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ANTIBRACKETS AND LOCALIZATION OF (PATH) INTEGRALS

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1. Recently a number of papers were published (for example, [1-3]), where exact evaluation of the phase space path integrals was studied using corresponding generalization [1] of the Duistermaat—Heckman localization formula [4] (DH-formula). In accordance with it, if on the compact manifold M provided with the symplectic structure $\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j$ the Hamiltonian $H(x)$ defines the action of the group $U(1) \sim S^1$, then

$$Z_0 = \int_M e^H(\omega)^N = \sum_{dH=0} \frac{e^H\sqrt{det\omega_{ij}}}{\sqrt{det\frac{\partial^2 H}{\partial x^i \partial x^j}}}. \qquad (1)$$

Using its path integral generalization, one can localize the phase space path integral into (finite-dimensional) integral over classical phase space.

This approach turns out convenient for a number of problems [2], in topological field theories particularly. It formed the basis for a conceptually new method of description of supersymmetric theories [3].

In the present letter we propose a simple method of the invariant description of DH-localization. For this, following [1-3] we present the integral (1) in the form

$$Z_0 = \int_M e^{H(x)} det\omega_{ij} d^{2N}x = \int_{\mathcal{M}} e^{H-F} d^{2N}x d^{2N}\theta, \qquad (2)$$

where θ^i are auxiliary Grassmannian fields ($p(\theta^i) = p(x^i) + 1$), which correspond to 1-forms dx^i, $\mathcal{M}$ is the supermanifold associated with the tangent bundle of M ($z^A = (x^i, \theta^i)$ are the local coordinates on M),

$$F(z) = -\frac{1}{2}\theta^i \omega_{ij} \theta^j. \qquad (3)$$

After that we shall define on $\mathcal{M}$ the *odd symplectic structure*. The corresponding odd Poisson brackets (antibrackets) give the Hamiltonian description (and natural interpretation) of the DH-localization without introduction of the additional structures, used in the cited papers.

Besides we show that the use of antibrackets gives the simple supersymmetrization method for the Hamiltonian systems, which define the isometries of the Riemannian metric on the their phase space.

Finally, the present constructions can be generalized straightforwardly to the case, if M is a symplectic *supermanifold*. Moreover, they are completely symmetrical according to the relation to initial and auxiliary coordinates.

All constructions presented in Letter relate to the finite-dimensional integrals over compact symplectic manifolds. One can accomplish their generalization for the path integrals by the lifting on the loop space analogously [1-3]. It does not principally change the presented description scheme.

Notice that it is naturally connected with the Batalin—Vilkovisky quantization formalism [5].

2. Let us provide the supermanifold $\mathcal{M}$, which we defined above, with odd symplectic structure

$$\Omega_1 = \omega_{ij} dx^i \wedge d\theta^j + \omega_{ij,k}\theta^j dx^i \wedge dx^k, \qquad (4)$$

where ω_{ij} corresponds to the symplectic structure on M.

The corresponding to (4) odd Poisson brackets (antibrackets)

$$\{f,g\}_1 = \frac{\partial_r f}{\partial z^A}\Omega_1^{AB}\frac{\partial_l g}{\partial z^B} \qquad (5)$$

are defined by the conditions:

$$\{x^i,x^j\}_1 = 0, \quad \{x^i,\theta^j\}_1 = -\{\theta^j,x^i\}_1 = \omega^{ij}, \quad \{\theta^i,\theta^j\}_1 = -\{\theta^j,\theta^i\}_1 = \frac{\partial\omega^{ij}}{\partial x^k}\theta^k \qquad (6)$$

where $\omega^{ij}\omega_{jk} = \delta^i_k$. The antibrackets (5-6) satisfy the Jacobi identity:

$$(-1)^{(p(f)+1)(p(h)+1)}\{f,\{g,h\}_1\}_1 + \text{cycl.perm.(f,g,h)} = 0. \qquad (7)$$

Let us map the functions on M to the odd functions on $\mathcal{M}$:

$$f(x) \to Q_f(z) = \{f(x), F(z)\}_1,$$

where F is defined by the expression (3). It puts the Hamiltonian dynamics $(H(x), \omega, M)$, into odd one $(Q, \Omega_1, \mathcal{M})$, where

$$Q = \{H,F\}_1, \qquad (8)$$

with the equation of motion

$$\frac{dx^i}{dt} = \{x^i,Q\}_1 = \{x^i,H_0\}_0 \equiv \xi^i_H, \quad \frac{d\theta^i}{dt} = \{\theta^i,Q\}_1 = \frac{\partial\xi^i_H}{\partial x^j}\theta^j. \qquad (9)$$

This dynamics is supersymmetric: from the closeness of ω follows $\{F,F\}_1 = 0$, and taking into account (8) we obtain the simplest superalgebra

$$\{H \pm F, H \pm F\}_1 = \pm 2Q,$$
$$\{H + F, H - F\}_1 = \{H \pm F, Q\}_1 = \{Q,Q\}_1 = 0. \qquad (10)$$

The following correspondence is obvious:

$$\begin{aligned}
\{H, \quad \}_1 &= \xi^i_H\frac{\partial}{\partial\theta^i} \to \imath_H - \text{the operator of interior product on } \xi_H;\\
\{F, \quad \}_1 &= \theta^i\frac{\partial}{\partial x^i} \to d - \text{the operator of exterior differentiation;}\\
\{Q, \quad \}_1 &= \xi^i_H\frac{\partial}{\partial x^i} + \xi^i_{H,k}\theta^k\frac{\partial}{\partial\theta^i} \to \mathcal{L}_H - \text{the Lie derivative along } \xi_H.
\end{aligned} \qquad (11)$$

Taking into account the Jacobi identity (7) we have:

$$\{H,F\}_1 = Q \to d\imath_H + \imath_H d = \mathcal{L}_H - \text{homotopy formula.}$$

As we see, the supersymmetry of $(Q, \Omega_1, \mathcal{M})$ corresponds to the equivariant differentiation $d_H = d + \imath_H$.

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