объединенный
институт
ядерных
исследований
дубна

E2-93-159

A.A.Vladimirov*[1]

A CLOSED EXPRESSION FOR THE UNIVERSAL R-MATRIX IN A NON-STANDARD QUANTUM DOUBLE

Submitted to «Modern Physics Letters A»

*Work supported in part by the Russian Foundation of Fundamental Research (grant 93-02-3827)
[1]E-mail: alvladim@theor.jinrc.dubna.su

1993

In papers [1, 2] we have modified the recipes of [3, 4] and developed a regular method for constructing a quantum double out of any invertible constant matrix solution R of the quantum Yang-Baxter equation (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}\,. \tag{1}$$

To illustrate the efficiency of the method, an R-matrix from the two-parameter class

$$\begin{pmatrix} 1 & p & -p & pq \\ 0 & 1 & 0 & q \\ 0 & 0 & 1 & -q \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

discovered by D.Gurevich (cited in [5]) and studied also in [6–12], has been taken as an input (actually, with $p = q = 1$). The result [2] is a new non-standard quantum double with four generators $\{b, g, v, h\}$ obeying the following relations:

$$\begin{gathered}
[g,b] = [h,b] = 2\sinh g\,, \quad [g,v] = [h,v] = -2\sinh h\,, \\
[b,v] = 2(\cosh g)v + 2(\cosh h)b\,, \quad [g,h] = 0\,, \\
\Delta(b) = e^{g}\otimes b + b\otimes e^{-g}\,, \quad \Delta(v) = e^{h}\otimes v + v\otimes e^{-h}\,, \\
\Delta(g) = g\otimes 1 + 1\otimes g\,,\ \Delta(h) = h\otimes 1 + 1\otimes h\,,\ S^{\pm 1}(g) = -g\,, \\
S^{\pm 1}(h) = -h\,,\ S^{\pm 1}(b) = -b \pm 2\sinh g\,,\ S^{\pm 1}(v) = -v \mp 2\sinh h\,.
\end{gathered} \tag{3}$$

A month later, Burdik and Hellinger [12] introduced a quantum double also related to R-matrix (2) in terms of generators $\{\tau, \pi, T, P\}$ and a parameter γ. It is not difficult to verify that their double is isomorphic to (3) due to the following identification:

$$\tau = e^{g}\,b\,, \quad \pi = \frac{1 - e^{-2g}}{\gamma}\,, \quad T = h\,, \quad P = \frac{\gamma}{2}\,e^{h}\,v\,. \tag{4}$$

The universal $\mathcal{R}$-matrix of the quantum double (3) is displayed in [2] as several terms of its power expansion in g and h (in [12] – as a power series in appropriately chosen combinations of generators). The main result of the present paper is an explicit formula for $\mathcal{R}$:

$$\mathcal{R} = \exp\left\{\frac{g\otimes 1 + 1\otimes h}{\sinh(g\otimes 1 + 1\otimes h)}(\sinh g\otimes v + b\otimes\sinh h)\right\}. \tag{5}$$

This has been guessed with the use of computer (namely, the symbolic calculation program FORM [13]) and then proved by hand. I believe that expanding (5) and taking (4) into account should eventually yield the power-series expression for $\mathcal{R}$ given in [12].

The key property of $\mathcal{R}$ to be proved is its quasicocommutativity [14]. For example, the $\mathcal{R}$-matrix (5) must obey

$$\mathcal{R}(e^h \otimes v + v \otimes e^{-h})\mathcal{R}^{-1} = e^{-h} \otimes v + v \otimes e^h . \tag{6}$$

Denoting $\mathcal{R} = \exp A$ we come to

$$2(v \otimes \sinh h - \sinh h \otimes v) = [A, \Delta(v)] + \frac{1}{2}[A,[A,\Delta(v)]] + \dots , \tag{7}$$

as it follows from the Hadamard formula. Denoting also

$$\Phi = \frac{z}{\sinh z}, \quad \Phi' = \frac{d}{dz}\frac{z}{\sinh z} \quad \text{with} \quad z = g \otimes 1 + 1 \otimes h , \tag{8}$$

we find

$$[A, \Delta(v)] = 2(v \otimes \sinh h - \sinh h \otimes v) - 2(\Phi + \Phi')D , \tag{9}$$

$$\begin{aligned} D = {} & \sinh(g \otimes 1 + 1 \otimes h)\,(v \otimes \sinh h - \sinh h \otimes v) \\ & + \sinh(h \otimes 1 + 1 \otimes h)\,(\sinh g \otimes v + b \otimes \sinh h) . \end{aligned} \tag{10}$$

From the relations

$$[g \otimes 1 + 1 \otimes h\, , \sinh g \otimes v + b \otimes \sinh h] = 0 , \tag{11}$$

$$[g \otimes 1 + 1 \otimes h\, , \sinh h \otimes v - v \otimes \sinh h] = 0 , \tag{12}$$

$$[g \otimes 1 + 1 \otimes h\, , D] = 0 , \tag{13}$$

$$[\sinh g \otimes v + b \otimes \sinh h\, , D] = 2\sinh(g \otimes 1 + 1 \otimes h)D , \tag{14}$$

we deduce

$$[A, \Phi] = [A, \Phi'] = [D, \Phi] = [D, \Phi'] = 0 , \tag{15}$$

$$[A\, , v \otimes \sinh h - \sinh h \otimes v] = 2\Phi D , \tag{16}$$

$$[A, D] = 2(g \otimes 1 + 1 \otimes h)D . \tag{17}$$

The last equality enables us to keep multiple commutators in (7) under control and sum them up, with a desired result.

There is no need of a special proof of the other requirements on $\mathcal{R}$ [14], because an iterative solution of (6) is unique in the Hopf algebra (3). Therefore, the universal $\mathcal{R}$-matrix (5) obeys QYBE.

It is also interesting to consider the reduced version of (3), that is the Hopf algebra with generators $\{v, h\}$ and relations

$$\begin{gathered} [v, h] = 2\sinh h , \\ \Delta(v) = e^h \otimes v + v \otimes e^{-h} , \quad \Delta(h) = h \otimes 1 + 1 \otimes h , \\ S^{\pm 1}(h) = -h , \quad S^{\pm 1}(v) = -v \mp 2\sinh h . \end{gathered} \tag{18}$$

Algebra (18) is a subalgebra of (3) and, at the same time, the quotient algebra with respect to the centre of (3). The latter is generated by the elements

$$\{\ h - g , \ \ (\sinh g)v + (\sinh h)b\ \} . \tag{19}$$

Simply speaking, (3) reduces to (18) by means of a substitution

$$g = h , \qquad b = -v . \tag{20}$$

Another way to get (18) is to begin with the R-matrix (2) and use the original Majid's procedure [4], instead of the above one [1, 2], to build a quasitriangular Hopf algebra. Recall [1] that Majid's approach is based on the $< T, L^{\pm} >= R^{\pm}$ duality whereas we proceed from $< L^-, L^+ >= R^{-1}$. In the $sl_q(2)$ case both procedures lead to the same result [1, 4], but in the case (2), due to $R^+ \equiv R_{12} = R_{21}^{-1} \equiv R^-$, the resulting Hopf algebras are substantially different.

By construction, the Hopf algebra (18) is quasitriangular (but is not a quantum double, of course). Its universal $\mathcal{R}$-matrix is obtained by substituting (20) into (5) and looks like

$$\mathcal{R} = \exp\left\{\Delta\left(\frac{h}{\sinh h}\right)(\sinh h \otimes v - v \otimes \sinh h)\right\} . \tag{21}$$

By the way, to prove (21) directly is easier than (5) because $[A,[A,\Delta(v)]]$ in eq. (7) vanishes in this case.

It is worth mentioning that the standard matrix format for an algebra (18) admits, analogously to $sl_q(2)$ [15-17], an exact exponential parametrization:

$$\begin{pmatrix} e^h & v \\ 0 & e^{-h} \end{pmatrix} = \exp \begin{pmatrix} h & y \\ 0 & -h \end{pmatrix}, \quad [y, h] = 2h, \tag{22}$$

where

$$v = \frac{\sinh h}{h} y + \cosh h - \sinh h - \frac{\sinh h}{h}. \tag{23}$$

A similar reparametrization,

$$v = \frac{\sinh h}{h} x, \tag{24}$$

transforms (18) into a Hopf algebra

$$[x, h] = 2h, \tag{25}$$

$$\Delta(h) = h \otimes 1 + 1 \otimes h, \tag{26}$$

$$\Delta(x) = \Delta\left(\frac{h}{\sinh h}\right)\left(e^h \otimes \frac{\sinh h}{h} x + \frac{\sinh h}{h} x \otimes e^{-h}\right), \tag{27}$$

$$S^{\pm 1}(h) = -h, \quad S^{\pm 1}(x) = -x + 2\left(h \frac{e^{\mp h}}{\sinh h} - 1\right), \tag{28}$$

which can be viewed as a deformation of the universal enveloping algebra of (25) treated as (trivial) Hopf algebra

$$[x, h] = 2h, \quad \Delta_0(h) = h \otimes 1 + 1 \otimes h, \quad \Delta_0(x) = x \otimes 1 + 1 \otimes x, \tag{29}$$

$$S_0(h) = -h, \quad S_0(x) = -x. \tag{30}$$

The universal $\mathcal{R}$-matrix takes the form

$$\mathcal{R} = \exp\left\{\Delta\left(\frac{h}{\sinh h}\right)\left(\frac{\sinh h}{h} \otimes \frac{\sinh h}{h}\right)(h \otimes x - x \otimes h)\right\}$$

$$= 1 \otimes 1 + h \otimes x - x \otimes h + \mathcal{O}(h^2). \tag{31}$$

According to Drinfeld [18], this can be interpreted as the quantization (with $\hbar = 1$) of the classical r-matrix

$$r = h \otimes x - x \otimes h. \tag{32}$$

It is proved in [18] that such a quantization exists and is unique. Our relations (27), (28) and (31) produce it in an explicit form.

Universal $\mathcal{R}$-matrix (31) obeys QYBE (1) in an abstract algebra (25) as well as in all its representations. For instance, to recover the R-matrix (2) with $p = q = 1$, one has to substitute into (31) the 2×2-matrices

$$x = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad h = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}. \tag{33}$$

In conclusion we should remark that in [11], where the problem of quantizing (25) was also studied, an explicit formula has been written for an invertible element F which, according to [18], deforms the coproduct,

$$\Delta(x) = F \Delta_0(x) F^{-1}, \tag{34}$$

and is related to universal $\mathcal{R}$-matrix by

$$\mathcal{R}_{12} = F_{21} F_{12}^{-1}. \tag{35}$$

However, a straightforward calculation shows that the r.h.s. of (35) with F given in [11] neither coincides with (31) nor obeys QYBE (1).

An open question is whether $\mathcal{R}$ (31) (and maybe also F in closed form) can be obtained by the very interesting direct method recently proposed [19] for evaluating quantum objects like $\mathcal{R}$ and F as functionals of the corresponding classical r-matrix.

Acknowledgments

I wish to thank L.Avdeev, I.Aref'eva, L.Faddeev, A.Isaev, P.Kulish, V.Lyakhovsky, V.Priezzhev, P.Pyatov and V.Tolstoy for helpful discussions.

References

[1] A. A. Vladimirov, JINR preprint E2-92-506; to appear in *MPLA*.

[2] A. A. Vladimirov, JINR preprint E2-93-34; to appear in *Z. Phys. C*.

[3] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, *Algebra i Analiz* **1**, Vol. 1, 178 (1989); English transl. in *Leningrad Math. J.* **1**, 193 (1990).

[4] S. Majid, *Int. J. Mod. Phys.* **A5**, 1 (1990).

[5] V. V. Lyubashenko, *Usp. Mat. Nauk* **41**, Vol. 5, 185 (1986).

[6] E. E. Demidov, Yu. I. Manin, E. E. Mukhin and D. V. Zhdanovich, *Progr. Theor. Phys. Suppl.* **102**, 203 (1990).

[7] S. Zakrzewski, *Lett. Math. Phys.* **22**, 287 (1991).

[8] H. Ewen, O. Ogievetsky and J. Wess, *Lett. Math. Phys.* **22**, 297 (1991).

[9] L. Hlavaty, *J. Phys.* **A25**, L63 (1992).

[10] J. Hietarinta, *Phys. Lett.* **A165**, 245 (1992).

[11] C. Ohn, *Lett. Math. Phys.* **25**, 85 (1992).

[12] Č. Burdik and P. Hellinger, Charles Univ. preprint PRA-HEP-93/2.

[13] J. A. M. Vermaseren, *Symbolic manipulation with FORM* (Amsterdam, 1991).

[14] V. G. Drinfeld, *Algebra i Analiz* **1**, Vol. 2, 30 (1989).

[15] S. P. Vokos, B. Zumino and J. Wess, *Z. Phys.* **C48**, 65 (1990).

[16] E. Corrigan, D. B. Fairlie, P. Fletcher and R. Sasaki, *J. Math. Phys.* **31**, 776 (1990).

[17] O. Ogievetsky and J. Wess, *Z. Phys.* **C50**, 123 (1991).

[18] V. G. Drinfeld, *DAN SSSR* **273**, 531 (1983); English transl. in *Sov. Math. Dokl.* **28**, 667 (1983).

[19] L. Freidel and J. M. Maillet, ENS Lyon preprint ENSLAPP 399/92.

**Received by Publishing Department
on May 7, 1993.**