сообщения
объединенного
института
ядерных
исследований
дубна

E2-93-292

A.D.Popov

# GEOMETRIC QUANTIZATION OF SYMPLECTIC MANIFOLDS WITH SYMMETRY

1993

## 1. Introduction

Constrained systems are often considered in physics and therefore their quantization deserves particular attention. To quantize such systems one usually uses the method of canonical quantization or the method of path integral quantization. The method of geometric quantization of Kostant and Souriau [1–4] is a generalization of the standard canonical quantization on the curved phase manifolds $M$. Geometric quantization of the constrained systems have been considered in Refs.[5–11]. In these papers it has been supposed that the $G$-invariant polarization $P$ exists on the symplectic manifold $(M, \omega)$. We consider a positive Kähler polarization corresponding to the complex structure $J$ on $M$ and impose the weakened form of the symmetry requirement. Namely, we describe the homomorphism $\tau$: $G \rightarrow Aut\mathcal{P}$ of the group $G$ into the group of the automorphisms of the bundle $\mathcal{P} = \mathcal{P}(M, Sp(2n, R))$ over $M$ and impose the condition of invariance of the complex structure $J$ under the group $\tau(G)$ (weak $G$-invariance). In this paper we present the geometric quantization scheme of the Kähler manifolds $(M, \omega, J)$ with weakly $G$-invariant complex structure $J$.

## 2. Marsden-Weinstein reduction

**2.1.** On a symplectic manifold $M$ with a 2-form $\omega$ for two arbitrary functions $f$ and $h$ one can define a Poisson bracket $\{f, h\} = \omega(X_f, X_h)$. Here a vector field $X_f$ is defined by the formula $X_f \rfloor \omega = -df$. A correspondence $f \rightarrow X_f$ maps a Lie algebra $C^\infty(M)$ of functions on $M$ (with the Poisson bracket) to the Lie algebra of the Hamiltonian vector fields on $M$ (with the ordinary commutator).

**2.2.** Let $G$ be a Lie group and $\alpha(G)$ a representation of $G$ in the group of the symplectomorphisms of $M$. Then a Lie algebra $\alpha_*(\mathcal{G})$ of the Lie group $\alpha(G)$ acts on $M$ as a Lie algebra of vector fields. This action is called a Hamiltonian action if to each vector field $X_\xi$ ($\xi \in \mathcal{G}$) one may correspond a function $\varphi_\xi \in C^\infty(M)$ by the formula

$$X_\xi \rfloor \omega = -d\varphi_\xi \tag{2.1}$$

**2.3.** Let $\mathcal{G}^*$ be a space dual to $\mathcal{G}$. Using functions $\varphi_\xi$ on $M$ one may define a momentum map $\varphi$: $M \rightarrow \mathcal{G}^*$ by the formula

$$< \varphi(x), \xi >= \varphi_\xi(x), \tag{2.2}$$

where $\xi \in \mathcal{G}$, $\varphi(x) \in \mathcal{G}^*$.

**2.4.** Let us consider a constraint set $\varphi_\xi = 0$, $\forall \xi \in \mathcal{G}$. In such a situation the reduced phase space $M_G$ is obtained as the quotient [12]

$$M_G = M_0/\alpha(G), \tag{2.3}$$

where $M_0 = \varphi^{-1}(0)$. For the description of conditions under which $M_G$ will be a manifold, see [6, 13]. It may be shown [12, 4] that there is a natural symplectic structure $\omega_G$ on the space $M_G$.

## 3. Geometric quantization

**3.1.** To quantize a classical system with the phase space $(M, \omega)$ means to construct an irreducible unitary Lie algebra representation

$$r : C^\infty(M) \to EndH_0 \tag{3.1}$$

of the algebra $C^\infty(M)$ in the algebra $EndH_0$ of linear self-adjoint operators in some complex Hilbert space $H_0$. To do this, it is necessary to introduce

i) a prequantization bundle $L$ over $M$;

ii) a polarization $P$ of $M$;

iii) a metaplectic structure on $M$.

**3.2.** We define the prequantization bundle $L$ over $M$ as a complex line bundle with the covariant derivative $\nabla$ (the connection) compatible with the Hermitian structure $<,>$ in fibres, the curvature 2-form $F_\nabla$ of which ($F_\nabla(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$) coincides with the symplectic 2-form $\omega$ on $M$.

**Proposition** [1-4]. *The prequantization bundle $L$ over $(M, \omega)$ exists if and only if the cohomology class of $\omega$ is integral.*

**3.3.** Take for the Hilbert space $H$ of prequantization the space $\mathcal{L}^2(M, L)$ defined as the completion of the space of smooth sections of $L$ over $M$ with compact supports with respect to the inner product

$$(s,t) = \int_D < s, t > \omega^n, \tag{3.2}$$

where $<,>$ is given by the Hermitian structure of $L$. Then we define the Kostant-Souriau prequantization $r : C^\infty(M) \to EndH$ by setting

$$r(f) = f - i\nabla_{X_f} \tag{3.3}$$

The introduced Hilbert space $H$ is too large to represent the phase space $(M, \omega)$ and we need a polarization.

**3.4.** Let $TM$ be a tangent bundle over $M$ and $T^C M = TM \otimes C$ its complexification. We call by a polarization of $(M, \omega)$ a subbundle $P \subset T^C M$ such that

i) a fibre $P_x \subset T_x^C M$ is a Lagrangian subspace in $T_x^C M$ for all $x \in M$, i.e. the restriction of $\omega$ to $P_x$ vanishes and $dim P_x = n$;

ii) a space of sections of the bundle $P$ is closed under the Lie bracket.

If $X \to \bar{X}$ is a complex conjugation then a subbundle $\bar{P}$ will also be a polarization. The polarization $P$ is called the Kähler polarization if $P \cap \bar{P} = 0$, i.e. $T_x^C M = P_x \oplus \bar{P}_x$ for any $x \in M$.

**3.5.** Let $P$ be a polarization $P \subset T^C M$ of a symplectic manifold $(M, \omega)$. Then we can introduce the Hilbert space of quantization

$$H_P = \left\{ \psi \in \mathcal{L}^2(M, L) : \quad \nabla_X \psi = 0, \ \forall X \in \Gamma(M, P) \right\}, \qquad (3.4)$$

where by $\Gamma(W, V)$ we denote the space of sections of a bundle $V$ over $W$.

**3.6.** The introduction of a metaplectic structure on $M$ is equivalent to the extention of the structure group of the bundle $TM$ from the symplectic group $Sp(2n, R)$ to the metaplectic group $Mp(2n, R)$ which is the connected double covering of $Sp(2n, R)$.

**Proposition** [1–4]. *A metaplectic structure on $M$ exists if and only if the 1st Chern class of $M$ is even.*

**3.7.** For the fixed positive Kähler polarization $P$ of $M$ (see [14]) we consider the spaces $P_x^*$ and $\bar{P}_x^*$ of the 1-forms dual to $P_x$ and $\bar{P}_x$. These spaces are sections of the bundles $P^*$ and $\bar{P}^*$ accordingly. Now let us introduce the determinant bundles $Det_P^* = \bigwedge^n P^*$ and $Det_P = \bigwedge^n \bar{P}^*$ [4].

The existence of the metaplectic structure on $(M, \omega)$ is equivalent to the existence of a line bundle $Pf_P$ over $M$ such that $(Pf_P)^2 = Det_P$ [3, 4]. Analogously, there exists $Pf_P^* : (Pf_P^*)^2 = Det_P^*$.

**3.8.** Let us now consider the bundle $L \otimes Pf_P$. A connection in $L$ can be extended to the connection in $L \otimes Pf_P$ (see [4]) and we obtain a covariant derivative $\nabla'$ acting on the sections $s'$ of the bundle $L \otimes Pf_P$. In a corrected quantization scheme, the $P$ wave functions are defined to be the smooth sections of $L \otimes Pf_P$ such that $\nabla'_X s' = 0, \quad \forall X \in P$.

## 4. $G$-invariant polarizations

**4.1** Let $M$ be a symplectic manifold possessing a connected Lie group $G$ of symmetries. Let $G \times M \to M$ be a Hamiltonian action of $G$ on $M$ and $\varphi : M \to \mathcal{G}^*$ be the associated momentum map (see Sect.2). There is a canonical representation of the Lie algebra $\mathcal{G}$ on smooth sections of $L$ given by the operators (see Sect.3.3):

$$r(\varphi_\xi) = \varphi_\xi - i\nabla_{X_\xi}, \tag{4.1}$$

where $\varphi_\xi \in C^\infty(M)$ and $X_\xi \in \alpha_*(\mathcal{G})$ correspond to $\xi \in \mathcal{G}$ (see (2.1)). We suppose that there exists a global action of $G$ on $L$ such that the induced action of $\mathcal{G}$ is given by (4.1).

**4.2** Consider the submanifolds $M_0 = \varphi^{-1}(0)$ and $M_G = M_0/\alpha(G)$ in $M$. Let $\chi$: $M_0 \to M_G$ be the projection map and $\eta$: $M_0 \to M$ be the inclusion map.

**Theorem**[6]. *There is unique line bundle $(L_G, \nabla^G)$ with connection $\nabla^G$ on $M_G$ such that*

$$\chi^* L_G = \eta^* L \quad \text{and} \quad \chi^* \nabla^G = \eta^* \nabla. \tag{4.2}$$

*The curvature of the connection $\nabla^G$ is the symplectic form $\omega_G$.*

Since the Hermitian inner product $<,>$ is $G$-invariant, there is a unique Hermitian inner product $< , >_G$ on $L_G$ such that $\chi^* < , >_G = \eta^* < , >$. Thus $L_G$, $\nabla^G$ and $< , >_G$ are prequantum data of $M_G$.

**4.3** Let $P$ be a polarization of $M$. It is clear that we may associate with $P$ a polarization $P_G$ of the reduced space $M_G$ if and only if the polarization $P$ is invariant with respect to the action of the group $G$ [6-12]. In particular, in [6] the following theorem has been proved:

**Theorem.** *Let $G$ be a connected compact Lie group, $M$ a Hamiltonian $G$-space and $P$ a $G$-invariant, positive definite Kähler polarization of $M$. Then there is canonically associated with $P$ a positive definite polarization $P_G$ of the reduced space $M_G$.*

**4.4.** Having polarizations $P$ and $P_G$, one may introduce the following Hilbert spaces

$$H_P^G = \{\psi \in H_P : \; r(\varphi_\xi)\psi = 0, \; \forall \xi \in \mathcal{G}\}$$

$$H_{P_G} = \left\{\phi \in \mathcal{L}^2(M_G, L_G) : \; \nabla_X^G \phi = 0, \; \forall X \in P_G\right\} \tag{4.3}$$

In [6–11] it has been proved that these spaces are isomorphic as vector spaces:

$$H_P^G \cong H_{P_G}.$$

We shall not discuss here the more difficult questions connected with the introduction of an inner product on $H_P^G$ and $H_{P_G}$ (see [4, 6–11]).

**4.5.** Generally speaking, it is difficult to properly correlate the quantization of the extended phase space $M$ and the reduced phase space $M_G$. In Sect.4.2-4.4 it was accomplished by requiring that the auxiliary structures on $(M, \omega)$ necessary for quantization (in particular — polarization $P$) be $G$-invariant. Then they can be projected to compatible quantization structures on $(M_G, \omega_G)$. But the condition of $G$-invariance of polarization $P$ does not always take place and we shall consider the possibility of its weakening.

## 5. Weakly $G$-invariant complex structures

**5.1.** Suppose we choose a positive Kähler polarization of the symplectic manifold $M$. This polarization is in the one-to-one correspondence with the complex structure $J$ on $M$. It consists of a linear operator $J$ from $TM$ to itself such that $J^2 = -1$ and

$$[JX, JY] = [X, Y] + J[JX, Y] + J[X, JY], \quad X, Y \in \Gamma(M, TM) \quad (5.1)$$

Condition (5.1) means that the almost complex structure $J$ is integrable and one may introduce the Kähler metric

$$g(X, Y) = \omega(X, JY) \quad (5.2)$$

with Kähler form $\omega$.

**5.2.** Let $f$ be a diffeomorphism of the manifold $M$ and $f_*$ an isomorphism of the tangent space $T_{f^{-1}(x)}M$ onto the tangent space $T_x M$. This isomorphism may be extended up to the isomorphism of the tensor algebra in $T_{f^{-1}(x)}M$ and the tensor algebra in $T_x M$ [15]. This isomorphism we shall denote by $\tilde{f}$. For any tensor field $B$ we shall define the tensor field $\tilde{f}B$ in the following way:

$$(\tilde{f}B)_x = \tilde{f}\left(B_{f^{-1}(x)}\right), \quad x \in M. \quad (5.3)$$

**5.3.** Let $X$ be a vector field on $M$ and $f_t = e^{tX}$ a one-parameter group of transformations, generated by $X$. For each $t$ we have an automorphism

$\tilde{f}_t$ of the tensor algebra on $M$. For any tensor field $B$ on $M$ a Lie derivative $\mathcal{L}_X B$ is defined by the formula [15]:

$$\mathcal{L}_X B = -\lim_{t \to 0} \frac{1}{t} \left[ \tilde{f}_t B - B \right]. \tag{5.4}$$

If the $k$-parameter subgroup $\alpha(G)$ of the symplectomorphisms group acts on $M$ then we have $k$ Hamiltonian vector fields $X_\xi$ ($\xi \in \mathcal{G}$), to which the one-parameter group of transformations $\tilde{f}_g$ ($g = exp(t\xi)$, $\alpha(g) \equiv f_g = exp(tX_\xi)$) corresponds. Thus, we may define the derivatives $\mathcal{L}_{X_\xi} B$ for any tensor field $B$.

**5.4.** Consider the principal $Sp(2n, R)$-bundle

$$p : \mathcal{P} \longrightarrow M \tag{5.5}$$

of symplectic frames on $M$ and the group $Aut\mathcal{P}$ of all automorphisms of $\mathcal{P}$ (which are bundle maps). A map $\lambda$ of $\mathcal{P}$ onto $\mathcal{P}$ will be called an automorphism of principal fibration if $\lambda(qb) = \lambda(q)b$ for every $q \in \mathcal{P}, b \in Sp(2n, R)$. Each automorphism $\lambda$ determines a transformation of the base $M = \mathcal{P}/Sp(2n, R)$; we shall denote this transformation by $p(\lambda)$.

The group of automorphisms of the principal fibration $\mathcal{P}$ determining an identity transformation of the base will be denoted by $Gauge\ Sp(2n, R)$. Gauge transformations of the space $\mathcal{P}$ are defined by smooth functions $\tau(x)$ on $M$ with values in the group $Sp(2n, R)$ and a set $Gauge\ Sp(2n, R)$ of all $\tau(x)$ may be identified with the space of sections of the associated bundle $\mathcal{P} \times_{Sp(2n,R)} Sp(2n, R) \to M$.

We have a homomorphism $\alpha$ of the connected Lie group $G$ into $Symp(M, \omega)$. The group $Symp(M, \omega)$ of the canonical transformations of the manifold $M$ is a subgroup in the group $p(Aut\mathcal{P})$ which preserves the symplectic form $\omega$. Thus we defined the action of two groups on $\mathcal{P}$: the action of the group $\alpha(G) \subset Symp(M, \omega) \subset Aut\mathcal{P}$ and the action of the group $Gauge\ Sp(2n, R) \subset Aut\mathcal{P}$. Let us also consider a group $Aut_G = \alpha(G) \times GaugeG$, where $GaugeG \subset GaugeSp(2n, R)$. The group $Aut_G$ is the group of pairs $(\alpha(g), \tau(x))$, where $\alpha(g) \in \alpha(G)$, $\tau(x) \in GaugeG$ and the product of pairs $(\alpha(g_1), \tau_1(x)) \in Aut_G$, $(\alpha(g_2), \tau_2(x)) \in Aut_G$ is a pair $(\alpha(g), \tau(x))$ given by formula

$$\alpha(g) = \alpha(g_1)\alpha(g_2), \quad \tau(x) = \tau_1(x)\tau_2(x). \tag{5.6}$$

5.5 The action of $\alpha(G)$ on $M$ induces an action on $J$. This action on the tensor of the complex structure is given by

$$\alpha(g)J := \tilde{f}_g J \tag{5.7}$$

Analogously, the action of $Gauge\ G$ on $\mathcal{P}$ induces the following action on $J \in Gauge\ Sp(2n, R)$:

$$\tilde{\tau} J := \tau J \tau^{-1} \Longleftrightarrow (\tilde{\tau} J)_x = \tau(x) J_x \tau^{-1}(x) \tag{5.8}$$

Finally, the action of the group $Aut_G$ on $J$ has the form

$$\gamma(g, \tau)J := \tilde{\tau}(\alpha(g)J). \tag{5.9}$$

Suppose that there exists a homomorphism $g \to \{\tau_g(x)\}$ of the group $G$ into the group $Gauge\ G \subset Gauge\ Sp(2n, R)$. Then we may define a homomorphism $\tau$ of the group $G$ into the group $Aut_G$ by the correspondence of an element

$$\tau(g) = (\alpha(g), \tau_g(x)) \in Aut_G \tag{5.10}$$

to the element $g \in G$ (cf. [16]). The action of $\tau(G)$ on $\mathcal{P}$ induces the following action of the group $\tau(G) \subset Aut_G$ on $J$:

$$\tau(g)J := \tilde{\tau}_g(\alpha(g)J) \Longleftrightarrow (\tau(g)J)_x = \tau_g(x)(\tilde{f}_g J)_x \tau_g^{-1}(x). \tag{5.11}$$

**5.6.** The usual $G$-invariance of the complex structure means (see [6, 17]) that $J$ is invariant under automorphisms $\alpha(g) \in \alpha(G)$, i.e.

$$\alpha(g)J = J, \quad \forall g \in G. \tag{5.12}$$

Locally the condition (5.12) is equivalent to the following condition:

$$\mathcal{L}_{X_\xi} J = 0, \quad \forall \xi \in \mathcal{G}, \tag{5.13}$$

where $\mathcal{L}_{X_\xi}$ is a Lie derivative along the Hamiltonian vector field $X_\xi \in \alpha_*(\mathcal{G})$ on $M$.

From (5.12) it follows that $G \subseteq U(n)$ and therefore

$$\tau(g)J = \tilde{\tau}_g(\alpha(g)J) = \tilde{\tau}_g J = \tau_g J \tau_g^{-1} = J, \quad \forall g \in G,$$

because $\tau_g(x) \subset G_x \subseteq (U(n))_x$. Thus, from the invariance of $J$ under $\alpha(G)$ follows the invariance under the group $\tau(G)$.

**5.7.** We would like to impose the weakened form of the symmetry requirement. We shall weaken the conditions (5.12) demanding the invariance of the complex structure $J$ only under the $\tau$-automorphisms:

$$\tau(g)J = J, \quad \forall g \in G. \tag{5.14}$$

We shall call condition (5.14) the weak $G$-invariance condition of the complex structure. We have already noticed that the weak G-invariance follows from the standard $G$-invariance, but the converse is not true.

To define the $\tau$-automorphisms one must require the existence of the fields $\tau_g(x)$ on $M$. Let us suppose that a symplectic connection $\hat{\nabla}$ is defined on $M$. Let us also suppose that there are $k = dimG$ covariantly constant tensors $W_\xi = \{W_\xi(x)^\nu_\mu\}$, $\hat{\nabla} W_\xi = 0$, $\xi \in \mathcal{G}$, which constitute a basis of the Lie algebra $\mathcal{G}_x \subset sp_x(2n, R)$ for every $x \in M$. Then for $g = exp(\xi)$ the function $\tau_g(x)$ may be expressed in the form

$$\tau_g(x) = exp(W_\xi(x)). \tag{5.15}$$

It is clear that in virtue of covariant constancy of the tensors $W_\xi$, all such functions $\tau_g$ are completely determined by their value at the point $x = 0$ and parametrized by the group manifold $G$.

**5.8.** Consider the transformation (5.7). Let us denote by $\alpha(K)$ the subgroup in $\alpha(G)$ under which the complex structure $J$ is invariant. The group $\alpha(K)$ is the image under the homomorphism $\alpha$ of the subgroup $K$ of the group $G$. The Lie algebra $\mathcal{G}$ may be decomposed in the following way

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{Q}, \tag{5.16}$$

where $\mathcal{K}$ is a Lie algebra of the Lie group $K$, and $\mathcal{Q}$ is a tangent space in the origin of the homogeneous space $Q = G/K$.

We shall number the subspace $\mathcal{K}$ in $\mathcal{G}$ by the indices $i, j, ... = 1, ..., k-l$, and the subspace $\mathcal{Q}$ in $\mathcal{G}$ — by the indices $\alpha, \beta, ... = 1, ..., l$. Then locally the condition (5.14) of the weak $G$-invariance may be written in the form:

$$\mathcal{L}_{X_i} J^\nu_\mu = 0, \quad \mathcal{L}_{X_\alpha} J^\nu_\mu = J^\lambda_\mu (W_\alpha)^\nu_\lambda - (W_\alpha)^\lambda_\mu J^\nu_\lambda, \tag{5.17}$$

where $(X_i, X_\alpha)$ are the Hamiltonian vector fields on $M$ constituting the basis of the subspaces $\alpha_*(\mathcal{K})$ and $\alpha_*(\mathcal{Q})$ in the Lie algebra $\alpha_*(\mathcal{G})$.

**5.9.** Taking the Lie derivative $\mathcal{L}_{X_\alpha}$ of the identity $J^2 = -1$, we have

$$(\mathcal{L}_{X_\alpha} J^\lambda_\mu)\, J^\nu_\lambda + J^\lambda_\mu\, (\mathcal{L}_{X_\alpha} J^\nu_\lambda) = 0. \tag{5.18}$$

Condition (5.18) means that $\mathcal{L}_{X_\alpha} J$ transforms the $(-i)$ eigenspace $T^{(0,1)}M$ of $J$ to the $(+i)$ eigenspace $T^{(1,0)}M$ and hence

$$\mathcal{L}_{X_\alpha} J \in \Omega^{0,1}(M, T^{(1,0)}M)$$

where $T^{(1,0)}M$ denotes the holomorphic vector bundle of the $(1,0)$ tangent vectors. Linearizing the condition $\omega(X, JY) + \omega(JX, Y) = 0$, we find that [18, 19]:

$$\mathcal{L}_{X_\alpha} J = B^{ab}_\alpha \omega_{bc} \frac{\partial}{\partial z^a} \otimes dz^c \tag{5.19}$$

where $\omega = \omega_{ab} dz^a \wedge dz^b$ is the Kähler form, $a, b, \ldots = 1, \ldots, n$, and

$$B_\alpha = B^{ab}_\alpha \frac{\partial}{\partial z^a} \otimes \frac{\partial}{\partial z^b} \tag{5.20}$$

are the $C^\infty$ sections of the bundle $T^{(1,0)}M \otimes T^{(1,0)}M$ over $M$. Tensors $B_\alpha$ are symmetric (i.e. $B^{ab}_\alpha = B^{ba}_\alpha$) and may also be decomposed with respect to the basis $\frac{\partial}{\partial x^\mu} \otimes \frac{\partial}{\partial x^\nu}$ with components

$$B^{\mu\nu}_\alpha = \{J^\lambda_\gamma (W_\alpha)^\nu_\lambda - (W_\alpha)^\lambda_\gamma J^\nu_\lambda\}\, \omega^{\gamma\mu}. \tag{5.21}$$

**5.10.** As already noted, the conditions (5.14) and (5.17) mean that there are $k$ tensor fields $W_\xi$ on the manifold $M$. In particular, there are $l = dimG/K$ tensor fields $W_\alpha = \{(W_\alpha)^\nu_\mu\}$ or, equivalently, $l$ fields $B_\alpha = \{B^{\mu\nu}_\alpha\}$, which are the global sections of the bundle

$$R \times_G G/K \longrightarrow M \tag{5.22}$$

over $M$, associated with the principal $G$-bundle $\mathcal{R}(M, G)$ over $M$.

Let us denote the total space of the bundle (5.22) by $Z$. From the purely differential geometric point of view, this space is the product

$$Z = M \times G/K, \tag{5.23}$$

i.e. the trivial bundle.

The conditions (5.14) (and (5.17)) mean that the complex structure $J$ depends on the $l = dimG/K$ parameters $t_\alpha$ (coordinates on $G/K$) and may be represented in the form

$$J = \tau_y^{-1} J_0 \, \tau_y, \tag{5.24}$$

where $J_0$ is a fixed (canonical) complex structure and the element $\tau_y \in GaugeG$ has been written out in (5.15). The complex structure $J$ given by this formula coincides with $J_0$ if and only if $\tau_y(x)$ belongs to the subgroup $K_x \subset G_x$. Hence, on $M$ we have the $l$-parametric family of covariantly constant complex structures which are weakly $G$-invariant by construction. Notice that the weak $G$-invariance of the complex structure introduced by us is analogous to the generalized $G$-invariance of the connection in the principal fibre bundles studied in the papers[16, 20].

## 6. Symplectic twistors

In Sect.5, the bundle (5.23) $\pi : Z \to M$ appeared when we were describing the weakly $G$-invariant complex structure $J$. Here, we shall describe this bundle in more details.

**6.1.** Denote by $R^{2n}$ the real vector space of dimension $2n$ with coordinates $(p,q) = (p_1, ..., p_n, q_1, ..., q_n)$ and the standard symplectic structure

$$\omega_0 = dp_a \wedge dq^a. \tag{6.1}$$

Let $S(R^{2n})$ be the space of Kähler structures on $R^{2n}$, i.e. complex structures $J$ on $R^{2n}$ compatible with the symplectic structure $\omega_0$. It means that $J \in End(R^{2n})$ belongs to the group $Sp(2n, R)$ of linear symplectic transformations of $R^{2n}$.

The space $S(R^{2n}) \cong Sp(2n, R)/U(n)$ of the Kähler structure is a Hermitean symmetric domain of dimension $n(n+1)/2$ which can be identified with the Siegal unit disc $S_n$. This disc consists of complex $n \times n$ matrices $D$ subject to

$$D^t = D, \quad I - D^+D >> 0 \quad \text{(positive definite)}, \tag{6.2}$$

where $D^t$ is the transposed matrix of $D$, $D^+$ — its Hermitian conjugate. For proof of the identification $Sp(2n, R)/U(n) \cong S_n$ note that the action of $Sp(2n, R)$ on $R^{2n} \simeq C^n$ can be given (in coordinates $z^a$, $\bar{z}^a$ on $C^n$) by the block matrices

$$g = \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \tag{6.3}$$

preserving the matrix $\begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$ of the symplectic form $\omega_0$: $g\omega_0 g^t = \omega_0$, i.e. subject to the relations

$$AB^t = B^t A, \quad AA^+ - BB^+ = I. \tag{6.4}$$

The group $Sp(2n, R)$ acts transitively on the Siegel disc $S_n$ by the fractional linear transformation

$$D \to (AD + B)(\bar{B}D + \bar{A})^{-1} \tag{6.5}$$

and the isotropy group of the point $D = 0$ is $U(n)$.

**6.2.**. Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. We introduce the bundle

$$\pi_0 : S(M) \to M \tag{6.6}$$

of almost Kähler structures on $M$ associated with the principal $Sp(2n, R)$-bundle $\mathcal{P} = \mathcal{P}(M, Sp(2n, R))$ over $M$ (symplectic frame bundle). The fibre $\pi_0^{-1}(x)$ in a point $x \in M$ is the space $S(T_x M)$ of Kähler structures on $T_x M$ defined above.

As in the Riemannian case, taking a symplectic connection on $M$ we can provide $S(M)$ with a natural almost complex structure. In fact, let us denote by $\hat{\nabla}$ the symplectic connection on $M$. It generates the splitting of the tangent bundle $TS(M)$ into the direct sum

$$TS(M) = \mathcal{H} \oplus \mathcal{V} \tag{6.7}$$

of horizontal and vertical subbundles of $TS(M)$. The fibre $\mathcal{V}_p$ in $p \in S(M)$ is tangent to the fibre $\pi_0^{-1}(\pi_0(p))$ of $S(M) \to M$ through the point $p$. Recall that the fibre of $S(M) \to M$ over $\pi_0(p)$ is identified with $S(T_x M) \approx Sp(2n, R)/U(n)$ so it has a natural complex structure $\mathcal{J}^v$. Hence we can define an almost complex structure $\mathcal{J}$ on $S(M)$ using the decomposition (6.7) by setting

$$\mathcal{J} = \mathcal{J}^h \oplus \mathcal{J}^v, \tag{6.8}$$

where $\mathcal{J}^h$ is an almost complex structure equal at a point $p \in S(M)$ to the complex structure $\mathcal{J}_p^h$ on $\mathcal{H}_p \approx T_{\pi_0(p)}M$ given by the point $p = (x, J_x)$.

**6.3.** In Sect.3.7 we introduced the half-form bundle $Pf_P$ over $M$ for the fixed positive Kähler polarization $P$ of $M$. Polarization $P$ is totally fixed by the complex structure $J$ on $M$. that is why we shall denote the half-form bundle by $Pf_J$. As fibres of the bundle $Pf_J \to M$ in a point $x \in M$ we have $Pf_J(x) \in C$.

We have the bundle $S(M) \to M$ of positive Kähler structures on $M$. Let us introduce the complex line bundle [14, 4]

$$Pf \to S(M) \tag{6.9}$$

over $S(M)$, which has fibre $Pf_J(x)$ at $(x, J(x)) \in S(M)$. The bundle (6.9) defines a metaplectic structure on $M$. The restriction of $Pf$ to the fibre of $S(M)$ over some $x \in M$ is simply the half-form bundle on the space $S(T_xM) \approx Sp(2n, R)/U(n)$ (see [14, 4]). The restriction of $Pf$ to the base of the bundle $S(M)$ is the half-form bundle $Pf_J$ introduced in Sect.3.7.

**6.4.** The space $S(M)$ has a natural almost complex structure $\mathcal{J}$; using it we can analyze the real geometry of $M$ through the complex geometry of $S(M)$. Unfortunately, this almost complex structure is almost never integrable (it is integrable $\Longleftrightarrow$ $M$ is conformally symplectic flat, cf. [21 24]), so $S(M)$ is only an almost complex manifold. However, it is possible to define the symplectic twistor bundle of $M$ as the bundle

$$\pi : \quad Z \to M \tag{6.10}$$

together with the fibre preserving map

$$j : \quad Z \to S(M)$$

over $M$. Otherwize, we define $Z$ as a subbundle in $S(M)$ with complex fibres. Each $z$ in the fibre $Z_x$ over $x = \pi(z)$ then defines a symplectic complex structure on $T_xM$ via the isomorphism with the complex vector space $T_zZ/V_z$.

Conditions of the integrability of the complex structure on $Z$ are more weak than on $S(M)$, and in [22-24] one may find a number of examples of the manifolds $M$ which are not conformally flat and to which the twistor spaces $Z$ with the integrable complex structure $\mathcal{J}$ correspond. Namely, in [22] it has been shown that the almost complex structure $\mathcal{J}$ on $Z$ is integrable if the curvature $R^\nabla$ and the torsion $T^\nabla$ of the symplectic connection $\hat{\nabla}$ on $M$ satisfy the equations

$$J_x^+ T_x^\nabla(J_x^- X, J_x^- X) = 0, \tag{6.11a}$$

$$J_x^+ R_x^\nabla (J_x^- X, J_x^- X) J_x^- = 0, \tag{6.11b}$$

for all $J_x \in Z_x = \pi^{-1}(x)$ and $X, Y \in T_x M$. Here $J^\pm = \frac{1}{2}(1 \mp iJ)$ are the associated projectors onto $\pm i$ eigenspaces of $J$. For examples of the manifolds, connection $\hat{\nabla}$ on which satisfies condition (6.11), see [21–24].

**6.5.** In this paper we shall consider the twistor manifolds $Z$ introduced in Sect.5 by describing the weakly $G$-invariant complex structures. A connected Lie group $G$ is a closed subgroup of $Sp(2n, R)$. Let us consider a $G$-invariant submanifold $Q$ of $S(R^{2n})$ with a $G$-invariant complex structure obtained by restriction to $Q$ of the complex structure $\mathcal{J}^\nu$ on $S(R^{2n})$. We also suppose that $Q$ is a reductive homogeneous space $G/K$, where $K$ is a subgroup in $G$. The Lie algebra $\mathcal{G}$ of $G$ may be split as $\mathcal{G} = \mathcal{K} \dotplus \mathcal{Q}$ then the complex structure $\mathcal{J}_Q$ on $Q$ is determined by the splitting

$$\mathcal{Q}^C = \mathcal{Q}^+ \dotplus \mathcal{Q}^-. \tag{6.12}$$

Let $\mathcal{R} \to M$ be a principal bundle over $M$ with a structure group $G$, which is a subbundle in the principal bundle $\mathcal{P} \to M$. We shall introduce a bundle

$$\pi : Z = \mathcal{R} \times_G Q \longrightarrow M \tag{6.13}$$

which is the associated bundle with a fibre $Q$. In this way we shall not consider the bundle $S(M)$ of all almost Kähler structures on $M$, but the subbundle $Z$ in $S(M)$.

## 7. Double twistor fibration

**7.1.** The twistor bundle introduced by us is trivial: $Z = M \times G/K$. The $\hat{\nabla}$-parallel complex structures $J$ on $M$ under consideration are parametrized by the space $G/K$. That is why we may define a projection

$$\rho : Z \to G/K \tag{7.1}$$

by correspondence a point $(0, J_{x=0})$ of the manifold $G/K$ to each point $(x, J_x)$ of the manifold $Z$, transfering $J_x$ $\hat{\nabla}$ -parallel to the origin $x = 0$, where all non-equivalent structures are parametrized by the manifold $G/K$. We shall denote a point $J \in G/K$ and the corresponding complex structure on $M$ by the same letter $J$. The fibre $\rho^{-1}(J)$ in a point $J \in G/K$ can be identified with the complex manifold $M_J = (M, J)$, i.e. $M$ provided with the complex structure corresponding to $J \in G/K$.

Thus, we have a double fibration

$$\begin{array}{ccc} Z & \xrightarrow{\rho} & G/K \\ \pi \downarrow & & \\ M & & \end{array} \tag{7.2}$$

which is crucial for our considerations. Double fibration similar to (7.2) arises naturally in different problems of the twistor theory. In particular, Hitchin has defined [17] a double fibration of the type (7.2) where $M$ is substituted by a $4n$-dimensional hyper-Kähler manifold and $G/K = SU(2)/U(1) = CP^1$. In this case $Z$ appears to be a $(2n+1)$-dimensional complex manifold and points of $M$ are identified with real holomorphic sections of $Z \to CP^1$. The main difference between these constructions and our diagram (7.2) is that the twistor spaces, considered in [17], consist of almost complex structures compatible with a Riemannian metric while ours are of complex structures compatible with a symplectic form. So it is natural to call the twistor space under consideration by a symplectic twistor space.

**7.2.** Let us describe the definition of the almost complex structure on $Z$ by the use of the structure of the double fibration (7.2). Consider the bundles $\pi^{-1}(TM)$ and $\rho^{-1}(TQ)$ over $Z$ which are the pullbacks of the tangent bundles of $M$ and $Q = G/K$ respectively. The projections $\pi$ and $\rho$ generate the natural bundle homomorphisms

$$\pi_* : TZ \to \pi^{-1}(TM), \quad \rho_* : TZ \to \rho^{-1}(TQ). \tag{7.3}$$

We call the kernel of $\pi_*$ the vertical subbundle $\mathcal{V}$ of $TZ$ and the kernel of $\rho_*$ the horizontal subbundle $\mathcal{H}$ of $TZ$. Note that the fibre $\mathcal{V}_z$ in a point $z \in Z$ is identified by $\rho$ with the tangent space space $T_JQ$ in the point $J = \rho(z) \in Q$ and so has the complex structure $\mathcal{J}_Q$ defined on $Q = G/K$. Now we can define an almost complex structure $\mathcal{J}$ on $Z$ in exactly the same way as in Sect.6. The projection $\rho$: $Z \to G/K$ will become a holomorphic map w.r. to this structure $\mathcal{J}$.

**7.3.** Let $L \to M$ be the prequantization line bundle over $M$ with the connection $\nabla$. Denote by $\tilde{L} \to Z$ the pull-back of $L$ to $Z$. Then $\tilde{L}$ is a holomorphic bundle. It is essentially Ward's construction from twistor theory (cf. [25]).

To prove the assertion denote by $\tilde{\nabla}$ the pull-back of the connection $\nabla$

to $\tilde{L}$ and define a $\bar{\partial}$-operator on section $\tilde{s}$ of $\tilde{L} \to Z$ by setting

$$\bar{\partial}\tilde{s} := \tilde{\nabla}^{(0,1)}\tilde{s}, \tag{7.4}$$

where $\tilde{\nabla}^{(0,1)}$ is the (0,1)-component of $\tilde{\nabla}$ w.r. to an almost complex structure $\mathcal{J}$ on $Z$ introduced above. The symplectic structure $\omega$ on $M$ being compatible with all Kähler structures on $M$ has the type (1,1) w.r. to any such structure, hence the curvature $F_\nabla$ also has the type (1,1) w.r. to any Kähler structure. According to the definition of an almost complex structure on $Z$ it means that the curvature $F_{\tilde{\nabla}}$ of the pulled-back connection $\tilde{\nabla}$ on $\tilde{L}$ has the type (1,1) w.r. to the almost complex structure of $Z$ It follows that

$$\bar{\partial}^2\tilde{s} = F^{(0,2)}_{\tilde{\nabla}}\tilde{s} = 0, \tag{7.5}$$

i.e. the introduced $\bar{\partial}$-operator satisfies the integrability condition of Newlander-Nirenberg.

**7.4.** In Sect.6.3 we have introduced the half-form bundle $Pf$ over $S(M)$. We have the map $j : Z \to S(M)$ (see Sect.6.4) and, hence, we have a pull-back bundle $j^*Pf \to Z$ over $Z$. We shall denote this line bundle by $\widetilde{Gh} := j^*Pf$

$$\widetilde{Gh} \to Z \tag{7.6}$$

which will be called, following the physical tradition, the ghost bundle of $Z$ (or the restricted half-forms bundle).

The restriction of $\widetilde{Gh}$ to the fibre of $Z$ over some $x \in M$ will be denoted by $\widetilde{Gh}_x$ . The restriction of $\widetilde{Gh}$ to the base $M$ of the bundle $Z$ we shall denote by $Gh_J$ (cf. with Sect.6.3).

Let us also introduce the product of the bundles $\tilde{L}$ and $\widetilde{Gh}$:

$$\tilde{L} \otimes \widetilde{Gh} \longrightarrow Z \tag{7.7}$$

The restriction of this bundle on $M$ will be the following bundle

$$L \otimes Gh_J \longrightarrow M, \tag{7.8}$$

and its restriction on fibre $Q_x$ $(x \in M)$ will be

$$\tilde{L}_x \otimes \widetilde{Gh}_x \longrightarrow Q_x \tag{7.9}$$

So, we have a double fibration (7.2). that is why we may transfer the bundle (7.9) to the point $x = 0$ and obtain a holomorphic line bundle

$$L' \otimes Gh' \longrightarrow Q, \tag{7.10}$$

where $L' := \tilde{L}_{x=0}$, $Gh' := \widetilde{Gh}_{x=0}$, $Q := Q_{x=0}$.

## 8. Fock bundle

**8.1.** Let us consider the space $H_J$ given by formula (3.4). It is the space of holomorphic (w.r. to the complex structure $J$) sections of the bundle $L$ over $M_J = (M, J)$, i.e. $M$ provided with the complex structure corresponding to $J \in G/K$. In other words, $H_J$ consists of holomorphic sections of $\tilde{L}$ over $\rho^{-1}(J) = M_J$:

$$H_J = \mathcal{O}(M_J, \tilde{L}). \tag{8.1}$$

In particular, we shall consider $H_0 := H_{J^0}$.

The space $G/K$ is a parameter space of Kähler polarization $J$. As $J$ varies over $Q = G/K$, the vector space $H_J$ defines a vector bundle

$$\tilde{H} \longrightarrow Q \tag{8.2}$$

over $Q$ with the fibres $H_J$ in points $J \in Q$. $\tilde{H}$ is a subbundle of the trivial Hilbert bundle with a total space $Q \times \mathcal{L}^2(M, L)$.

**8.2.** The space $\tilde{H}$ coincides with the space of sections holomorphic w.r. to the almost complex structure $J$ on $Z$ of the bundle $\tilde{L}$ over $Z$: $\tilde{H} = \mathcal{O}(Z, \tilde{L})$. The bundle (8.2) is the homogeneous $G$-bundle over $G/K$ (see, e.g., [26]) and we may use this for its description.

Let us identify the group $G$ with its image $\alpha(G)$ in $Symp(M, \omega)$ on homomorphism $\alpha$.

We denote by $K$ a subgroup in $G$ preserving the complex structure $J_0$. Formula (4.1) gives the representation of the Lie algebra $\mathcal{K}$ of this group in $H_0$ from which, by the exponentiating, we may obtain the representation:

$$K \ni k \longrightarrow R(k): \quad H_0 \longrightarrow H_0 \tag{8.3}$$

of the group $K$ in $H_0 := H_{J_0}$.

Now consider the product space $G \times H_0$. The group $K$ acts on $(g, \psi) \in G \times H_0$ as follows:

$$k \circ (g, \psi) = (gk, R(k^{-1})\psi). \tag{8.4}$$

This allows us to define the complex vector bundle over $G/K$:

$$\tilde{H} = G \times_K H_0 \longrightarrow G/K \tag{8.5}$$

with the fibres $H_J$ in the points $J \in G/K$.

The Fock bundles like (8.2) and (8.5) were introduced and have been considered in the papers [27, 18, 19]. In these papers the connection in the Fock bundles has been defined; moreover the Bowick-Rajeev approach [27] differs from the approach of Hitchin [18] and Axelrod-Pietra-Witten [19].

**8.3.** We have a bundle $L$ over $M$ and a connection $\nabla$ in $L$ with a curvature $F_\nabla = \omega$. Because $M$ is the Kähler manifold with a complex structure $J$ then a canonical polarization $P$ is defined by

$$P_x = \left\{ Y_x \in T_x^{C} M : J_x Y_x = -iY_x, \quad x \in M \right\}. \tag{8.6}$$

In Sect.3.7 the anticanonical bundle $Det_P$ of antiholomorphic $n$-form on $M$ has been described. Because the Kähler form $\omega$ defines a cohomology class $[\omega] \in H^1(M, T^*)$ where $T \equiv T^{(1,0)}M$, then there exists a number $\lambda$ such that

$$Det_P \cong L^\lambda. \tag{8.7}$$

The curvature of the bundle $Det_P$, the Ricci form,

$$R = \frac{1}{2\pi} R_{a\bar{b}} dz^a \wedge d\bar{z}^b, \tag{8.8}$$

represents the first Chern class of $M$, and from (8.7) this is cohomologous to $\lambda\omega$. Since the Ricci form $R$ and $\lambda\omega$ are cohomologous, we may define a real function $\Phi$ characterized by

$$R_{a\bar{b}} + \lambda\omega_{a\bar{b}} = -2i \frac{\partial^2 \Phi}{\partial z^a \partial \bar{z}^b} \tag{8.9}$$

and normalized by the condition that its integral over $M$ is zero.

The (0,1)-forms

$$\theta_a = \left( 2iB_a^{ab} \omega_{bc} \frac{\partial \Phi}{\partial z^a} + (\nabla_a B_a^{ab}) \omega_{bc} \right) d\bar{z}^c \tag{8.10}$$

are $\bar{\partial}$-closed [18] and at least locally there are functions $f_a$ such that

$$\bar{\partial} f_a = \theta_a. \tag{8.11}$$

**8.4.** We have a holomorphic fibration $Z \xrightarrow{p} Q'$ over some open set $Q' \subset Q$ with coordinates $(t_1, ..., t_l)$ such that each fibre is a complex manifold. A covariant derivative in the Fock bundle $\tilde{H} \to Q'$ in coordinates $t_a$ on $Q'$ may be written in the form [18, 19]:

$$D = dt^a \left[ \frac{\partial}{\partial t^a} + \frac{1}{(2+\lambda)} \left( i\nabla_a (B_a^{ab} \nabla_b) - 2B_a^{ab} \frac{\partial \Phi}{\partial z^a} \nabla_b - f_a \right) \right], \quad (8.12)$$

where $\lambda$, $B_a$, $\Phi$ and $f_a$ were introduced earlier. Because the holonomy of this connection is given by a scalar multiplication, $D$ is projectively flat (see [ 18, 19]). Let us consider the bundle $Gh' \to Q$ introduced in Sect.7.4. Take the tensor product of $\tilde{H}$ with a line bundle $Gh' \to Q$, connection of which has the opposite curvature to that of $\tilde{H}$. By certain conditions, the tensor product $\tilde{H} \otimes Gh'$ will have a flat connection $D'$ (see [18, 19]) and we may introduce into consideration the bundle

$$\mathcal{F} = \left\{ \Psi \in \Gamma(Q, \tilde{H} \otimes Gh') : \quad D'\Psi = 0 \right\} \quad (8.13)$$

of covariantly constant sections of the bundle $\tilde{H} \otimes Gh'$.

**8.5.** On the space $\mathcal{F}$ one may define a metaplectic representation of the Lie group $G \subset Sp(2n, R)$. Using the fact that $\tilde{H} \otimes Gh'$ is a homogeneous $G$-bundle, the action of $g \in G$ on $\Psi \in \Gamma(Q, \tilde{H} \otimes Gh')$ may be defined as [26]

$$(g\Psi)(J) = g(\Psi(g^{-1}J)), \quad (8.14)$$

where $G$ acts canonically on $J \in G/K$ from the left and the action on $\Psi(g^{-1}J) \in H_{g^{-1}J}$ is described in [26].

For a physical Fock space $\mathcal{F}_G$ corresponding to the quantization of the manifold $M_G = \varphi^{-1}(0)/\alpha(G)$ one should take

$$\mathcal{F}_G = \{\Psi \in \mathcal{F} : \quad g\Psi = \Psi, \quad \forall g \in G\} \quad (8.15)$$

Thus, we described the physical Fock space of the system with the first class constraints as a $G$-invariant subspace in the space of covariantly constant sections of the Fock bundle over the space of weakly $G$-invariant complex structures on the phase manifold $M$.

## Acknowledgements

Author is grateful to D.V.Alekseevsky, M.V.Karasev, M. de León, A.G.Sergeev, J.Sniatycki, S.L.Woronowicz and T.Wurzbacher for valuable remarks and useful comments.

## References

1. B.Kostant, Lect.Notes Math. 1970, v.170, 87-208.
2. J.M.Souriau, Structure des systemes dynamiques. Paris: Dunod, 1970.
3. J.Sniatycki, Geometric quantization and quantum mechanics, Berlin: Springer, 1980.
4. N.M.J.Woodhouse, Geometric quantization, 2nd ed. Clarendon: Oxford, 1992.
5. M.J.Gotay, J.M.Nester and G.Hinds, J.Math.Phys. 1978, v.19, N 11, 2388-2399.
6. V.Guillemin and S.Sternberg, Invent.Math. 1982, v.67, 515-538.
7. J.Sniatycki, Lect.Notes Math. 1983, v.1037, 301-344.
8. A.Ashtekar and M.Stillerman, J.Math.Phys. 1986, v.27, N 5, 1319-1330.
9. M.J.Gotay, J.Math.Phys. 1986, v.27, N 8, 2051-2066.
10. M.Blau, Phys.Lett. 1988, v.205B, N 4, 525-529; Class.Quantum Grav. 1988, v.5, 1033-1044.
11. G.M.Tuynman, J.Math.Phys. 1990, v.31, N 1, 83-90; P.Schaller and G.Schwarz, J.Math.Phys. 1990, v.31, N 10, 2366-2377.
12. J.E.Marsden and A.Weinstein, Rep.Math.Phys. 1974, v.5, 121-130.
13. J.M.Arms, J.E.Marsden and V.Moncrief, Commun.Math.Phys. 1981, v.78, 455-478; J.Sniatycki and A.Weinstein, Lett.Math.Phys. 1983, v.7, N 2, 155-161; R.Sjamaar and E.Lerman, Ann.Math. 1991, v.134, 375-422.

14. R.J.Blattner, Lect. Notes Math. 1977, v.570, 11-45.

15. S.Kobayashi and K.Nomizu. Foundations of differential geometry, Interscience: New York, v.1, 1963, v.2, 1969.

16. A.S.Schwarz, Commun.Math.Phys. 1977, v.56, 79-86.

17. N.J.Hitchin, Contemp. Math. 1986, v.58, 157-178; N.J.Hitchin, A.Karlhede, U.Lindström and M.Roček, Commun.Math.Phys. 1987, v.108, 535-589.

18. N.J.Hitchin, Commun.Math.Phys. 1990, v.131, 347-380.

19. S.Axelrod, S.D.Pietra and E.Witten, J.Diff.Geom. 1991, v.33, 787-902.

20. P.Forgacs and N.S.Manton, Commun. Math. Phys. 1980, v.72, 15-35; J.Harnad, S.Snider and L.Vinet, J.Math.Phys. 1980, v.21, N 12, 2719-2724; J.Harnad, S.Snider and J.Tafel, Lett.Math.Phys. 1980, v.4, 107-113; M.Molelekoa, J.Math.Phys. 1985, v.26, N 1, 192-197; M.Legare and J.Harnad, J.Math. Phys. 1984, v.25, N 5, 1542-1547; 1986, v.27, N 2, 620-626.

21. L.Berard Bergery and T.Ochiai, On the generalization of the construction of twistor spaces. In: Glob. Riemannian Geom. Symp., eds. I.J.Willmore and N.J.Hitchin, 1984, N.Y., pp.52-59.

22. N.R. O'Brian and J.H.Rawnsley, Ann.Global Anal.Geom. 1985, v.3, N 1, 29-58.

23. F.E.Burstall, Lond.Math.Soc.Lect.Notes Series, 1990, v.156, 53-70.

24. D.V.Alekseevsky and M.M.Graev, J.Geom. and Phys. 1993, v.10, 203-229; Russian Acad.Sci.Izv.Math. 1993, v.40, N 1, 1-31.

25. R.S.Ward, Phys.Lett. 1977, v.61A, 81-82; M.F.Atiyah, N.J.Hitchin and I.M.Singer, Proc.R.Soc.Lond. 1978, v.A362, 425-461.

26. V.V.Gorbacevič and A.L.Oniščik, Itogi nauki i tekn. Sovr. probl. matem. Fund. napr., v.20, VINITI, Moscow, 1988, 103-240.

27. M.J.Bowick and S.G.Rajeev, Nucl.Phys. 1987, v.B293, 348-384; 1988, v.B296, 1007-1033.

**Received by Publishing Department
on July 27, 1993.**