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CLASSICAL DYNAMICS OF RIGID STRING FROM WILLMORE FUNCTIONAL

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1 Introduction

Polyakov-Kleinert (P-K) rigid string model [1, 2] has been recently widely considered by researchers working in fields ranging from particle physics [3] and cosmology [4, 5] through condensed matter [6] and biophysics [7]. For a recent review, e.g. see [3, 8].

In spite of the above interest of various groups of researchers in the rigid string model, very little progress has been achieved to date in our theoretical understanding of this model. Indeed, unlike the Nambu-Goto (N-G) model [9] which is thoroughly studied both classically and quantum mechanically, the rigid string model is not well understood even at the classical level. If the traditional methods developed for N-G model are applied to rigid string, then, at the classical level, the rigid string equations of motion are nonlinear in any gauge. The above nonlinearity precludes the use of conventional quantization methods developed for N-G model. Accordingly, for the rigid string model there is no analog of the Virasoro algebra which allows to determine the critical dimension of this model.

In view of situation just described, most of our knowledge about the rigid string is based on rather inconclusive numerical simulation results which employ the discretized lattice version of P-K string [10]. These simulations typically involve only study of the Euclidean version of the P-K rigid string. In case of Minkowski space-time this model is having an additional ghost states [2, 11]. Because of these states, the rigid string model has been severely criticized recently [12]. The authors of Ref. 12 had come to the conclusion that "either fourth derivative kinetic term most be quantized with an indefinite norm... or with energy unbound from below."

The rigid string action is given by [1, 2]

$$A = \gamma \iint dS + \alpha \iint H^2 dS \tag{1.1}$$

where α and γ are some constants. In (1.1), integration takes place over the string world surface S which has extrinsic mean curvature H. For simplicity we confine ourselves to considering the three-dimensional space-time and to the Euclidean version of this model. In order to rid of the boundary conditions we shall treat closed string world surfaces which are encountered, for example, in string vacuum functional.

The equations of motion for action (1.1) written in terms of the string coordinates are very complicated. They stand for a system of nonlinear partial differential equations of the fourth order [13]. Except for one model example [14], nothing is known about solutions of such system of equations. However, varying the functional (1.1) one can arrive at quite simple equations relating basic geometrical invariants of the string world surface, its Gauss curvature K and mean curvature H. We shall be working mainly with equations of such type.

The main result of our work can be formulated as follows. We are going to demonstrate that the effects of N-G term in the total action A (the first term in equation (1.1)) could be accounted for by considering the truncated action which contains only the second term in (1.1) provided that this term is considered not in flat but in curved space-time. Thus truncated action is known in the literature as Willmore functional [15] and, whence, we shall call string model, based on such truncated action, as Willmore string. Reduction of the variational problem for the rigid string action A to that for Willmore string is advantageous for number of reasons.

First, because the action (1.1) is two dimensional extension of the action used for particles with curvature-dependent action [16, 17], it is logically natural to search for methods which extend those developed for particles to that used for strings. In Ref. 18 the non-relativistic quantum mechanics of point-like particles is formulated

on the surface of 3-sphere S^3 embedded in R^4. Such reformulation leads to the emergence of spin for initially spinless particle. Because the statistical mechanics of spinless particles is the same as fully flexible polymers, the presence of spin for such particles leads to the effective rigidification of initially flexible polymers [19]. Alternatively, such rigidification could be achieved if the above "particle" moves in the presence of (in general) (non)abelian monopole gauge field [20]. In this case the rigidification mechanism lies in the replacement of ordinary derivatives in "flexible particle" Hamiltonian by covariant ones causing our particle to move in the effective gauge (gravity) field. The present work can be viewed as extension of the above ideas to the case of two dimensional objects, e.g., rigid strings.

Second, in modern geometry there had been accumulated considerable amount of results related to Willmore functionals, e.g. see Ref. [21, 22], so that our understanding of rigid strings (at least at the classical level), in view of results of our work, will depend to large extent on appropriate interpretation and utilization of the already accumulated knowledge.

The layout of the paper is as follows. In Section 2 we provide auxiliary facts from classical differential geometry of surfaces in order to arrive at Willmore's "equation of motion" connecting Gaussian curvature K with mean curvature H in flat embedding space-time. In Section 3 we extend the above results to the case when the embedding space is the space of constant curvature. In the context of quantum field theories the problem of embedding of the corresponding field-theoretic model into space of constant negative curvature was recently considered in [23] in connection with improved infrared regularization of QCD. In our case we study the embedding with different purpose. By doing so we are hoping to apply some results about the Nambu-Goto string dynamics in the de Sitter universe to the rigid string model in the Minkowski space-time. Finally, in Section 4 (Conclusion) we provide a brief summary and discussion of possible future applications of the obtained results. In Appendix a more simple one-dimensional version of our problem is considered.

Instead of surfaces in action (1.1) we are dealing here with curves.

2 Normal variations of the surfaces

For the completeness we give here the basic equations from the classical differential geometry of the surfaces that will be required in the following [24].

Let $x^\mu(u^1, u^2)$, $\mu = 1, 2, 3$ be a parametric representation of the surface M in the three dimensional Euclidean space E^3 and n^μ is a unit normal to the surface. Intrinsic differential geometry of the surface is defined by the induced metric or the first quadratic differential form of the surface

$$g_{i,j}(u^1, u^2) = x^\mu_{,i} x^\mu_{,j}, \quad x^\mu_{,i} \equiv \frac{\partial x^\mu(u^1, u^2)}{\partial u^i}, \quad i, j = 1, 2. \tag{2.1}$$

The central point of the surface theory is the derivation equations of Gauss

$$x^\mu_{,ij} = \Gamma^k_{ij}\, x^\mu_{,k} + b_{ij} n^\mu \tag{2.2}$$

and Weingarten

$$n^\mu_{,i} = -\, b_{ij} g^{jk} x^\mu_{,k}. \tag{2.3}$$

Here Γ^k_{ij} are the Christoffel symbols for the metric g_{ij} [24], g^{ij} is an inverse matrix to g_{ij}, b_{ij} are the coefficients of the second quadratic form of the surface that determines its external curvature ($b_{ij} = b_{ji}$).

Equations (2.2) and (2.3) describe the motion of the basis $\{x^\mu_{,1}\, x^\mu_{,2}, n^\mu\}$ along the surface. The compatibility conditions of these linear equations are given by the Gauss equation

$$R_{ijkl} = b_{ik} b_{jl} - b_{il} b_{jk} \tag{2.4}$$

and by the Codazzi equations

$$b_{ij;k} - b_{ik;j} = 0, \quad i, j, k = 1, 2. \tag{2.5}$$

The semicolon means the covariant differentiation with respect to the metric tensor g_{ij} in (2.1) and R_{ijkl} is the Riemann curvature tensor [24].

When the equations (2.4) and (2.5) are satisfied by given tensors g_{ij} and b_{ij} then the derivation equations (2.2) and (2.3) can be integrated and their corresponding solution $x^\mu(u^1, u^2)$ determines the surface up to its motion in E^3 as a whole.

The important geometrical invariants of the surface are its Gaussian curvature

$$K = -R/2, \quad R = g^{il} g^{jk} R_{ijkl} \tag{2.6}$$

and its mean curvature

$$H = \frac{1}{2} b_{ij} g^{ij} = \frac{1}{2} b^i_i. \tag{2.7}$$

For physical applications dealing with closed surfaces it is sufficient to consider normal variations of the surface that are defined as follows. For a given surface M with a position vector $x^\mu(u^1, u^2)$ we form the surface $\bar{M}$ *parallel* to M putting

$$\bar{x}^\mu = x^\mu + t\, f\, n^\mu, \quad -\varepsilon < t < \varepsilon, \tag{2.8}$$

where $f(u^1, u^2)$ is a sufficiently smooth function given on M. We denote by δ the operator $\partial/\partial t\,|_{t=0}$. Thus $\delta x^\mu = f n^\mu$. For simplicity, we shall omit the bar at the argument of δ.

From the definition (2.8) we obtain

$$\delta\, x^\mu_{,i} = f_{,i} n^\mu + f\, n^\mu_{,i}, \tag{2.9}$$

$$\delta\, x^\mu_{,ij} = f_{,ij} n^\mu + f_{,i} n^\mu_{,j} + f_{,j} n^\mu_{,i} + f n^\mu_{,ij}. \tag{2.10}$$

The variation of the metric tensor (2.1) is given by

$$\delta g_{ij} = \delta x^\mu_{,i} x^\mu_{,j} + x^\mu_{,i} \delta x^\mu_{,j} = f(n^\mu_{,i} x^\mu_{,j} + n^\mu_{,j} x^\mu_{,i}). \tag{2.11}$$

By making use of the Weingarten derivation equation (2.3) the last equation can be rewritten as

$$\delta g_{ij} = -2 f b_{ij}. \tag{2.12}$$

By varying the definition

$$g_{ij}g^{jk} = \delta_i^k$$

we have

$$\delta g_{ij}g^{jk} + g_{ij}\delta g^{jk} = 0.$$

Whence,

$$\delta g^{lk} = -g^{li}g^{jk}\delta g_{ij} = 2fb^{lk}. \qquad (2.13)$$

Denoting, as usual, by g a determinant of the metric tensor, $g = \det(g_{ij})$, we can write

$$\delta\sqrt{g} = \left(\frac{\partial}{\partial x^\mu_{,i}}\sqrt{g}\right)\delta x^\mu_{,i} = \sqrt{g}g^{im}x^\mu_{,m}n^\mu_{,i}f = -2\sqrt{g}Hf. \qquad (2.14)$$

From (2.2) it follows that

$$b_{ij} = n^\mu x^\mu_{,ij}. \qquad (2.15)$$

Hence

$$\begin{aligned}\delta b_{ij} &= \delta n^\mu x^\mu_{,ij} + n^\mu \delta x^\mu_{,ij} = \\ &= \Gamma^k_{ij}\delta n^\mu x^\mu_{,k} + b_{ij}\delta n^\mu n^\mu + n^\mu \delta\, x^\mu_{,ij}.\end{aligned} \qquad (2.16)$$

By varying the equalities following from the definition of the normal n^μ

$$n^\mu n_\mu = 1, \quad n^\mu x^\mu_{,i} = 0 \qquad (2.17)$$

we get

$$\delta n^\mu n^\mu = 0, \qquad (2.18)$$

$$\delta n^\mu x^\mu_{,i} = -n^\mu \delta x^\mu_{,i} = -f_{,i}. \qquad (2.19)$$

In addition, one can write

$$n^\mu n^\mu_{,ij} = -n^\mu_{,i} n^\mu_{,j} = -b_{ik}\, b_{jl}\, g^{kl}. \qquad (2.20)$$

Finally the variation of the second quadratic form is given by

$$\delta b_{ij} = f_{;ij} - f\, b_{ik}\, b_{jl} g^{kl}. \qquad (2.21)$$

Now we can calculate the variation of H^2:

$$\begin{aligned}\delta H^2 &= H\,\delta\,(b_{ij}\,g^{ij}) = H\,g^{ij}\,\delta b_{ij} + H\,b_{ij}\,\delta g^{ij} = \\ &= H\,(\Delta f + f\,b^i_k b^k_i),\end{aligned} \qquad (2.22)$$

where Δ is the Laplace-Beltrami operator given on the surface M. From the Gauss equation (2.4) it follows that

$$R = b^i_k b^k_i - b^i_i b^k_k = b^i_k b^k_i - 4\,H^2. \qquad (2.23)$$

Thus the variation δH^2 acquires the final form

$$\delta\, H^2 = H\,[\Delta f + f\,(R + 4\,H^2)]. \qquad (2.24)$$

Now we can derive the Euler-Lagrange equation following from the vanishing of the normal variation of the rigid string action (1.1)

$$\delta A = \delta \iint (\gamma + \alpha\,H^2)\,dS = 0, \quad dS = \sqrt{g}\,du^1\,du^2.$$

By making use of eqs. (2.14) and (2.24) we obtain

$$\delta\,A = \iint dS\,\{[-2\,\gamma\,H + \alpha\,(2H^3 + H\,R)]\,f + \alpha\,H\Delta f\} = 0. \qquad (2.25)$$

On the closed surfaces the Laplace-Beltrami operator Δ is selfadjoint operator [25]

$$\int dS\,\varphi\Delta f = \int dS\,f\Delta\varphi,$$

therefore the variation δA in (2.25) can be rewritten as follows

$$\delta\,A = \int dS\,[-2\gamma H + \alpha\,(\,\Delta H + 2H^3 + H\,R\,)]\,f = 0.$$

Due to the arbitrariness of the function $f(u^1,\,u^2)$ we arrive at the equation of motion

$$-\,2\,\gamma\,H + \alpha(\,\Delta + 2H^3 + \quad RH) = 0. \qquad (2.26)$$

We gave here quite detailed derivation of eq. (2.26) that is rather well known in literature [15, 26] in the case of a Euclidean ambient space E^3. We shall use the methods, just described, in the next section for deriving the equation on the geometrical invariants H and R when the string world surface is placed in a space-time of a constant curvature S^3.

3 Willmore surfaces in a space of a constant curvature

Here we show that the equation of motion (2.26) can be derived by considering the Willmore surfaces in a space of a constant curvature S^3. The Willmore surfaces are extremals of the Willmore functional

$$W = \iint dS\, H^2. \tag{3.1}$$

By making use of the Weierstraß coordinates z^α, $\alpha = 1, ..., 4$ [24, 27] the three-dimensional sphere S^3 with radius a can be represented as a hypersurface in the four dimensional Euclidean space E^4

$$\sum_{\alpha=1}^{4} z^\alpha z^\alpha = a^2. \tag{3.2}$$

Let $z^\alpha(u^1, u^2)$, $\alpha = 1, 2, 3, 4$ is a parametric representation of the surface M embedded into S^3 in term of the Weierstraß coordinates obeying (3.2). The natural unit normal to this surface in E^4 is $z^\alpha(u^1, u^2)$ and let n^α be the second unit normal to this surface

$$\sum_{\alpha=1}^{4} n^\alpha n^\alpha = 1, \quad \sum_{\alpha=1}^{4} n^\alpha z^\alpha = 0, \quad \sum_{\alpha=1}^{4} n^\alpha z^\alpha_{,i} = 0. \tag{3.3}$$

The important advantage of the Weierstraß coordinates in the problem under consideration is the following. The basic equations for the surface embedded into S^3 are very simple, they are almost

the same as in the Euclidean ambient space. For the metric tensor on M we have now

$$g_{ij} = \sum_{\alpha=1}^{4} z^\alpha_{,i} z^\alpha_{,j} = -\sum_{\alpha=1}^{4} z^\alpha_{,ij} z^\alpha. \tag{3.4}$$

The derivation equations (2.2) and (2.3) become [24]

$$z^\alpha_{,ij} = \Gamma^k_{ij} z^\alpha_{,k} + b_{ij} n^\alpha - \frac{g_{ij}}{a^2} z^\alpha, \tag{3.5}$$

$$n^\alpha_{,i} = -b_{ij} g^{jk} z^\alpha_{,k}. \tag{3.6}$$

The Gauss equation (2.4) now reads

$$R_{ijkl} = b_{ik} b_{jl} - b_{il} b_{jk} + \frac{1}{a^2}(g_{ik} g_{jl} - g_{il} g_{jk}). \tag{3.7}$$

The Codazzi equations (2.5) keep their form.

The normal variation in terms of the Weierstraß coordinates is defined as follows

$$\begin{aligned} \bar{z}^\alpha &= z^\alpha + t f n^\alpha, \quad -\varepsilon < t < \varepsilon, \\ \delta z^\alpha &= f n^\alpha, \\ \delta z^\alpha_{,i} &= f_{,i} n^\alpha + f n^\alpha_{,i}, \\ \delta z^\alpha_{,ij} &= f_{,ij} n^\alpha + f_{,i} n^\alpha_{,j} + f_{,j} n^\alpha_{,i} + f n^\alpha_{,ij}. \end{aligned} \tag{3.8}$$

By making use of (3.2) and (3.3) one can easily convinced that such a variation does not take out from S^3.

For variation of the metric tensor (3.4) we have obviously the same equations (2.12) – (2.14). From (3.5) it follows that the coefficients of the second fundamental form b_{ij} are defined by

$$b_{ij} = \sum_{\alpha=1}^{4} n^\alpha z^\alpha_{,ij}. \tag{3.9}$$

Therefore,

$$\begin{aligned} \delta b_{ij} &= \delta n^\alpha z^\alpha_{,ij} + n^\alpha \delta z^\alpha_{,ij} = \\ &= \delta n^\alpha \left(\Gamma^k_{ij} z^\alpha_{,k} + b_{ij} n^\alpha - \frac{g_{ij}}{a^2} z^\alpha \right) + n^\alpha \delta z^\alpha_{,ij}. \end{aligned} \tag{3.10}$$

For simplicity we omit here and in the following the sign of summation with respect to repeated indices.

From (3.2), (3.3) and (3.8) it follows that

$$\delta n^\alpha n^\alpha = 0, \quad n^\alpha_{;i} n^\alpha = 0, \quad \delta n^\alpha z^\alpha = -n^\alpha \delta z^\alpha = -f,$$
$$\delta n^\alpha z^\alpha_{,i} = -n^\alpha \delta z^\alpha_{,i} = -f_{,i}, \tag{3.11}$$
$$n^\alpha \delta z^\alpha_{,ij} = f_{,ij} + f n^\alpha n^\alpha_{,ij} = f_{,ij} - f n^\alpha_{,i} n^\alpha_{,j}.$$

Now equation (3.10) becomes

$$\delta b_{ij} = f_{,ij} - \Gamma^k_{ij} f_{,k} + \frac{g_{ij}}{a^2} f - f n^\alpha_{,i} n^\alpha_{,j}. \tag{3.12}$$

With allowance of (3.6) we obtain

$$\delta b_{ij} = f_{;ij} + \frac{g_{ij}}{a^2} f - f\, b^k_i b^l_j\, z^\alpha_{,k} z^\alpha_{,l} =$$
$$= f_{;ij} + f \left(\frac{g_{ij}}{a^2} - b^k_i b_{jk} \right) \tag{3.13}$$

By making use of the Gauss equation (3.7) we deduce now instead of (2.23)

$$R = b^i_k\, b^k_i - 4\,H^2 - \frac{2}{a^2}. \tag{3.14}$$

Whence

$$\delta\, H^2 = H \left[\Delta f + f \left(R + 4H^2 + \frac{4}{a^2} \right) \right]. \tag{3.15}$$

Taking into account that

$$\delta\, dS = -2H\, f\, dS \tag{3.16}$$

we can write

$$\delta\, W = \iint dS \left[\Delta H + 2H^3 + H \left(R + \frac{4}{a^2} \right) \right] f. \tag{3.17}$$

Therefore the equation of motion for the Willmore string in S^3 is

$$\Delta H + 2H^3 + H \left(R + \frac{4}{a^2} \right) = 0. \tag{3.18}$$

Thus it has the same form as (2.26) if we put

$$\frac{\gamma}{\alpha} = -\frac{2}{a^2}. \tag{3.19}$$

This result is in a complete agreement with an analogous relation in the one-dimensional version of the problem under consideration (see eq. (A.5) in Appendix and take into account that the sectional curvature G in the case of the sphere (3.2) is equal to $1/a^2$).

We would like to note here the following. In spite of the Willmore surfaces in spaces with curvature have been considered in a number of mathematical papers [28 – 30] nevertheless a simple derivation of eq. (3.18) in a correct form are given here actually for the first time. Indeed, if we try to apply the final equation (3.13) from Ref. [28] for embedding two-dimensional Willmore surface into the S^3 we obtain instead of the second term $4/a^2$ in brackets in (3.18) the wrong expression $3/a^2$. In paper [29] a more general functional as compared with (3.1) has been considered. For closed surfaces it reads

$$W_1 = \iint (H^2 + \tilde{K})\, dS, \tag{3.20}$$

where $\tilde{K}$ is a constant sectional curvature of the ambient space. Taking into account that the subtraction from W_1 of the functional

$$-\,\tilde{K} \iint dS \tag{3.21}$$

results in the additional term in the Euler-Lagrange equation $+2\,\tilde{K}\,H$ one obtains from eq. (14) of paper [29] our result (3.18). In the same time, the final equations (5.43) and (5.46) in paper [30] cannot be compared directly with our eq. (3.18). In order to do this they should be combined with the Gauss equation.

The result obtained here (eq. (3.18)) can be generalized directly in the following two ways. At first, if we consider the Willmore surfaces not in the S^3 but in the three dimensional manifold of a constant negative curvature (the pseudosphere with imaginary radius ia) then eq. (3.18) becomes

$$\Delta H + 2H^3 + H\left(R - \frac{4}{a^2}\right) = 0. \tag{3.22}$$

Instead of (3.19) we have in this case

$$\frac{\gamma}{\alpha} = \frac{2}{a^2}. \tag{3.23}$$

Secondly, we can generalize our result to the d-dimensional hypersurfaces in the S^{d+1} determined by a functional

$$W_2 = \iint H^m \, dS, \quad H = \frac{1}{d} g^{ij} b_{ij}, \quad i,j = 1,2,\ldots,d, \quad m > 0. \tag{3.24}$$

In this case equation (3.18) becomes

$$\Delta H^{m-1} + d^2\left(1 - \frac{1}{m}\right) H^{m+1} + H^{m-1}\left(R + \frac{d^2}{a^2}\right) = 0. \tag{3.25}$$

When the Willmore d-dimensional surface is embedded into the sphere S^{d+n+1} with $n > 1$ then one arrives at the n equations relating internal and external characteristics of this surface in a complete analogy with one-dimensional case (see Appendix).

4 Conclusion

Recently the string model based only on the second term in eq. (1.1) has been considered in paper [31]. It was called as a spontaneous string alluding to the fact that in this case the Nambu-Goto term with nonzero string tension can be generated spontaneously due to the quantum fluctuations. We have proposed here other classical scenario for this situation.

As a special solution to eq. (3.18) we can consider the minimal surfaces in S^3 with $H = 0$. There are some new results about

these solutions obtained under consideration of the usual Nambu-Goto string in the space-time of a constant curvature [21, 22, 27, 32]. In particular, authors of Ref. [32] arrive at the conclusion that the dynamics of the Nambu-Goto string in the de Sitter space-time should be unstable. This instability turns out to be a direct consequence of the unboundness of the Hamiltonian of the Sinh-Gordon equation that describes minimal surfaces in the three-dimensional de Sitter universe [27]. The relationship between the rigid string in flat space-time and the Willmore string in S^3 enables us to argue that the same instability should take place in the rigid string model in flat space-time. In mathematics another relation between minimal surfaces in S^3 and the Willmore surfaces in R^3 is known [15, 21]. Applying a stereographic projection to minimal surfaces in S^3 one obtains Willmore surfaces in R^3. Whence, we can conclude that the Willmore string in R^3 is unstable also.

And the final note concerns a modified version of the Willmore functional in S^3. From the physical point of view it is desirable to preserve the conformal invariance of this functional in the case of ambient space with a nonzero curvature too. To this end one has to use a modified form of it given in (3.20).

Appendix

We consider here more simple version of our problem, i.e. one dimensional version of it. Let us introduce two functionals defined on the curves $x^\mu(s)$:

$$F_1 = m\int ds + \alpha \int k^2 ds, \tag{A.1}$$

$$F_2 = \int k^2 ds, \tag{A.2}$$

where k is a curvature of the curve. First functional we shall consider in the Euclidean space E^n and the second one in the n-dimensional manifold of constant sectional curvature G. When $n = 2$ it has been shown in the book [33] that the Euler-Lagrange equations are identical for these two problems. This result can be generalized easily to arbitrary n. By making use the results of papers [34] we can write the corresponding equations of motions. In the first case we have

$$\begin{aligned} 2\,k_{ss} + k^3 - 2k\tau^2 - \frac{m}{\alpha}k &= 0, \\ k^2\tau &= \text{const}, \\ k_i &= 0, \quad i \neq 1, 2. \end{aligned} \tag{A.3}$$

Here subscribe s means differentiation with respect to the curve length, τ is the torsion of the curve and k_i, $i = 3, 4, \ldots, d-1$ are the higher curvatures of the curve. For the functional F_2 the Euler-Lagrange equations read

$$\begin{aligned} 2\,k_{ss} + k^3 - 2k\tau^2 + 2kG &= 0, \\ k^2\tau &= \text{const}, \\ k_i &= 0, \quad i \neq 1, 2. \end{aligned} \tag{A.4}$$

Thus we get identical systems if we put

$$\frac{m}{\alpha} = -2G. \tag{A.5}$$

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