: this method depends on a choice of the finite difference representation of the terms with spatial derivatives which enter into initial continuous space *Lagrangian* . Analogously, the continuous space approach meets the problem how to choose the form factor.

2.3.5. In fact, when introducing the cut-off into the Hamiltonian (a realistic regularization), – and such a cut-off cannot be introduced manifestly Lorentz invariantly, for the Hamiltonian itself is not Lorentz invariant – one meets the formidable problem of choosing form factors so that not to destroy the Lorentz invariance of the theory ([8], items 1.4.-1.4.4.).

Maybe, this problem is more easy to be solved using the lattice approach (item 2.3.4.).

2.3.6. At present, almost nobody uses a realistic regularisation methods – but mostly formalistic ones, choosing them so that to ensure the Lorentz invariance, the gauge invariance, etc., of the theory from the very beginning.

2.3.6.1. *We should like to stress here that a formalistic regularization leads to loosing a contact with the Shroedinger equation*

$$(H - i\frac{\partial}{\partial t})\Omega = 0 \qquad (2.10)$$

which is the foundation of any QFT model (in eq. (2.10) H is the Hamiltonian of the model and Ω is the wave function). Thus, the formalistic regularization approach, however Lorentz invariant, is fraught with creating formidable problems.

SUMMARY

Our derivation of the QED Hamiltonian is obviously preferable to other derivations we know.

a) Our derivation is throughout gauge invariant; it entirely avoids the problem of fixing the gauge (see ,e.g, [3-6]). We do not impose restrictions like $\partial_\mu A_\mu = 0$, or $div\mathbf{A} = 0$, or $A_0 = 0$, etc., on the components of the vector potential. Introduction of the variables $\mathbf{B}(\mathbf{j})$ and $\psi_1(\mathbf{j})$, eq. (1.4), though related to the formulas determining the De-Witt gauge, is by no means connected with any fixation of the gauge.

b) Our derivation entirely avoids problems connected with the equality $\pi_4 \equiv \partial\mathcal{L}/\partial\dot{A}_4 = 0$ – see [3-6].

c) Our derivation of the QED Hamiltonian takes account of the zero momentum mode of the vector potential, see item **2.2.1.** which was disregarded in old considerations [3-6].

d) Equations (1.13) and (1.16) exhibit the QED Hamiltonian dependence on the derivatives with respect to the longitudinal degrees of freedom variables which was also disregarded in old considerations.

e) Our derivation reveals the fact that the QED Hamiltonian is gauge invariant, item 1.5. The Hamiltonian depends on gauge invariant variables $q(\mathbf{n}, \lambda), \lambda = 1, 2, \mathbf{q}(0)$ and the spinor variables $\psi_2(\mathbf{j}), \mathbf{j} \in R$ which are gauge invariant up to a spatially independent phase. As for the gauge dependent variables

$q(\mathbf{n},3)$, the Hamiltonian depends only on the derivatives $\frac{\partial}{\partial q(\mathbf{n},3)}$ which are gauge invariant, see eq. (1.17).

The consideration of this work essentially repeats the continuous space consideration of ref. [13] but for the defect which is noticed in items 2.3.1.-2 and which is inherent to continuous space QED treatment.

Our important conclusion is that it is impossible to give the continuous space consideration of QED, perfect from the point of view of gauge invariance (while our lattice consideration of QED seems to be irreproachable from this point of view).

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APPENDIX. ANOTHER DERIVATION OF EQ. (1.13)

Here we give a new derivation of the lattice QED Hamiltonian. This time we start with the Lagrange function L_1, eq. (1.6) and the classical Hamiltonian H_1, eq. (1.11). The standard prescription [3] gives the quantum Hamiltonian

$$H_{QED,l} = T + U(\mathbf{B}) + D(\psi_1, \mathbf{B}), \qquad (A1)$$

$$T = -\frac{1}{2}[(\frac{\partial}{\partial \mathbf{q}(0)})^2 + \sum_{\mathbf{n}\in R, \lambda=1,2,3; \mathbf{n}\neq 0} \frac{\partial}{\partial q(\mathbf{n},\lambda)_{\psi_1}} \frac{\partial}{\partial q(-\mathbf{n},\lambda)_{\psi_1}}]. \qquad (A2)$$

The operators ψ_1^* and ψ_1 satisfy the same commutation relations as the operators ψ_2^* and ψ_2, see eq. (1.12).

Now let us introduce a new spinor variable ψ_2 using eq. (1.9). The operators ψ_2 and ψ_2^* do depend on the variables $q(\mathbf{n},3), \mathbf{n} \in R$, so that one has

$$[\frac{\partial}{\partial q(\mathbf{n},\lambda)_{\psi_1}}, \psi_2(\mathbf{j})] = \psi_2(\mathbf{j}) \frac{-ieaF^{(\mathbf{j}\cdot\mathbf{n})}}{N(\mathbf{n})\sqrt{V}} \delta_{\lambda,3}. \qquad (A3)$$

This formula, the analogous formula for the function ψ_2^* and eq. (1.12) enable one to get the equality

$$\frac{\partial}{\partial q(\mathbf{n},\lambda)_{\psi_1}} = \frac{\partial}{\partial q(\mathbf{n},\lambda)_{\psi_2}} + \frac{iea\rho(\mathbf{n})}{N(\mathbf{n})\sqrt{V}} \delta_{\lambda,3}. \qquad (A4)$$

It follows from this equation and eq. (1.10a) that the Hamiltonians (A1-A2) and (1.13) do coincide.

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