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THE HEAT KERNEL EXPANSION ON A CONE AND QUANTUM FIELDS NEAR COSMIC STRINGS

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1 Introduction

In the last years a considerable number of works was devoted to the quantum theory on space times with conical singularities [1]-[12]. The line element on a conical space can be written in a form like on the plane in polar coordinates

$$ds^2 = dr^2 + r^2 d\varphi^2 \quad , \quad 0 \leq r \leq \infty \tag{1.1}$$

but with a polar angle φ ranging from 0 to an arbitrary positive parameter α. Besides, the cone (1.1) can be considered as a space whose curvature is completely concentrated at the apex $r = 0$ and looks like a delta function [13]. Singularities of that sort arise at the points on the world sheet of idealized cosmic strings with zero thickness [13],[14]. Even if the string space-time is flat out the world sheet, its topology is non-trivial and therefore the spectrum of vacuum fluctuations gets modified as compared to the case of the Minkowsky space. This effect has been investigated by many authors [3]-[6] who have determined the expectation value of the renormalized energy momentum tensor. Also of interest is quantum theory on orbifold factors of the Riemannian manifolds [9]-[12] where conical singularities appear at fixed points of the corresponding isometry groups.

In the present paper we investigate the global effects of vacuum polarization around a cosmic string by using the trace of the heat kernel on the cone (1.1) that is shown to look essentially different at asymptotically small values of the proper time as compared to the plane heat kernel. For this reason, the effective action obtained on its base includes a surface divergent functional given on the string world sheet. It is interesting that these surface infinities can be removed by renormalization of the string tension rendering finite the total renormalized energy.

This indicates a close analogy with quantum theory on manifolds with boundaries [15],[16] where similar divergent terms appear on boundary surfaces giving rise to renormalization of bare surface gravitational actions. The analogy can be continued further to demonstrate that the total renormalized energy is finite owing to cancellation of the non-integrable divergence in the energy density with a surface counterterm resulting from the bare string tension.

The remainder of this paper is organized as follows. In Section 2, an asymptotic expansion of the trace of the heat kernel on a cone in powers of the proper time is found. To characterize the effect of singularity at the apex, a more general problem is worth to be set up. In its framework the diagonal part of the kernel is considered as a functional and the heat coefficients turn out to have a delta function behavior at the cone tip. It is used in Section 3 to derive the renormalized effective action, including a surface term, and the total energy of a self-interacting scalar field around infinitely thin straight string. The approach by Critchley, Dowker and Kennedy [15] is explored then in Section 4 to reconcile our result with calculations [3]-[6] that have demonstrated a non-integrable character of the renormalized energy density. Conclusions and remarks are presented in Section 5.

The effects of the curvature are partially considered in Appendix A for the case of the sphere with two conical singularities at its poles and some exact results concerning the generalized zeta-function are presented.

2 The heat kernel on a cone

The heat kernel K_α of the Laplace-Beltrami operator Δ_α on the cone (1.1) is a solution of the Schrödinger-like equation

$$(\partial/\partial s + \Delta_\alpha(x))\, K_\alpha(x,x',s) = 0 \tag{2.1}$$

with the boundary condition

$$K_\alpha(x,x',0) = \delta_\alpha(x,x') \quad .$$

(δ_α is the delta function on (1.1)).

Let us define the diagonal part of the heat kernel $K_\alpha(x,x,s)$ as a functional on the functions $f(r,\varphi)$ integrable on the cone and such that the product $rf(r,\varphi)$ is an infinitely differentiable function at zero radius $r=0$. Then, as it will be shown below, the following expansion

$$Tr\,(e^{-s\Delta_\alpha}f) \equiv \int \sqrt{g(x)}\; d^2x\; K_\alpha(x,x,s)f(x) = \frac{1}{4\pi s}\sum_{n=0}^{\infty} a_{\alpha,n}(f)s^{n/2} + ES \tag{2.2}$$

as $s \to 0$ holds, where $\sqrt{g(x)}\; d^2x = rdrd\phi$ is the integration measure on the cone (1.1), $a_{\alpha,n}(f)$ are functionals on the chosen space of functions and ES means the terms that vanish exponentially as $s \to 0$. It is interesting to mention that this series includes half-integer powers of the proper time s and has therefore the form of an expansion on a two-dimensional manifold with a boundary [17].

The kernel K_α can be constructed explicitly if the eigenfunctions and eigenvalues of the Laplace-Beltrami operator on the cone are known. We shall consider the Friedrichs self-adjoint extension of the operator Δ_α so far as in this case it is positive. This corresponds to the wave functions regular at the conical singularity [8].

The solution of the problem (2.1) can be given then in an integral form by using the heat kernel $K_{2\pi} \equiv K$ of the Laplace operator Δ on the plane [1],[2]

$$K_\alpha(x, x', s) = \frac{i}{2\alpha} \int_C \cot\left(\pi\alpha^{-1} w\right) K\left(x(w), x', s\right) dw \quad , \tag{2.3}$$

where $x(w) = (r\cos(\varphi + w), r\sin(\varphi + w))$, $x' = (r'\cos\varphi', r'\sin\varphi')$ and

$$K(x, x', s) = \frac{1}{4\pi s} \exp\left(-\frac{(x - x')^2}{4s}\right) \quad . \tag{2.4}$$

The contour C in (2.3) has two branches, one in the upper half complex plane of the parameter w going from $(-\pi - \Delta\varphi + i\infty)$ to $(\pi - \Delta\varphi + i\infty)$ and the other in the lower half-plane from $(\pi - \Delta\varphi - i\infty)$ to $(-\pi - \Delta\varphi - i\infty)$, see Appendix A and Fig.1.b. in [2]. It follows, in particular, form the representation (2.3) that K_α can also be written as an infinite periodicity sum

$$K_\alpha(x, x', s) = \sum_{m=-\infty}^{\infty} K_\infty(x(m\alpha), x', s) \tag{2.5}$$

of the heat kernel $K_\infty = \lim_{\alpha \to \infty} K_\alpha$ on an infinitely-sheeted Riemann surface [1].

It is useful to represent (2.3) for $|\Delta\varphi| = |\varphi - \varphi'| < \pi$ in a bit different form

$$K_\alpha(x, x', s) = K(x, x', s) + \frac{i}{2\alpha} \int_\Gamma \cot\left(\pi\alpha^{-1} w\right) K\left(x(w), x', s\right) dw \quad , \tag{2.6}$$

by explicitly writing the contribution of the plane heat kernel. In the remaining integral contour Γ consists now of two curves, going from $(-\pi - \Delta\varphi + i\infty)$ to $(-\pi - \Delta\varphi - i\infty)$ and from $(\pi - \Delta\varphi - i\infty)$ to $(\pi - \Delta\varphi + i\infty)$ and intersecting the real axis between the poles of the integrand $-\alpha$, 0 and 0, α respectively. The equation (2.6) can easily be obtained from (2.3) by transformating of the contour C.

Let us return now to the asymptotic expansion (2.2) and calculate the heat coefficients $a_{\alpha,n}(f)$. It turns out that $a_{\alpha,0}(f)$ is determined by the heat kernel on a plane due to the first term in (2.6) and simply is

$$a_{\alpha,0}(f) = \int \sqrt{g}\, d^2x f(x) \quad , \tag{2.7}$$

whereas the other coefficients $a_{\alpha,n}(f)$, $n \geq 1$, result from the integral term in (2.6). To evaluate them, when the singularity at $r = 0$ is taken into account, we restrict the integration in (2.6) by a final part Γ_R of the contour Γ of a size R, passing then from Γ_R to Γ. One can thus write for the difference

$$\int \sqrt{g}\, d^2x\, (K_\alpha(x,x,s) - K(x,x,s))\, f(x) =$$

$$= \lim_{\Gamma_R \to \Gamma} \frac{i}{8\pi\alpha s} \int_0^\infty r dr\, f_0(r) \int_{\Gamma_R} \cot\left(\pi w \alpha^{-1}\right) \exp\left(-\frac{r^2 \sin^2 w/2}{s}\right) dw \quad , \tag{2.8}$$

$f_0(r) \equiv \int_0^\alpha d\varphi\, f(r,\varphi)$, and change the order of integration. So far as Γ can be chosen so that $Re(\sin^2 w/2) > 0$, then as $s \to 0$ the following expansion

$$\int_0^\infty r dr\, f_0(r) \exp\left(-\frac{r^2 \sin^2 w/2}{s}\right) = \frac{1}{2}\sum_{n=o}^{\infty} \frac{\Gamma\left((n+1)/2\right)}{n!(\sin^2 w/2)^{(n+1)/2}} \frac{d^n(r f_0(r))}{dr^n}|_{r=0}\, s^{\frac{n+1}{2}} + ES \tag{2.9}$$

holds ($\Gamma(x)$ denotes the gamma-function). The sign in the square root of $\sin^2 w/2$ in (2.9) has to be chosen from the conditions that are determined by the properties of the integral over r

$$\left(\sin^2 w/2\right)^{1/2} = \sin w/2, \qquad Re \sin w/2 \; > \; 0, \tag{2.10}$$

$$\left(\sin^2 w/2\right)^{1/2} = -\sin w/2, \qquad Re \sin w/2 \; < \; 0,$$

where the upper one is valid for the right part of Γ; whereas the lower, for the left. By using (2.9) it is not difficult to show now that the action of the other functionals $a_{\alpha,n}$ on the considered space of functions gives

$$a_{\alpha,n}(f) = \frac{\Gamma(n/2)}{(n-1)!} C_n(\alpha) \int_0^\alpha d\varphi\, \frac{d^{n-1}\left(r f(r,\varphi)\right)}{dr^{n-1}}|_{r=0} \quad , \quad n \geq 1 \quad , \tag{2.11}$$

where $C_n(\alpha)$ are the following integrals

$$C_n(\alpha) = \frac{i}{4\alpha} \int_\Gamma \cot(\pi w \alpha^{-1}) (\sin^2 w/2)^{-n/2} dw \quad . \tag{2.12}$$

For $\alpha = 2\pi$ the integrand in (2.12) is a 2π-periodic function of w and Γ can be deformed so as the contributions of both its parts to cancel each other. In this case all $C_n(\alpha)$ and consequently $a_{\alpha,n}(f)$ for $n \geq 1$ turn out to be zero leaving the only contribution in (2.12) provided by the kernel $K(x, x', s)$.

It is important that according to (2.11) the heat coefficients $a_{\alpha,n}(f)$, $n \geq 1$ act like a delta function and don't depend on the behavior of f at regular points on a cone. They would never appear if the integration over the cone in (2.2) were stopped short before the point $r = 0$, by no matter how close.

Integrals of the type (2.12) have been discussed in [3]. For even values of indices $n = 2k$ they can be converted to the following form

$$C_{2k}(\alpha) = \frac{i}{4\alpha} \oint \cot(\pi w \alpha^{-1})(\sin^2 w/2)^{-2k} dw \tag{2.13}$$

and represented in terms of polynomials of the order $2k$ in powers of α^{-1}. We list here the values of the first two ones for $k = 1, 2$

$$C_2(\alpha) = \frac{1}{6}\left(\left(2\pi\alpha^{-1}\right)^2 - 1\right) \quad , \tag{2.14}$$

$$C_4(\alpha) = \frac{1}{90}\left(\left(2\pi\alpha^{-1}\right)^2 - 1\right)\left(\left(2\pi\alpha^{-1}\right)^2 + 11\right) \tag{2.15}$$

to be required for the further analysis. However, as for the odd indices, the quantities $C_{2k+1}(\alpha)$ can be given only in an integral form, see [3].

For a particular but important case when the function in (2.2) is assumed to be equal to unity in a domain of V of the cone including its apex and zero at other points the series is truncated and one gets the expression exact up to the ES terms

$$Tr\left(e^{-s\Delta_\alpha} f_V\right) = \frac{1}{4\pi s}\left(V + \alpha C_2(\alpha)\, s\right) + ES \quad . \tag{2.16}$$

This result can be immediately checked for certain values $\alpha = 2\pi n^{-1}$ $(n = 2, 3, ...)$ when $K_\alpha(x, x', s)$ is explicitly presented as a finite periodicity sum of $K(x, x', s)$.

It is worth also to point out that expression (2.2) can trivially be generalized to the heat kernel on the space product of a cone and a smooth manifold. For instance, if the latter is the $d-2$-dimensional Euclidean space R^{d-2} with the Laplace operator Δ_{d-2}, one can write, by using (2.16),

$$Tr\left(e^{-s(\Delta_{d-2}+\Delta_\alpha)}\right) = \frac{1}{(4\pi s)^{d/2}}\left(\Omega_d + \Sigma_{d-2}\alpha C_2(\alpha)\, s\right) + ES \quad , \tag{2.17}$$

where $\Omega_d = \Sigma_{d-2} V$ is the volume of the total space and effect of the conical singularities consists in appearing of the "surface" term proportional to the volume Σ_{d-2} of the hypersurface $r = 0$.

So far as the space is non-compact, Ω_d and Σ_d are to be infinite and thus (2.17) has to be treated in a regularized sense like (2.16). In this case the ES terms are significant. If L is a typical size of the space (the length at which the integrals are cut off), then ES terms in (2.17) can be shown to be of the order $s^{-(d-2)/2} \exp(-L^2/s)$. From now on we drop ES as negligible in the limit $L \to \infty$ we are interested in.

3 Quantum field near cosmic string

Let us consider a quantum scalar field near a cosmic string being in the flat space-time. For simplicity we confine the following analysis to the case of an infinitely thin straight string that is at rest along the z axis. The metric around it can be written in the form

$$ds^2 = dt^2 - dz^2 - dr^2 - r^2 d\varphi^2 \quad , \quad 0 \leq \varphi \leq \alpha \tag{3.1}$$

and it is a solution of the Einstein equations [13],[14]

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu} \tag{3.2}$$

where the energy-momentum tensor of the string, $T_{\mu\nu}$, has only two non-zero components

$$T_{tt} = -T_{zz} = \mu \delta_2(r) \quad , \quad \int_0^\alpha d\varphi \int_0^\infty r dr \; \delta_2(r) = 1 \quad . \tag{3.3}$$

($\delta_2(r)$ can be represented with the help of the one-sided delta function; $\delta_2(r) =$ $= (\alpha r)^{-1} \delta(r + 0)$.) From (3.2) the string tension μ turns out to be immediately related to the polar angle deficit [13]-[14]

$$\mu = \frac{1}{8\pi G}(2\pi - \alpha) \quad . \tag{3.4}$$

In this Section we concern the global effects of the vacuum polarization on the string space-time (3.1) that are displayed in the integral quantities like the effective action W or the ground energy $E_0(\alpha)$ of a quantum field around the string.

As for $E_0(\alpha)$, two different ways can be used to calculate this quantity. The first one is to obtain $E_0(\alpha)$ as the integral of the renormalized energy density $< \hat{T}_{00}(x) >^{\alpha}_{sub}$.

However this method cannot be applied immediately so far as the renormalized energy momentum tensor has a non-integrable infinity at the string axis [3]-[6] and an additional regularization will be shown in the next Section to be needed. Here we consider another way based on the thermodynamical relation between the internal energy $E_{\beta^{-1}}$ of the system at a temperature β^{-1} and the partition function Z_β

$$E_{\beta^{-1}} = < \hat{H} >_\beta = -\frac{\partial}{\partial\beta} \log Z_\beta \tag{3.5}$$

where $\hat{H}$ is the Hamiltonian. In such approach the ground energy E_0 is the energy at zero temperature $E_0 = \lim_{\beta\to\infty} E_{\beta^{-1}}$ and to get it in the one-loop approximation the equation (2.17) for the trace can be used.

The partition function is known to be represented in the form of a functional integral by passing to an imaginary time. In particular, for a self-interacting scalar field ϕ around the string with a potential $V(\phi)$ one has

$$Z_\beta = Tr(e^{-\beta\hat{H}}) = \int D\phi e^{-S_e[\phi]} \quad , \tag{3.6}$$

where $D\phi$ is an integration measure and the action in the exponential

$$S_e[\phi] = \int \sqrt{g} d^4x \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + V(\phi) \right) \tag{3.7}$$

is given on the Euclidean section of the space-time around the string

$$ds^2 = d\tau^2 + dz^2 + dr^2 + r^2 d\varphi^2 \quad , \quad 0 \leq \varphi \leq \alpha \, , \quad 0 \leq \tau \leq \beta \tag{3.8}$$

with periodicity in τ.

Another quantity we are interested in is the effective action W that can be also defined for a finite temperature with the help of the partition function

$$W = -\log Z_\beta \quad , \tag{3.9}$$

by taking next the limit $\beta \to \infty$. Its variations coincide with the thermal average of the functional δS_e that is interpreted as a quantum operator

$$\delta W = Z_\beta^{-1} \int D\phi \; \delta S_e[\phi] e^{-S_e[\phi]} = < \delta\hat{S}_e >_\beta \quad . \tag{3.10}$$

Strictly speaking, W is an Euclidean form of the effective action but transition to the convenient definition [16],[24] doesn't make any difficulty for the static space like (3.1).

To obtain W and $E_0(\alpha)$ in the one loop approximation let us consider the system in a large but finite volume Ω_4 of the space (3.8) that includes the string and expand the action S_e near its minimum φ up to the second order terms

$$S_e[\varphi + \phi'] = S_e[\varphi] + \frac{1}{2}\int \sqrt{g} d^4x \phi' [\Delta + M^2]\phi'$$

$$M^2 \equiv V''(\varphi)$$

(Δ denotes the Laplace operator on (3.8)). Assuming φ to be a constant configuration and by integrating in (3.6) over ϕ' one gets

$$W[\varphi] = \Omega_4 \ V(\varphi) + \frac{\hbar}{2} \log\det(\Delta + M^2) + O(\hbar^2) \quad , \tag{3.11}$$

$$E_0 = \frac{\partial}{\partial \beta} W \qquad (\beta \to \infty) \tag{3.12}$$

where $\Omega_4 = \beta \int dv$ and the Planck constant $\hbar$ is introduced explicitly to emphasize the quantum corrections.

The second term in (3.11) is ultraviolet divergent and to get a finite expression we have to renormalize the effective action $W[\varphi]$. To this end the dimensional regularization [19], for instance, can be used. It suggests in our case that the space (3.8) has to be changed to the space product $R^{d-2} \otimes Cone$, passing to arbitrary values of the parameter d. The quantity $\log\det(\Delta + M^2)$ can be evaluated then with the help of the following representation [21]

$$\log\det(\Delta + M^2) = Tr\log(\Delta + M^2) = -\int_0^{\infty} \frac{ds}{s} Tr\left(e^{-s\Delta}\right) e^{-M^2 s} \quad , \tag{3.13}$$

where at low temperature ($\beta \to \infty$) the equation (2.17) for the trace of the heat kernel on $R^{d-2} \otimes Cone$ is valid. Besides, in (2.17) the "regularized" volumes Ω_d, and Σ_{d-2} are expressed through the physical volume $\Omega_4 = \beta \int dv$ and the area $\Sigma_2 = \beta \int dz$ of the surface $r = 0$

$$\Omega_d = \nu^{\epsilon} \Omega_4 \quad , \quad \Sigma_{d-2} = \nu^{\epsilon} \Sigma_2 \quad , \quad \epsilon \equiv 4 - d \quad . \tag{3.14}$$

Here an additional parameter ν with the mass dimension is introduced to adjust the dimensions of the left and right sides of these equalities. After the integration in (3.13) the regularized $\log\det(\triangle + M^2)$ at $\beta \to \infty$ reads

$$\log\det(\triangle + M^2) = -\Omega_4 \left(\frac{4\pi\nu^2}{M^2}\right)^{\epsilon/2} \frac{M^4}{16\pi^2}\Gamma(\epsilon/2 - 2)$$

$$- \Sigma_2 \left(\frac{4\pi\nu^2}{M^2}\right)^{\epsilon/2} \frac{M^2}{16\pi^2}\alpha C_2(\alpha)\Gamma(\epsilon/2 - 1) \quad . \tag{3.15}$$

From (3.11) and (3.15) one can see that in our case as distinct from the theory in the Minkowsky space an additional surface term proportional to Σ_2 appears in the effective action W. After passing from the metric (3.8) to (3.1) the new term in W is represented by an integral over the string world sheet. Consequently it is worth to unify it with the string action $\mu\Sigma_2$ that can be added to W. This gives rise to a surface effective action in the total functional.

To investigate now the renormalization let us consider a simple model of a real scalar field with the potential

$$V(\varphi) = \frac{m^2}{2}\varphi^2 + \frac{\lambda}{24}\varphi^4 \quad , \quad m^2\,, \lambda > 0 \quad . \tag{3.16}$$

In this case the total regularized one-loop effective action expressed through the bare parameters λ_B, m_B, μ_B looks as follows

$$W_{tot}[\varphi] = W[\varphi] + W_{B,surf}[\varphi] =$$

$$= \Omega_d \left(\frac{m_B^2}{2}\varphi_B^2 + \frac{\lambda_B}{24}\varphi_B^4\right) + \frac{\hbar}{2}\log\det(\triangle + M^2) + O(\hbar^2) + \Sigma_{d-2}\left(\mu_B + \sigma_B\varphi_B^2\right) \tag{3.17}$$

where apart from the bare string action an additional term $\sigma_B\varphi_B^2$ is included in the bare surface functional $W_{B,surf} = \Sigma_{d-2}(\mu_B + \sigma_B\varphi_B^2)$ to eliminate the corresponding divergence. To remove

$$\textit{the pole part of}\quad \log\det(\triangle + M^2) =$$

$$= \frac{1}{\epsilon}\left[-\Omega_4\frac{1}{16\pi^2}(m^2 + \lambda\varphi^2/2)^2 + \Sigma_2\frac{\alpha C_2(\alpha)}{16\pi^2}(m^2 + \lambda\varphi^2/2)\right] \quad ,$$

from the functional $W[\varphi]$ the bare constants $\lambda_B, m_B^2, \mu_B, \sigma_B$ have to be expressed through the renormalized ones $\lambda, m^2, \mu, \sigma$

$$\lambda_B = \nu^\epsilon\left(\lambda + \frac{\hbar}{\epsilon}\frac{3\lambda^2}{16\pi^2} + O(\hbar^2)\right) \quad , \tag{3.18}$$

$$m_B^2 = m^2\left(1 + \frac{\hbar}{\epsilon}\frac{\lambda}{16\pi^2} + O(\hbar^2)\right) \quad , \tag{3.19}$$

$$\mu_B = \nu^{-\epsilon}\left(\mu - \frac{\hbar}{\epsilon}\frac{m^2\alpha C_2(\alpha)}{16\pi^2} + O(\hbar^2)\right) \quad , \tag{3.20}$$

$$\sigma_B = \sigma - \frac{\hbar}{\epsilon}\frac{\lambda^2\alpha C_2(\alpha)}{32\pi^2} + O(\hbar^2) \quad . \tag{3.21}$$

The above definitions correspond to a renormalization recipe in which the finite parts of the counterterms are assumed to be equal to zero [19]. As for the bare field φ_B, it is related to the renormalized one φ by the equality $\varphi_B = \nu^{-\epsilon/2}\varphi$ because any counterterms in φ_B can always be removed, shifting the variable of integration in (3.6). Differentiating the equations (3.18)-(3.21) with respect to the parameter ν it is easy to get apart from the standard renormgroup equations for λ and m^2 the new ones for the string tension μ and σ.

The total one-loop effective action W_{tot} written in terms of the renormalized quantities defined by (3.18)-(3.21) can be represented as a sum of the volume and surface parts

$$W_{tot}[\varphi] = W_{vol}[\varphi] + W_{surf}[\varphi] = \Omega_4\ V_{eff}(\varphi) + \Sigma_2\ \mu(\varphi) \tag{3.22}$$

where for a constant argument the functional $W_{vol}[\varphi]$ is expressed through the renormalized effective potential of the system $V_{eff}[\varphi]$ that looks the same as in the Minkowsky space. This fact can be used to fix the value of the renormalization parameter ν. For instance, in the considered model one can identify m with the physical mass, that is equivalent to the following normalization condition [20]

$$V''_{eff}(0) = m^2 \tag{3.23}$$

at the minimum $\varphi = 0$ of V_{eff}. It gives the relation $4\pi\nu^2 = m^2\exp(1/2+\gamma)$, where γ is the Euler constant, and

$$V_{eff}(\varphi) = \frac{m^2}{2}\varphi^2 + \frac{\lambda}{24}\varphi^4 + \frac{\hbar}{64\pi^2}\left(m^2 + \lambda\varphi^2/2\right)^2\left(\log\left(\frac{m^2+\lambda\varphi^2/2}{m^2}\right) - \frac{1}{2}\right) \quad . \tag{3.24}$$

Besides, it results to the renormalized surface effective action that can be represented as follows

$$W_{surf}[\varphi] = \Sigma_2\mu(\varphi) = \mu + \sigma\varphi^2 - \frac{\hbar}{32\pi^2}\alpha C_2(\alpha)\left(m^2 + \lambda\varphi^2/2\right)\left(\log\left(\frac{m^2+\lambda\varphi^2/2}{m^2}\right) - \frac{1}{2}\right) \quad . \tag{3.25}$$

Finally, the total renormalized energy of the string and quantum field can be obtained in accordance with (3.12) replacing there the functional $W[\varphi]$ to the total renormalized effective action $W_{tot}[\varphi]$. This is the same as if we changed the definition (3.6) of the partition function Z_β and added to the functional $S_e[\phi]$ the surface action $S_{surf}[\varphi] = \int d\tau dz(\mu + \sigma\phi^2)$. Thus, after subtracting the vacuum energy $E_{0,Mink} = \int dv V_{eff}(0)$ in the Minkowsky space, we come to result

$$E_{tot} = \frac{\partial}{\partial\beta} W_{tot}[0] - E_{0,Mink} = \mu(0) \int dz \quad . \tag{3.26}$$

This quantity is taken at the minimum $\varphi = 0$ of the potential $V_{eff}(\varphi)$ that corresponds to a field configuration with zero average field strength $< \hat{\phi} >= 0$ [20]. It follows from (3.26) that renormalized energy per unite length turns out to be finite and equal to

$$\mu(0) = \mu + \frac{\hbar}{64\pi^2} m^2 \alpha C_2(\alpha) + O(\hbar^2) \quad . \tag{3.27}$$

So far as $\mu(0)$ occurs from the surface functional (3.25) the non-zero value of E_{tot} is completely provided by the energy density at the string axis.

The constant $\mu(0)$ should be considered as an effective tension of the string that includes quantum corrections to the classical tension μ related with the parameter α by (3.4). It is interesting that the renormalized surface action $W_{surf}[\varphi]$ depends on φ even if $\sigma = 0$ and therefore in general case the effective tension $\mu(\varphi)$ varies if the average value of the field $< \hat{\phi} >= \varphi$ changes that happens in the case of a phase transition.

4 Total energy end energy density

Until now we dealt with the integral quantities like the effective action and total energy using for their calculation the trace of the heat kernel. The surface terms appearing in these quantities have a global origin: they would have not arisen, if we had excluded, from the integrals over the space-time, the region around the string world sheet. The local renormalized energy momentum tensor near the cosmic string was calculated by a number of authors [3]-[6]. Let us find out a connection between their and our results and demonstrate that the local non-integrable divergence in the average energy density arising as the string is approached can be removed by a suitable renormalization of the bare string tension so that the total energy turns out to be finite. What we are going to do is to explore the same approach as used in [15] in quantum theory with boundaries.

We consider a real massless scalar field for which the energy density can be obtained in the closed form [3]

$$< \hat{T}_{00}(x) >^{\alpha}_{sub} = \frac{1}{16\pi^2 r^4}\left(2(1-4\xi)C_2(\alpha) - C_4(\alpha)\right) \qquad (4.1)$$

expressed in terms of the polynomials (2.14),(2.15). The value $\xi = 1/6$ corresponds to a conformally invariant field. The local energy is evaluated in a standard way from the Green function $G^{\alpha}(x,x') = i^{-1} < T(\hat{\phi}(x), \hat{\phi}(x')) >$ as a coincidence limit

$$< \hat{T}_{00}(x) >^{\alpha}_{sub} = \lim_{x' \to x} \mathcal{T}_{00} G^{\alpha}_{sub}(x,x') \qquad (4.2)$$

where $\mathcal{T}_{00}$ denotes a second order differential operator [22] depending on the type of field and the divergences are removed by subtracting from the Minkowsky Green function $G^{\alpha=2\pi}$ from G^{α}

$$G^{\alpha}_{sub}(x,x') = G^{\alpha}(x,x') - G^{\alpha=2\pi}(x,x') \quad . \qquad (4.3)$$

It is obvious that the local divergence of the energy density (4.1) at $r = 0$ can be regularized if we restrict the domain of integration in the total energy by the values of coordinates $r \geq r_0$ where r_0 is a positive small parameter that can be treated as the string radius. Moreover the regularization suggested also makes finite the surface term in the effective action. To see this it is worth to use the equation (2.6), which gives, instead of (2.17),

$$Tr\left(e^{-s(\Delta_\alpha+\Delta_2)}\right)_{r_0} \equiv \int_{r_0}^{\infty} r dr \int_0^{\alpha} d\varphi \; K_\alpha(x,x,s) Tr\left(e^{-s\Delta}\right) =$$

$$= \frac{1}{(4\pi s)^2}\left(\Omega_4 + \Sigma_2 \frac{is}{4}\int_{\Gamma} \frac{\cot(\pi\alpha^{-1}w)}{\sin^2 w/2} \exp\left(-\frac{\mathrm{r}_0^2 \sin^2 w/2}{s}\right) dw\right) \quad . \qquad (4.4)$$

Then for a free scalar field the total one-loop effective action (in the case when $\sigma_B = 0$) can be defined like (3.17) and takes the form

$$W_{tot} = \frac{\hbar}{2}\log\det(\Delta + m^2)_{r_0} + \mu_B \Sigma_2 = -\frac{\hbar}{2}\int_0^{\infty} \frac{ds}{s} Tr\left(e^{-s(\Delta_\alpha+\Delta_2)}\right)_{r_0} e^{-m^2 s} + \mu_B \Sigma_2 =$$

$$\equiv W_{vol} + W_{r_0,surf} \qquad (4.5)$$

It is separated into the volume part W_{vol} proportional to Ω_4 and the surface part $W_{r_0,surf}$ given on the world sheet Σ_2. As distinct from W_{vol} developing the standard divergences, the surface action $W_{r_0,surf}$ now turns out to be finite while $r_0 \neq 0$ and in the massless case $m^2 = 0$ its expression results from (4.4)

$$W_{r_0,surf} = \Sigma_2 \left(\mu_B - \frac{i\hbar}{8} \int_0^\infty \frac{ds}{(4\pi s)^2} \int_\Gamma \frac{\cot(\pi\alpha^{-1}w)}{\sin^2 w/2} \exp\left(-\frac{r_0^2 \sin^2 w/2}{s} \right) dw \right) =$$

$$= \left(\mu_B - \hbar \frac{\alpha C_4(\alpha)}{32\pi^2 r_0^2} \right) \Sigma_2 \quad . \tag{4.6}$$

It follows from (4.6) that the divergence in $W_{r_0,surf}$ at $r_0 \to 0$ can be removed by replacing as before the bare string tension μ_B by the renormalized μ

$$\mu_B = \mu + \frac{\hbar}{32\pi^2 r_0^2} \alpha C_4(\alpha) \quad . \tag{4.7}$$

Taking this into account one can write the local renormalized energy as the sum

$$< \hat{T}_{00} >^\alpha_{r_0,ren} = T_{00,B} + < \hat{T}_{00} >^\alpha_{r_0,sub} \tag{4.8}$$

of the string energy $T_{00,B} = \mu_B \delta_2(r)$ concentrated at the string axis and the renormalized energy density of the quantum field in the domain $r \geq r_0$. Two densities, $< \hat{T}_{00} >^\alpha_{sub}$ given by (4.1) and $< \hat{T}_{00} >^\alpha_{r_0,sub}$ coincide everywhere except the region near the string. To demonstrate this let us calculate the classical energy-momentum tensor of the field in this domain defined by the functional differentiation of the action that we take in the same form as in [15]

$$S = -\frac{1}{2} \int_{r \geq r_0} d^4x \sqrt{-g}\ \phi(x)[\Box + \xi R]\phi(x) \tag{4.9}$$

where $\Box = \sqrt{-g}^{\ -1} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu$ is the D'Alambertian and R is the scalar curvature. The variation of this functional δS under changing the metric $\delta g^{\mu\nu}$ consists of two parts

$$\delta S = \frac{1}{2} \int_{r \geq r_0} d^4x \sqrt{-g}\ T_{\mu\nu}(x) \delta g^{\mu\nu}(x) + \delta_{surf} S \tag{4.10}$$

where $T_{\mu\nu}$ stands for the normal expression of the stress tensor of a scalar field [21] and an additional surface term arises due to the restriction of the domain of integration

$$\delta_{surf} S = -\frac{1}{2} \int_{r=r_0} d\sigma^r \left[\phi^2 (g_{\mu\nu} \delta g^{\mu\nu}{}_{;r} + g_{r\mu} \delta g^{\mu\sigma}{}_{;\sigma}) \right.$$

$$\left. + \left((1/4 - \xi)(\phi^2)_{,r}\, g_{\mu\nu} + (\xi - 1/2)(\phi^2)_{,\nu}\, g_{\mu r} \right) \delta g^{\mu\nu} \right] \tag{4.11}$$

($d\sigma^r$ is the area element). So far as there is no real boundary of the space on the surface $r = r_0$, the variations of the metric $\delta g^{\mu\nu}|_{r=r_0}$ don't vanish on it. They are independent of its normal derivatives on the surface and thus the last ones can be ignored. As a result, $\delta_{surf} S$ produces the additional term in the energy density

$$T_{00,surf} = \frac{2}{\sqrt{-g}} \frac{\delta_{surf} S}{\delta g^{00}} = (1/4 - \xi)\delta(r - r_0)\frac{d}{dr}(\phi)^2 \tag{4.12}$$

giving rise to the distinction between the average density in the domain, $< \hat{T}_{00} >^{\alpha}_{r_0,sub}$, and the local energy (4.1)

$$< \hat{T}_{00} >^{\alpha}_{r_0,sub} = < \hat{T}_{00} >^{\alpha}_{sub} + i(1/4 - \xi)\delta(r - r_0) \lim_{x \to x'} \left(\frac{d}{dr} + \frac{d}{dr'} \right) G^{\alpha}_{sub}(x, x') \tag{4.13}$$

($\delta(r - r_0)$ is the one-sided delta-function). For its calculation the proper-time representation for the Green function [23],[24] written in the form

$$G = -(\Box + m^2)^{-1} = -\int_0^{\infty} ds \; e^{-(\Box+m^2)s}$$

can be used. It gives, together with (2.6), the subtracted Green function at $t = t', z = z'$ and $\varphi = \varphi'$ by the integral

$$G^{\alpha}_{sub}(r, r') = -\frac{i}{16\pi^2} \int_0^{\infty} \frac{ds}{s^2} \frac{i}{2\alpha} \int_{\Gamma} \cot(\pi\alpha^{-1}w) \exp\left(-\frac{r^2 + r'^2 - 2rr' \cos w}{4s} \right) dw \tag{4.14}$$

which can be substituted into (4.13) to obtain

$$< \hat{T}_{00} >^{\alpha}_{r_0,sub} = < \hat{T}_{00} >^{\alpha}_{sub} - (1/4 - \xi) \frac{C_2(\alpha)}{4\pi r_0^3} \delta(r - r_0) \quad . \tag{4.15}$$

Integrating now the renormalized quantity (4.8) over the space

$$E_{tot} = \int < \hat{T}_{00} >^{\alpha}_{r_0,ren} \; dv = \left[\mu_B + \int_{r_0}^{\infty} r\,dr \int_0^{\alpha} d\varphi < \hat{T}_{00} >^{\alpha}_{r_0,sub} \right] \int dz \tag{4.16}$$

and using (4.7),(4.15) we find that the counterterm in the bare tension μ_B cancels exactly the term proportional to r_0^{-2} in the integrated energy of the field rendering finite the renormalized total energy at $r_0 \to 0$

$$E_{tot} = \mu \int dz \quad . \tag{4.17}$$

This shows explicitly that finiteness of the total energy derived in the previous Section is a consequence of renormalization of the bare string tension. There is also quantitative agreement between (4.16) and equation (3.26) where for zero mass $\mu(0) = \mu$ and in both cases the parameter μ has to be identified with the classical string tension.

5 Conclusions and remarks

In this work a close analogy between quantum theory on the space with conical singularities and quantum theory with boundaries was outlined. In both cases the one loop quantum corrections result to divergent surface functionals in the effective actions. The renormalization of these functionals can be used to remove non-integrable divergence in the energy density and to obtain the finite total energy of the system. However, this analysis concerns the idealized objects, strings and boundaries of zero thickness. In effect one might expect that for the real string of a finite size the divergent terms on its world sheet give large but still finite contributions to the renormalized energy.

In the theory with boundaries the surface actions are known to essentially depend on which of the boundary conditions, Dirichlet or Neumann, are imposed. As for the string case, we used the finite boundary condition on the string axis taken in Section 2 and others possibilities are worth to be investigated as well. For example, the possibility of logarithmically divergent conditions has been pointed out in [8] in connection with the self-adjoint extensions of the Laplace operator on a cone. A hypothesis has been made there that effects of the true interaction of the cosmic string with the field can be taken into account by choosing one of the suitable extensions provided we are interested in what happens in sufficiently large length scales.

It is to be also mentioned that our consideration was virtually confined to conical singularities in the flat space and incorporation of the curvature effects represents an interesting problem.

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Appendix A. The zeta-function on a cone and on a sphere with conical singularities

The generalized zeta-function on a cone can also be considered as a functional on the chosen space of functions and related with the heat kernel via the Mellin transform [18]

$$\zeta_\alpha(z,f)=\frac{1}{\Gamma(z)}\int_0^\infty s^{z-1}e^{-m^2s}Tr\left(e^{-s\Delta_\alpha}f\right) \quad , \tag{A.1}$$

with a mass m providing convergence of the integral as $s\to\infty$. As the cone is a non-compact space, the convenient zeta-function, that is introduced through the trace of the heat kernel [18], can be defined as $\zeta_\alpha(z,f_V)$ with the help of the function f_V used in Section 2. It follows immediately from (2.16) that

$$\zeta_\alpha(z,f_V)=\frac{1}{4\pi}\left(\frac{V(m^2)^{1-z}}{z-1}+a_{\alpha,2}(f_V)(m^2)^{-z}\right) \quad . \tag{A.2}$$

In particular, taking into consideration (2.14) one gets from (A.2) at $m=0$ the finite expression that doesn't depend on the volume V

$$\zeta_\alpha(0)=\frac{\alpha}{24\pi}\left(\left(\frac{2\pi}{\alpha}\right)^2-1\right) \quad . \tag{A.3}$$

To take in our analysis the effects of curvature of the space, although simple ones, it is worth to compare $\zeta_\alpha(0)$ at $m=0$ with the zeta-function of the Laplace operator on the unit "sphere" with two conical singularities at "south" and "north" poles, where the corresponding line element reads

$$ds^2=\cos^2\chi d\varphi^2+d\chi^2, \quad 0\le\varphi\le\alpha, \quad |\chi|\le\pi/2 \tag{A.4}$$

and takes the form (1.1) as $|\chi|\to\pi/2$. Everywhere at the other points the metric (A.4) is regular and the space has a finite constant curvature, the same as the curvature of the ordinary unit sphere.

This example is interesting since the spectrum of the Laplace-Beltrami operator on (A.4) can be calculated exactly. It is determined by two non-negative integers n and m

$$\lambda_{n,m}=\left(n+(2\pi\alpha^{-1})m\right)\left(n+(2\pi\alpha^{-1})m+1\right) \tag{A.5}$$

with the double degeneracy for $m \neq 0$. Using the same transformations that were carried out in [11] for the case of four-dimensional analog of this "sphere", one can represent the zeta-function for $z \to 0, -1, -2, ..$ by the series

$$\zeta_\alpha^{sphere}(z) = \sum_{n,m} \lambda_{n,m}^{-z} =$$

$$= \frac{\alpha}{\pi} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\Gamma(z+k)\Gamma(2z+2k+2n-1)}{2^{2k}k!\,\Gamma(z)\Gamma(2z+2k)} \zeta_R(2z+2k+2n-1, 1/2) \frac{B_{2n}}{(2n)!} \left(\frac{2\pi}{\alpha}\right)^{2n} \quad (A.6)$$

where ζ_R is the Riemannian zeta-function and B_{2n} are the Bernulli numbers. It is not difficult to show in particular that for $z = 0$ it is given by the simple expression

$$\zeta_\alpha^{sphere}(0) = \frac{\alpha}{6\pi}\left[1 + \frac{1}{2}\left(\left(\frac{2\pi}{\alpha}\right)^2 - 1\right)\right] \quad . \quad (A.7)$$

Apart from the contribution [18]

$$\frac{\alpha}{6\pi} = -\frac{1}{4\pi}\int \sqrt{g} d^2x \, \frac{R}{6},$$

determined at the points where the metric (A.4) is regular by the scalar curvature of the sphere $R = -2$, it contains also an additional term, appearing because of the conical singularities at $|\chi| = \pi/2$ and equal exactly to the doubled value of the zeta-function on a cone $2\zeta_\alpha(0)$.

This result could be anticipated in advance, by taking into account that near each of the points $\chi = \pm\pi/2$ the heat kernel expansion on the space (A.4) can be approximated by the expansion on a cone (2.2).

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