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OCTONIONS, SELF-DUALITY AND STRINGS

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1. Introduction

Recently, many supersymmetric solutions to the low-energy heterotic string field equations have been found [1-5]. Among them there are singular "d=2" solutions of the papers [2] which correspond to the solution of massless field equations outside a fundamental macroscopic string, and smooth solutions without source terms [3-5], which correspond to soliton solutions to string theory. There have been obtained "d=6" solutions [3,4] describing fivebrane solitons in heterotic and type II string theories, and "d=2" solutions [5] describing a solitonic string in ten-dimensional space-time. All "d=2" solutions correspond to the solutions of two-dimensional string theory (see, e.g., [6,7]) in which strings replace the usual pointlike world-sheet fields.

In constructing the above-mentioned solutions, the instanton, monopole and vortex solutions of the Self-Dual Yang-Mills (SDYM) equations in $n = 4$ [2-4] and solutions of the generalized SDYM equations in $n > 4$ dimensions [5] have been used. In [8-14], new classes of solutions to the SDYM equations in dimensions $n \geq 4$ have been found. In this paper, we shall show that these solutions can be used in constructing the string "d=2" and membrane "d=3" soliton-type solutions to the low-energy equations of motion of the heterotic string.

2. Supersymmetry in d dimensions

The low-energy action for the bosonic degrees of freedom of the heterotic string is given by

$$S = \frac{1}{2k^2}\int d^{10}x\,\sqrt{-g}\,e^{-2\phi}\left\{R + 4\partial_M\phi\partial^M\phi - \frac{1}{3}H_{MNP}H^{MNP} + \right.$$
$$\left. +\alpha'\left(R_{MNPQ}R^{MNPQ} - trF_{MN}F^{MN}\right)\right\}, \quad (1)$$

plus higher-order α' corrections [1]. Here

$$H = dB + \alpha'(\omega_3^L - \omega_3^Y),$$

and the Bianchi identity

$$dH = \alpha'(trR\wedge R - trF\wedge F), \quad (2)$$

$$\bar{G}_{\bar{m}\bar{n}\bar{p}\bar{q}} \equiv \frac{1}{8}(\delta_{\bar{m}\bar{p}}\delta_{\bar{n}\bar{q}} - \delta_{\bar{m}\bar{q}}\delta_{\bar{n}\bar{p}}) + \frac{1}{8}H_{\bar{m}\bar{n}\bar{p}\bar{q}}. \quad (7b)$$

To simplify the notation, we shall not below mark the indices in R^8 by bar, but everywhere we shall point out in which space R^7 or R^8 our tensors are defined.

The tensors G and $\bar{G}$ project an arbitrary antisymmetric tensor T_{mn} in R^8 onto the orthogonal 21- and 7-dimensional subspaces of the 28-dimensional vector space of the two-index antisymmetric tensors in R^8:

$$T_{mn} = \frac{1}{2}(\delta_{mp}\delta_{nq} - \delta_{mq}\delta_{np})T_{pq} =$$

$$= (G_{mnpq} + \bar{G}_{mnpq})T_{pq} = G_{mnpq}T_{pq} + \bar{G}_{mnpq}T_{pq} = T^+_{mn} + T^-_{mn}, \quad (8)$$

$$T^+_{mn} \in so(7), \quad T^-_{mn} \in \mathcal{P}, \quad so(7) \oplus \mathcal{P} = so(8),$$

Here $\mathcal{P}$ is an orthogonal complement to the algebra $so(7)$ in $so(8)$.

It is not hard to show that [15]

$$H_{mnrs}G_{rspq} = -2G_{mnpq}, \quad (9a)$$

$$H_{mnrs}\bar{G}_{rspq} = 4\bar{G}_{mnpq}. \quad (9b)$$

4. Projectors and spinors

The Clifford generators Γ_A in the space $R^{9,1}$ can be represented by the real 32×32 matrices [1]. So the Majorana spinors have 32 real components and the Weyl condition $\Gamma_{11}\epsilon_+ = \epsilon_+$ leaves 16 components.

Let the space $R^{9,1}$ be a direct product $R^{9,1} = R^{2,1} \times R^{7,0}$. Then Γ-matrices may be chosen in the form

$$\Gamma_i = \tau_i \otimes \gamma_8, \qquad \Gamma_m = 1_2 \otimes \gamma_m, \quad (10a)$$

where $\tau_1 = \sigma_1$, $\tau_2 = i\sigma_2$, $\tau_3 = \sigma_3$, σ_i are the Pauli matrices and

$$\gamma_m = \begin{pmatrix} 0 & \beta_m \\ -\beta_m & 0 \end{pmatrix}, \qquad \gamma_8 = \begin{pmatrix} 0 & 1_8 \\ 1_8 & 0 \end{pmatrix},$$

$$(\beta_m)^p_q = f^p_{mq}, \quad (\beta_m)^n_8 = -(\beta_m)^8_n = \delta^n_m. \quad (10b)$$

Here f^p_{mq} are the octonionic structure constants introduced in Sect.3.

It is not difficult to see that

$$\gamma_9 = \gamma_1 \cdots \gamma_8 = \begin{pmatrix} 1_8 & 0 \\ 0 & -1_8 \end{pmatrix},$$

and the Majorana-Weyl spinor ϵ_+ in $R^{9,1}$ can be represented as

$$\epsilon_+ = \psi \otimes \eta_+, \quad (11)$$

where $\psi \in R^2$ and $\eta_+ \in R^8$ is defined as an eigenvector of γ_9: $\gamma_9\eta_\pm = \pm\eta_\pm$.

By a direct calculation, one can easily check [15] that

$$g_{mnpq}\gamma_{pq}\eta_+ = 0, \quad (12)$$

where $\gamma_{pq} = \gamma_{[p}\gamma_{q]}$ and g_{mnpq} is the projector onto the subalgebra g_2 in $so(7)$, $m, n, p, ... = 1, ..., 7$.

If the space $R^{9,1}$ is represented in the form $R^{1,1} \times R^{8,0}$, then Γ-matrices may be chosen in the form

$$\Gamma_i = \tau_i \otimes 1, \qquad \Gamma_m = \tau_3 \otimes \gamma_m, \quad (13)$$

where $i, j, ... = 1, 2$; $m, n, ... = 1, ..., 8$; $\tau_1 = \sigma_1$, $\tau_2 = i\sigma_2$, $\tau_3 = \sigma_3$, and the matrices γ_m have been introduced in (10b).

As before, ϵ_+ is given by (11) and it is not difficult to show that [15, 16, 5]

$$G_{mnpq}\gamma_{pq}\eta_+ = 0, \quad (14)$$

where G_{mnpq} is the projector on the subalgebra $so(7)$ in $so(8)$.

5. Ansatz for g_{MN} and H_{MNP}

We consider the reduction to $d < 10$ dimensions and suppose that in $n = 10 - d$ dimensional subspace the completely antisymmetric four-index tensor $\mathcal{E}_{\mu\nu\lambda\sigma}$ is defined. This tensor is defined through the constant tensor $\mathcal{E}_{mnpq}$ in R^n and through the vielbein h^m_μ:

$$\mathcal{E}_{\mu\nu\lambda\sigma} = h^m_\mu h^n_\nu h^p_\lambda h^q_\sigma \mathcal{E}_{mnpq}, \quad g_{\mu\nu} = h^m_\mu h^n_\nu \delta_{mn}. \quad (15)$$

The explicit form of tensors $\mathcal{E}_{mnpq}$ in R^7 and R^8 has been described in Sect.3.

Following the papers [2-5], for H_{MNP} and g_{MN} we shall consider the ansatz

$$H_{\mu\nu\lambda} = A\mathcal{E}^{\sigma}_{\mu\nu\lambda}\partial_\sigma\phi, \quad g_{ij} = diag(-1,1,...,1), \quad g_{\mu\nu} = e^{2B\phi}\delta_{\mu\nu}, \quad (16)$$

where A and B are constant parameters; ϕ is a function of $x_\mu \in R^n$. The other components of g_{MN} and H_{MNP} are equal to zero.

From (16) it follows that $h^m_\mu = e^{B\phi}\delta^m_\mu$ and we can calculate the components ω^{AB}_M of the Levi-Civita connection

$$\omega^{mn}_\mu = B(\delta^m_\mu\delta^{\sigma n} - \delta^n_\mu\delta^{\sigma m})\partial_\sigma\phi. \quad (17)$$

All the other components of ω^{AB}_μ are equal to zero.

Let ϵ_+ be a constant spinor (see(11)). Substitute (16) and (17) into (3a). It is easily seen that for $n = 7$ Eqs.(3a) reduce to the identities (12) if

$$A = \frac{1}{6}C, \quad B = \frac{1}{3}C, \quad (18a)$$

where C is an arbitrary constant. For $n = 8$ Eqs.(3a) reduce to the identities (14) if

$$A = \frac{1}{8}D, \quad B = \frac{3}{8}D, \quad (18b)$$

where D is an arbitrary constant. The function ϕ remains arbitrary.

Now substitute $H_{\mu\nu\lambda}$ from (16) into Eqs.(3b) and use the definition of γ-matrices and identities (12), (14). We obtain that Eqs.(3b) are satisfied identically if $C = \frac{3}{2}$ and $D = \frac{8}{7}$. Thus, for the ansatz (16) Eqs.(3a) and (3b) are satisfied if

$$n = 7, \quad A = \frac{1}{4}, \quad B = \frac{1}{2}, \quad (19a)$$

$$n = 8, \quad A = \frac{1}{7}, \quad B = \frac{3}{7}. \quad (19b)$$

6. Self-duality

Let us consider $n = 7$. Here we suppose that only the components $F_{\mu\nu}$ of F are nonzero. Using (12), it is not difficult to rewrite (3c) in the form

$$\left(F_{\mu\nu} + \frac{1}{2}h_{\mu\nu\lambda\sigma}F^{\lambda\sigma}\right)\gamma^{\mu\nu}\eta_+ = 0, \quad (20)$$

where $\mu,\nu,... = 1,...,7$. Therefore, Eqs.(3c) reduce to the equations

$$h_{\mu\nu\lambda\sigma}F^{\lambda\sigma} = -2F_{\mu\nu}. \quad (21)$$

Equations (21) generalize the usual self-duality equations in $n = 4$ [16]. Notice that for the conformally flat metric of the ansatz (16) these equations coincide with the equations in the Euclidean space R^7; therefore, below we shall not distinguish the indices $\mu,\nu,...$ from the indices $m,n,...$ and shall write them lowered.

To solve the generalized SDYM equations, in [12] the following ansatz

$$A_\mu = \frac{3}{2}\, g_{\mu\nu mn}\, x_\nu\, W_{mn}(u), \quad (22)$$

has been suggested. Here W_{mn} take values in an arbitrary Lie algebra $\mathcal{G}$ of the Lie group G, $W_{mn}(u) = -W_{nm}(u)$, $u = \rho^2 + x_\mu x_\mu$, $\rho = const$. For the explicit form of the tensor $g_{\mu\nu mn}$ see (5a).

Equations (21) for the ansatz (22) reduce to the special case of the Rouhani-Ward (RW) equations (see [17,12,14]):

$$S_{mnpqrs}\frac{dW_{rs}}{du} = -[W_{mn}, W_{pq}]. \quad (23a)$$

Here

$$S_{mnpqrs} = (\delta_{mp}\delta_{n[r}\delta_{s]q} - \delta_{np}\delta_{m[r}\delta_{s]q} + \delta_{nq}\delta_{m[r}\delta_{s]p} - \delta_{mp}\delta_{n[r}\delta_{s]p}) \quad (23b)$$

are the structure constants of the group $SO(7)$. Each solution of the RW equations (23) gives the solution of the SDYM equations (21).

The simplest solution of these equations has the form

$$W_{mn} = \frac{1}{u}\gamma_{[m}\gamma_{n]}, \quad (24)$$

where γ_m have been introduced in formula (10b) and $\gamma_{[m}\gamma_{n]} \equiv \frac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m)$. A solution of Eqs.(21), corresponding to the solution (24) of Eqs.(23a), has the form

$$F_{\mu\nu} = -\frac{3}{2u^2}\, g_{\mu\nu mn}\left\{(u+\rho^2)\,\delta_{m[r}\delta_{s]n} + \right.$$
$$\left. +2x_m\delta_{n[r}x_{s]} - 3x_k x_{[m}\, g_{n]krs}\right\}\gamma_{rs}. \quad (25)$$

Now, let us consider $n = 8$. As before, we suppose that only the components $F_{\mu\nu}$ of F are nonzero $(\mu, \nu, ... = 1, ...8)$. Using the identities (14), Eqs. (3c) can be rewritten in the form

$$\left(F_{\mu\nu} + \frac{1}{2}\, G_{\mu\nu\lambda\sigma}\, F^{\lambda\sigma}\right) \gamma^{\mu\nu}\, \eta_+ = 0. \qquad (26)$$

Thus, Eqs.(3c) reduce to the generalized SDYM equations in R^8:

$$H_{\mu\nu\lambda\sigma}\, F^{\lambda\sigma} = -2F_{\mu\nu}. \qquad (27)$$

For the conformally flat metric of the ansatz (16) these equations coincide with the equations in the Euclidean space R^8. So we shall write all the indices lowered and shall not tell $\mu, \nu, ...$ from $m, n, ...$.

The following ansatz for the YM fields in R^8

$$A_\mu = \frac{4}{3}\, G_{\mu\nu mn}\, x_\nu\, W_{mn}(u) \qquad (28)$$

has been considered in ref.[12]. The explicit form of $G_{\mu\nu mn}$ is given in (7a); $W_{mn}(u) = -W_{nm}(u)$ is antisymmetric $\mathcal{G}$-valued tensor, $u = \rho^2 + x_\mu x_\mu, \rho = const.$

For the ansatz (28) the self-duality equations (27) are reduced to the special case of the RW equations, described as before by (23), but now the indices $m, n, ...$ must change from 1 to 8 and S_{mnpqrs} must be the structure constants of $SO(8)$. Each solution of these equations leads to the solution of Eqs.(27).

The simplest solution (24) of the RW equations gives the Fairlie-Nuyts-Fubini-Nicolai (FNFN) solution [11]:

$$F_{\mu\nu} = -\frac{2}{9u^2}\, G_{\mu\nu mn} \left(\frac{6(u+\rho^2)}{u^2}\, \gamma_{mn} + x_k\, G_{rsk[m}x_{n]}\, \gamma_{rs}\right). \qquad (29)$$

7. Bianchi identities

We have solved the conditions (3a)–(3c) of preservation at least one supersymmetry in d=2 and d=3 dimensions. It fixes the parameters A and B but the function $\phi(x^\mu)$ remains arbitrary. Now, if one substitutes the ansätze (16), (22) and (28) into the Bianchi identities (2), then obtains the equations on ϕ.

By direct calculations it is not hard to show that for the ansatz (16) $trR \wedge R = 0$. To solve the equations

$$dH = -\alpha'\, trF \wedge F, \qquad (30)$$

that follow from the Bianchi identities, one ought to use the explicit form of solutions of the generalized SDYM equations. The explicit solutions $F_{\mu\nu}$ of the generalized SDYM equations in the $n > 4$ dimensions are known only in the simplest cases (see [12-14]). So in [5] the FNFN solution (29) has been substituted into (30) and the explicit form of function ϕ has been found. Solutions of Eqs.(30) for $n = 4$ (ordinary self-duality) have been described in [3,4]. Here we shall describe the simplest solution for $n = 7$.

Thus, let us consider $d = 3$ and $n = 7$. Let us take the solution of Eqs.(21) given by (22), (24) and (25). By direct calculations we obtain that

$$tr\ F \wedge F = -\frac{3}{4}\, \frac{(u+\rho^2)}{u^4}\, h_{\mu\nu\lambda\sigma}\, dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\sigma.$$

Put $\psi := e^{\phi/2}$. Then, from (30) it follows that ψ satisfies the differential equation

$$\frac{d^2\psi}{du^2} + \frac{7}{2(u-\rho^2)}\, \frac{d\psi}{du} = -\frac{21}{}\, \alpha'\, \frac{(u+\rho^2)^2}{(u-\rho^2)u^4}.$$

The solution has the form

$$e^{\phi/2} = e^{\phi_0/2} + \frac{7}{8}\frac{\alpha'}{(\tau+\rho^2)} + \alpha' f(\tau, \rho),$$

where $f(\tau, \rho)$ is the function of $\tau = x_\mu x_\mu$ and $f(\tau, 0) = 0$. This function has a very complicated form, that is why we do not write out it here.

Thus, we have shown that solutions of the RW equations can give new soliton-type solutions of the low-energy heterotic string field equations.

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