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LAGRANGIAN FORMULATION OF SOME q-DEFORMED SYSTEMS

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The concept of deformation has played a notable part in the development of modern theoretical physics. The most familiar and cardinal examples of such class of deformed physical theories are presumed to be quantum mechanics and special theory of relativity with deformation parameters as Planck's constant ($\hbar$) and speed of light (c), respectively [1,2]. A key feature of thorough understanding of these theories is the emergence of fundamental constants of nature, namely; $\hbar$ and c. It is conjectured that the deformation of groups, based on the quasi-triangular Hopf algebras [3,4], together with the ideas of noncommutative geometry might provide a " fundamental length " in the context of space-time quantization [5] which would have close kinship with the dimensionless deformation parameter (q) of the deformed groups [6]. Some attempts have recently been made to associate "q" with relativistic quantities [7] and length of compactification [8] in the context of some concrete physical examples. In addition, these deformed (so-called quantum) groups have also been treated as gauge groups for the development of the q-deformed gauge theories [9].

It is interesting endeavour to apply ideas of quantum groups in a cogent way to some known physical systems [7-10]. The purpose of the present paper is to develop the Lagrangian formulation for some known physical systems by exploiting the basic ingredients of the quantum group $GL_q(2)$ and corresponding differential calculus [11] discussed on the quantum phase space [12]. We obtain q-deformed Legendre transformations and relevant Euler-Lagrange equations of motion for free non-relativistic particle, harmonic oscillator and relativistic particle on a quantum line\which are consistent with the q-deformed Hamilton's equations of motion. One of the salient features of our approach is that the equations of motion for a given q-deformed physical system remain *the same* as that of its undeformed (classical) counterpart but the momentum, velocity and force etc.depend on the deformation parameter q. It is fascinating to find that, mass and metric in the case of the q-deformed relativistic particle, turn out to be non-commutative objects on the quantum world-line embedded in a D-dimensional undeformed flat Minkowski space.

We start off with the free motion of a non-relativistic particle on a quantum-line [12] characterized by coordinate generator $x(t)$ and momentum generator $p(t)$ that satisfy following relationship on this line [1]

$$x(t)\, p(t) = q\, p(t)\, x(t), \tag{1}$$

where the q-trajectory of the particle, moving on a q-deformed cotangent manifold, is parameterized by a real commuting variable t. It is straightforward to check that above relation is form-invariant under following $GL_q(2)$ transformations

$$\begin{pmatrix} x \\ p \end{pmatrix} \rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ p \end{pmatrix}, \tag{2}$$

if we assume the commutativity of the phase variables with the elements a,b,c and d of the 2×2 $GL_q(2)$ matrix obeying following braiding relations in rows and columns:

$$ab = qba; \quad cd = qdc; \quad ac = qca; \quad bc = cb,$$

$$bd = qdb; \quad ad - da = (q - q^{-1})\, bc. \tag{3}$$

[1]Note that the definition of the quantum-line present in ref.[12] would be obtained from (1) by replacement: $q \rightarrow q^{-1}$.

To develop the Lagrangian formulation for a given classical system, it is essential to discuss its dynamics in the tangent (velocity phase) space. The second order Lagrangian (L_s) describing the free motion ($m\ddot{x} = 0$) in this space is as follows

$$L_s = \frac{q}{1+q^2}\, m\, \dot{x}^2, \tag{4}$$

where, in addition to $x(t)$ and $p(t)$, t-independent mass parameter m, is also a hermitian element of an algebra with involution (i.e. $|q| = 1$) and $\dot{x} = \frac{dx}{dt}$.

The most basic geometrical object in classical mechanics is the non-degenerate and closed two-form symplectic structure, defined on a symplectic (cotangent) manifold. The covariant and the contravariant q-symplectic metrices that reduce to their classical canonical counterparts in the limit $q \rightarrow 1$, are [see, e.g. ref. 2]:

$$\Omega_{AB}(q) = \begin{pmatrix} 0, & -q^{-1/2} \\ q^{1/2}, & 0 \end{pmatrix} \quad \text{and} \quad \Omega^{AB}(q) = \begin{pmatrix} 0, & q^{-1/2} \\ -q^{1/2}, & 0 \end{pmatrix}. \tag{5}$$

The first-order Lagrangian (L_f) describing the free motion ($\dot{p} = 0$) can be obtained by exploiting the covariant metric $\Omega_{AB}(q)$ in the following Legendre transformations

$$L_f = z^A\, \bar{\Omega}_{AB}(q)\, \dot{z}^B - H \equiv q^{1/2}\, p\, \dot{x} - H, \tag{6}$$

where $z^A = (x, p)$; H is the Hamiltonian function defined on the cotangent manifold and the general expression [see, e.g. ref.8] $\bar{\Omega}_{AB}(z) = \int_0^1 \Omega_{AB}(\alpha z)\alpha d\alpha$ reduces in our case to $\bar{\Omega}_{AB}(q) = \frac{1}{2}\Omega_{AB}(q)$. The definition of the canonical momentum (p) crucially depends on the choice of the symplectic metric and (non)commutativity of velocity ($\dot{x}$) and momentum (p) in the Legendre transformations (6). For the first- and second-order Lagrangians, the consistent expression for this quantity is as follows

$$p = q^{-3/2} \left(\frac{\partial L_{(f,s)}}{\partial \dot{x}} \right) \equiv q^{1/2}\, m\, \dot{x}, \tag{7}$$

where on-shell noncommuatative relations $\dot{x}\, m = q\, m\, \dot{x}$ and $\dot{x}\, p = q\, p\, \dot{x}$,emerging from the $GL_q(2)$ invariant quantum-line (1), have been used in the derivation of (7). Furthermore, consistent with these non-commutative relations, following rule of the differentiation has been invoked [11]

$$\frac{\partial (y^r\, \dot{x}^s)}{\partial \dot{x}} = y^r\, \dot{x}^{s-1}\, q^r\, \frac{(1 - q^{2s})}{(1 - q^2)}, \tag{8}$$

where $y = m, p$ and $r, s \in \mathcal{Z}$ are real numbers but not fractions. The Hamiltonian function H, describing the motion in the cotangent space, can be obtained by the Legendre transformation ($H = q^{1/2}\, p\, \dot{x} - L_s$) and equation (7). This is expressed in terms of the noncommutative mass parameter (m) and momentum (p) as [12]:

$$H = \frac{q^2}{1+q^2}\, p\, m^{-1}\, p. \tag{9}$$

The contravariant symplectic metric of (5) is used in the computation of the the q-deformed Poisson-brackets present in the Hamilton's equations of motion. For instance, $\dot{x} = \{x, H\}_q = \Omega^{AB}\partial_A x \partial_B H \equiv q^{-1/2}\ m^{-1}\ p$ and $\dot{p} = \{p, H\}_q = 0$ result in due to the $GL_q(2)$ invariant differential calculus defined on the phase space. All the on-shell, associative and non-commutative relations, resulting from the defining quantum-line equation (1), are listed below:

$$\begin{aligned} \dot{x}\, p &= q\, p\, \dot{x}; \quad p(t)\, m = q\, m\, p(t), \\ \dot{x}\, m &= q\, m\dot{x}; \quad x\, m = q\, m\, x; \quad \dot{x}\, x = x\, \dot{x}. \end{aligned} \tag{10}$$

In the computation of the Hamilton's equations ($\dot{x}$ and $\dot{p}$), the Hamiltonian is firstly recast in the monomial form $m^r p^s$ ($r, s \in \mathcal{Z}$ are real numbers but not fractions) and then, following differentiation rule is used:

$$\frac{\partial(m^r p^s)}{\partial p} = m^r\ p^{s-1}\ q^r\ \frac{(1-q^{2s})}{(1-q^2)}. \tag{11}$$

One of the key features of our discussion is that the q-dependence appears only in the expressions for velocity and momentum but the equations of motion remain the same as that in the undeformed case. In addition to the description of quantization on a quantum-line, it has been demonstrated in ref.[12] that the solutions of equations of motion respect $GL_q(2)$ invariance at arbitrary time t.

The Hamilton's equations of motion can be derived by requiring the invariance of the action ($S = \int L_f dt$) in the framework of the principle of the least action, as illustrated below

$$\delta S = 0 = \int \Big(q^{1/2}\ \delta p\ \dot{x} + q^{1/2}\ p\ \delta\dot{x} - \delta x\ \frac{\partial H}{\partial x} - \delta p\ \frac{\partial H}{\partial p} \Big) dt, \tag{12}$$

where Hamiltonian is assumed to possess no explicit time dependence . Now taking all the variations to the left side by exploiting following on-shell q-commutation relations resulting from (1)

$$\delta\dot{x}\ p = q\ p\ \delta\dot{x}; \qquad \dot{x}\ \delta p = q\ \delta p\ \dot{x}, \tag{13}$$

and dropping off the total derivative term by choosing appropriate boundary conditions on the transformation parameters, we obtain following equations of motion

$$\dot{x} = q^{-1/2}\ \frac{\partial H}{\partial p} \quad \text{and} \quad \dot{p} = -\ q^{1/2}\ \frac{\partial H}{\partial x}, \tag{14}$$

which are in total agreement with the choice of the contravariant symplectic metric (5) and the q-deformed Poisson-brackets.

To derive the deformed Euler-Lagrange equations of motion, it is instructive to consider the q-deformed harmonic oscillator on the quantum-line (1). The Hamiltonian, first- and second-order Lagrangians for this system are

$$\begin{aligned} H^{osc} &= \frac{q^2}{1+q^2}\, p\, m^{-1}\, p + \frac{q^{-2}\,\omega^2}{1+q^2}\ x\ m\ x, \\ L_f^{osc} &= q^{1/2}\, p\, \dot{x} - \frac{q^2}{1+q^2}\, p\, m^{-1}\, p - \frac{q^{-2}\,\omega^2}{1+q^2}\ x\ m\ x, \\ L_s^{osc} &= \frac{q}{1+q^2}\, m\, \dot{x}^2 - \frac{q^{-2}\,\omega^2}{1+q^2}\ x\ m\ x, \end{aligned} \tag{15}$$

where frequency ω is a commuting number. All the q-commutation relations (10) are valid in this case as well, because, the Hamilton's equations of motion $\dot{x} = \{x, H^{osc}\}_q = q^{-1/2}\ m^{-1}\ p$ and $\dot{p} = \{p, H^{osc}\}_q = -\ \omega^2\ q^{1/2}\ m\ x$ do not spoil these relations. Moreover, the extra q-commutation relations $x\ \dot{p} = q\ \dot{p}\ x$ and $\dot{p}\ p = p\ \dot{p}$ are automatically satisfied due to (10). The expression for the canonical momentum (p) is same as (7) and, consistent with the Hamilton's equations, the Euler-Lagrange equation of motion ($\ddot{x} = -\omega^2\ x$) is:

$$q^{-3/2}\ \frac{d}{dt}\Big(\frac{\partial L^{osc}_{(f,s)}}{\partial \dot{x}}\Big) = q^{1/2}\Big(\frac{\partial L^{osc}_{(f,s)}}{\partial x}\Big). \tag{16}$$

The Hamiltonian (H_v), describing the motion of a classical q- deformed particle moving under the influence of a potential $V(x)$ is as follows:

$$H_v = \frac{q^2}{1+q^2}\ p\ m^{-1}\ p\ + V(x). \tag{17}$$

The first- and second-order Lagrangians can be derived in analogous manner as in (15). All the non-commutative relations listed above and the Euler-Lagrange equation of motion, remain the same if the potential $V(x)$ obeys:

$$x\ \frac{\partial V}{\partial x} = q\ \frac{\partial V}{\partial x}\ x. \tag{18}$$

This requirement implies that the force on the system is a non-commutative object. Equation (18) is satisfied by the harmonic oscillator potential due to (10). The $GL_q(2)$ invariant evolution equations, quantization and oscillator realisations have been discussed in ref.[12].

With the above background, we shall dwell a bit more on the free motion of a q-deformed relativistic particle on a quantum world-line parameterized by a commuting evolution parameter τ. A quantum world-line, traced out by the free motion of a q-relativistic particle in a D-dimensional flat Minkowski space, must be a Lorentz scalar. It can be readily seen that the following Lorentz invariant scalar product

$$x_\mu(\tau)\ p^\mu(\tau) = q\ p_\mu(\tau)\ x^\mu(\tau), \tag{19}$$

defined in terms of the D-dimensional coordinate generator x_μ and momentum generator p_μ,remains invariant under following transformations

$$\begin{aligned} x_\mu &\rightarrow a\ x_\mu + b\ p_\mu, \\ p_\mu &\rightarrow c\ x_\mu + d\ p_\mu, \end{aligned} \tag{20}$$

if we assume the commutativity of the phase variables with the elements a,b,c,and d of a $2 \times 2GL_q(2)$ matrix obeying the braiding relations (3). In the definition of the q-world-line (19), the repeated indices are summed over ($\mu = 0, 1, 2.........D-1$) and $GL_q(2)$ symmetry transformations (20) are implied for each component-pairs of the phase variables: $(x_o, p_o), (x_1, p_1), ...(x_{D-1}, p_{D-1})$. It will be noticed that another combination of the phase variables,namely; $x_\mu\ p_\nu = q\ p_\mu\ x_\nu$ is also component-wise $GL_q(2)$ invariant and Lorentz invariant. However, this relation is not a Lorentz scalar and, therefore, is not suitable for the definition of a quantum world-line. Moreover, the classical limit $q \rightarrow 1$ of the latter relation does not yield the undeformed relations between phase variables of the undeformed Minkowski space.

The other undeformed relations between phase variables with different Greek indices are $x_\mu\, x_\nu = x_\nu\, x_\mu$ and $p_\mu\, p_\nu = p_\nu\, p_\mu$. Consistent with the definition of the quantum world-line (19), the coordinate and momenta with different indices are assumed to satisfy: $x_\mu\, p_\nu = q\, p_\nu\, x_\mu$. All the relations quoted in this paragraph are associative and Lorentz invariant but *not* $GL_q(2)$ invariant. It can be seen that the requirement of the $GL_q(2)$ invariance of $x_\mu\, x_\nu = x_\nu\, x_\mu$ and $p_\mu\, p_\nu = p_\nu\, p_\mu$ entails a $GL_q(2)$ invariant relation $p_\mu\, x_\nu - p_\nu\, x_\mu = q\,(x_\nu\, p_\mu - x_\mu p_\nu)$. However, the latter relation is not consistent with the defining quantum world-line relation (19) because it becomes zero for $\mu = \nu$. Thus, the Lorentz invariance and $GL_q(2)$ invariance *cannot* be respected *together* in the general case of the q-commutation relations.

The Hamiltonian, describing the free motion of the q-relativistic particle, is as follows

$$\mathcal{H} = \frac{q}{1+q^2}\,\left(p_\mu\, e\, p^\mu - m\, e\, m\right), \tag{21}$$

where e is the einbein (metric) and all the variables, except mass parameter m, are function of τ. The Hamilton's equations of motion, derived from the generalized form of the Poisson-bracket and the symplectic metric (5), are as follows

$$\begin{aligned} \dot{x}_\mu &= \{x_\mu, \mathcal{H}\}_q = q^{1/2}\, e\, p_\mu, \\ \dot{p}_\mu &= \{p_\mu, \mathcal{H}\}_q = 0, \end{aligned} \tag{22}$$

where $\dot{x}_\mu = \frac{dx_\mu}{d\tau}$. The defining quantum world-line equation (19), together with (22) and associativity requirements, leads to the validity of following on-shell non-commutative relations:

$$\begin{aligned} \dot{x}_\mu\, p^\mu &= q\, p_\mu\, \dot{x}^\mu; \quad \ddot{x}_\mu\, p^\mu = q\, p_\mu\, \ddot{x}^\mu; \quad e\, m = qm\, e; \quad \dot{x}_\mu\, m = q\, m\, \dot{x}_\mu, \\ e\, p_\mu &= q\, p_\mu\, e; \quad x_\mu\, m = q\, m\, x_\mu; \quad e\, \dot{x}_\mu = q\, \dot{x}_\mu\, e; \quad p_\mu\, m = m\, p_\mu. \end{aligned} \tag{23}$$

It can be seen that these relations are consistent with $x_\mu x_\nu = x_\nu x_\mu; p_\mu p_\nu = p_\nu p_\mu; x_\mu p_\nu = q p_\nu x_\mu$ if we assume $e\, x_\mu = q\, x_\mu\, e$ and use on-shell conditions (22). The first- and second-order Lagrangians can be derived from the Hamiltonian (21), as listed below:

$$\begin{aligned} L_F &= q^{1/2}\, p_\mu \dot{x}^\mu - \frac{q}{1+q^2}\,\left(p_\mu\, e\, p^\mu - m\, e\, m\right), \\ L_S &= \frac{q^2}{1+q^2}\, e^{-1}\,(\dot{x}_\mu)^2 + \frac{q}{1+q^2}\, m\, e\, m. \end{aligned} \tag{24}$$

Analogous to equation (7) and consistent with (22), the expression for the canonical momenta (p_μ) is:

$$p_\mu = q^{-3/2}\left(\frac{\partial L_{(F,S)}}{\partial \dot{x}^\mu}\right) \equiv q^{-1/2}\, e^{-1}\, \dot{x}_\mu. \tag{25}$$

The q-differentiation [11] of the second-order Lagrangian L_S with respect to the multiplier field "e" yields

$$\frac{q^4}{1+q^2}\left[m^2 - q^{-1} e^{-1}\, \dot{x}_\mu\, e^{-1} \dot{x}^\mu\right] = 0, \tag{26}$$

which leads to the mass-shell condition for the q-deformed free particle as follows:

$$p_\mu\, p^\mu - m^2 = 0. \tag{27}$$

This equation is one of the Casimir invariants of the Poincaré group corresponding to undeformed Minkowski space. The eigen value of this operator and Pauli-Lubanski vector would designate the eigen states, that would be needed for the representation theory of the Poincaré group. This constraint condition is also in neat conformity with the recent discussion [13] of Klein-Gordon equation and Dirac-equation derived from the q-deformation of the Dirac-γ matrices. Furthermore, equation (26) yields following relationship amongst einbein, velocity and mass parameter:

$$e^{-2} = m^2\; (\dot{x}_\mu\, \dot{x}^\mu)^{-1}. \tag{28}$$

The computation of e and e^{-1} from (28) is a bit tricky because of the non-commutativity of velocity and mass. A nice and simple way to compute these is firstly start with

$$e^{-1} = f(q)\, m\, (\dot{x}^2)^{-1/2}, \tag{29}$$

and require validity of (28). Using q-commutation relations (23), the second order Lagrangian L_S can be recast in various forms where e^{-1} and e would occupy different positions in its first and second terms. Requirement of the equality of the resulting Lagrangians leads to the determination of $f(q)$ to be $q^{1/2}$ if substitution (29) is made[2]. Ultimately, following q-deformed Lagrangian with square root is obtained from the second-order Lagrangian L_S:

$$L_0 = q^{1/2}\, m\, (\dot{x}_\mu\, \dot{x}^\mu)^{1/2}. \tag{30}$$

The action $A = q^{1/2} m \int_{\tau_1}^{\tau_2} d\tau\, (\dot{x}^2)^{1/2} \equiv q^{1/2}\, m\, \int_1^2 ds$ corresponding to (30) and, proportional to the path length $ds = (dx_\mu dx^\mu)^{1/2}$, is invariant under undeformed reparametrization transformations $\tau \rightarrow f(\tau)$ where $f(\tau)$ is monotonically varying function of τ. The definition of the canonical momentum (25) is correct in the case of above Lagrangian too. This can be seen (with $(\dot{x}^2)^{1/2}\, m = q\, m\, (\dot{x}^2)^{1/2}$) as follows:

$$p_\mu = q^{-3/2}\, \frac{\partial (\dot{x})^2}{\partial \dot{x}^\mu}\, \frac{\partial (\dot{x}^2)^{1/2}}{\partial \dot{x}^2}\, \frac{\partial [q^{1/2} m (\dot{x}_\mu\, \dot{x}^\mu)^{1/2}]}{\partial (\dot{x}^2)^{1/2}} \equiv \dot{x}_\mu\, (\dot{x}^2)^{-1/2}\, m. \tag{31}$$

It should be emphasized here that, while computing q-derivative of the q-variables with fractional power, following rule has to be invoked

$$\frac{\partial}{\partial x}(x^{r/s}) = \frac{(1-q^{2r})}{(1-q^{2s})}\; x^{(r/s)-1}, \tag{32}$$

[2]We have also used $\dot{x}_\mu (\dot{x}^2)^{1/2} = (\dot{x}^2)^{1/2} \dot{x}_\mu$ which results in from (29), $e\dot{x}_\mu = q\, \dot{x}_\mu e$ and $\dot{x}_\mu m = q\, m\, \dot{x}_\mu$.

where r is *not* divisible by s ($r, s \in \mathcal{Z}$). Furthermore, the mass-shell condition (27) is satisfied for both the left chain rule as well as the right chain rule of differentiation, implemented in the computation of (31).

The equation of motion $\dot{p}_\mu = 0$, resulting from (31), leads to the following expression

$$\ddot{x}_\mu\,(\dot{x}^2) - \dot{x}_\mu(\dot{x}_\nu\ddot{x}^\nu) = 0, \tag{33}$$

which corresponds to the equation of motion in the undeformed case. It is difficult to extract out the evolution equation at arbitrary τ from (33). The viable alternative is to parameterize the evolution equation in terms of the path length "s"[14]. In the purview of this change of parameterization, the canonical momentum $p_\mu = m\,\frac{dx_\mu}{ds} \equiv m\,\dot{x}_\mu(\dot{x}^2)^{-1/2}$, leads to the equation of motion $\left(m\,\frac{d^2x_\mu}{ds^2} = 0\right)$. The evolution equations

$$\begin{aligned} x_\mu(s) &= x_\mu(o) + m^{-1}\,p_\mu(o)\,s, \\ p_\mu(s) &= p_\mu(o), \end{aligned} \tag{34}$$

respect the "$GL_q(2)$-invariance" (i.e. $x_\mu(s)\;p^\mu(s) = q\;p_\mu(s)\;x^\mu(s)$) at arbitrary path-length s because ($s\,p_\mu(o) = q\,p_\mu(o)\,s$) and ($s\,m = q\,m\,s$)[3]. It is worth pointing out that, in contrast to the commutativity of τ, the path length s is a non- commutative parameter which turns out to be handy only in the description of the evolution equations.

It is important to pin-point here that, unlike the non-relativistic cases where $p\,m = q\,m\,p$,one obtains $p_\mu\,m = m\,p_\mu$ in the case of q-deformed relativistic free particle. The correctness of these relations can be checked by using on-shell q-commutation relations (10),(23) and substitutions: $p = q^{1/2}\,m\,\dot{x}; p_\mu = q^{-1/2}e^{-1}\dot{x}_\mu$. In fact, the space part of $p_\mu m = q^{-1/2}(e^{-1}\,\dot{x}_\mu\,m)$ reduces to $(m\,\dot{x}\,m)$ in the (one-dimensional) non-relativistic limit which corresponds to $\dot{x}_\mu \to \dot{x}$, $(\dot{x}^2)^{-1/2} \to 1$ and $e^{-1} = q^{1/2}\,m\,(\dot{x}^2)^{-1/2} \to q^{1/2}\,m$. Now, due to (10), it is clear that ($m\,\dot{x}\,m = q\,m\,m\,\dot{x}$) yields the non-relativistic relation $p\,m = q\,m\,p$. This conclusion can also be drawn from the relativistic relation $p_\mu\,e^{-1} = q\,e^{-1}\,p_\mu$ because in the non-relativistic limit: $p_\mu \to p$ and $e^{-1} = q^{1/2}\,m\,(\dot{x}^2)^{-1/2} \to q^{1/2}\,m$. The commutativity of mass parameter m and momenta p_μ in the case of the q-relativistic particle is primarily due to the existence of mass-shell condition (27).

All the three Lagrangians of (24) and (30) are equivalent and are endowed with gauge and reparametrization symmetries. To illustrate this, we shall concentrate on the first order Lagrangian L_F. It is obvious that the q- canonical momentum (Π_e) with respect to the multiplier field $e(\tau)$ is zero. Thus, $\Pi_e \approx 0$ is the primary constraint. The secondary constraint $\Pi_e^{(1)}$ can be obtained by requiring the consistency of the primary constraint under time evolution, generated by the Hamiltonian $\mathcal{H}$. This is given by

$$\Pi_e^{(1)} = \{\Pi_e, \mathcal{H}\}_q = -\frac{q^{1/2}q^4}{1+q^2}\,(p^2 - m^2) \approx 0, \tag{35}$$

which amounts to the validity of the mass-shell condition. Both these constraints are first class in the language of Dirac and there are no tertiary constraints. The gauge symmetry

[3]These q-commutation relations are obtained due to (29), $e\;p_\mu = q\;p_\mu\;e$, $(\dot{x}^2)^{1/2}\;m = q\;m\;(\dot{x}^2)^{1/2}$, $p_\mu\;m = m\;p_\mu$ and commutativity of τ.

transformations generated by these constarints are as follows

$$\delta x^\mu = q^{1/2}\,\xi\,p^\mu; \qquad \delta p^\mu = 0; \qquad \delta e = q^2\,\dot{\xi} \tag{36}$$

where ξ is the non-commutative gauge transformation parameter. (This can be seen by the application of the transformations (36) and requiring the validity of (22) on the q-deformed world-line (19) which yields : $\xi\;p_\mu = q\;p_\mu\;\xi$). As per our convention, all the symmetry transformations are firstly taken to the left and then substitutions (36) are made. The quasi-invariance of the Lagrangian is succinctly expressed as follows

$$\delta L_F = \frac{d}{d\tau}\left[\frac{\xi(p^2 + q^2m^2)}{(1+q^2)}\right], \tag{37}$$

where the chain rule $\frac{d\,p^2}{d\,\tau} = \frac{\partial p^2}{\partial p^\mu}\frac{\partial p^\mu}{\partial \tau} = (1+q^2)p_\mu\dot{p}^\mu$ has been used. Even if we do not take the symmetry variations to the left side in all the terms of L_F but exploit the non-commutatitivity of ξ, then also, we end up with the transformation (37).

In addition to the gauge symmetry, the first-order Lagrangian is also endowed with following reparametrization symmetry transformations

$$\delta_r\,x_\mu = \epsilon\,\dot{x}_\mu; \qquad \delta_r\,p_\mu = \epsilon\,\dot{p}_\mu; \qquad \delta_r\,e = \frac{d}{d\tau}(\epsilon e), \tag{38}$$

emerging due to the one-dimensional diffeomorphism $\tau \to \tau - \epsilon(\tau)$ with commuting infinitesimal transformation parameter ϵ. (This is because of the fact that $\delta_r x_\mu\;p^\mu = q\;p_\mu\;\delta_r x^\mu$ with $\dot{x}_\mu\;p^\mu = q\;p_\mu\;\dot{x}^\mu$ leads to $p_\mu\epsilon = \epsilon\;p_\mu$). In fact, the first order Lagrangian undergoes following change under (38):

$$\delta_r L_F = \frac{d}{d\tau}\Big(\epsilon\;L_F\Big). \tag{39}$$

In the usual undeformed ($q = 1$) case of the free relativistic particle, the gauge (36) and the reparametrization (38) symmetries are equivalent on-shell with the identification $\xi = \epsilon e$ [15]. However, in the deformed case these are not equivalent because the transformations of the einbein field, in spite of the above identification, are not equal unless $q = \pm 1$. This discrepancy might manifest itself at very high energy and might turn out to have some significant implications in the study of non-commutative geometry and space-time structure at this energy scale.

The q-deformed free relativistic particle presents a prototype example of q-deformed Abelian gauge theories. In addition to the explicit derivation of the Noether's theorem, it would be worthwhile to develop a q-deformed BRST formalism to quantize this system on a quantum world-line. It seems, there would not be any principal difficulties in the extension of our results for the discussion of q-deformed spinning relativistic particle where the ideas of the quantum group $GL_q(1|1)$ would play a prominent role. Furthermore, it would be interesting to generalize the second order Lagrangian L_S to q-deformed string action and discuss various subtleties involved in it. We hope to come to these problems in future.

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