объединенный
институт
ядерных
исследований
дубна

E2-93-141

T.A.Ivanova

SOME SOLUTIONS OF THE KLEIN-GORDON AND DIRAC EQUATIONS IN THE EXTERNAL YANG-MILLS FIELDS

Submitted to «Europhysics Letters»

1993

1. Introduction

The consideration of solutions of the equations for the massless scalar and spinor fields interacting with the external nontrivial Yang-Mills fields is of interest [1,2]. The dynamics of nonabelian Yang-Mills (YM) and Higgs fields is described by the complicated nonlinear equations having nontrivial solutions [2-5]. The well-known explicit solutions of the YM equations are the instanton solutions obtained with the help of the Corrigan-Fairlie-'t Hooft-Wilczek (CFHW) ansatz (see [2-4]). Solutions of the Klein-Gordon and Dirac equations in these background YM fields have been described in [1,2].

A new class of self-dual solutions of the YM equations, obtained by the generalization of the CFHW ansatz to an arbitrary gauge group, has been described in [6-8]. In [9-14], the CFHW ansatz has been generalized to the spaces of dimension greater than four. In this paper, we shall construct some classes of massless solutions of the Klein-Gordon and Dirac equations in the background YM fields described in [6-8, 12-14].

2. Solutions of the Yang-Mills equations

Let us consider the Euclidean space R^n with the metric δ_{ab}, $a, b,... = 1,...,n$. Let A_a be the YM potentials with values in the semisimple Lie algebra $\mathcal{G}$ of the Lie group G and

$$F_{ab} = \partial_a A_b - \partial_b A_a + [A_a, A_b]$$

be the curvature tensor for A_a.

The YM equations for the gauge potentials A_a have the form

$$\partial_a F_{ab} + [A_a, F_{ab}] = O. \tag{1}$$

The summation convention is adopted throughout.

Let us suppose that in the space R^n with metric δ_{ab} there are q constant tensors $J^1_{ab},...,J^q_{ab}$ that are antisymmetric in indices a and b and obey the relations

$$J^\alpha_{ac} J^\beta_{bc} = \delta^{\alpha\beta}\delta_{ab} + \Sigma^{\alpha\beta}_{ab}, \tag{2}$$

where $\Sigma^{\alpha\beta}_{ab}$ are some constant antisymmetric in a and b tensors, $\alpha, \beta, ... = 1, ..., q$. From (2) it follows that

$$J^\alpha_{ac}J^\beta_{cb} + J^\beta_{ac}J^\alpha_{cb} = -2\delta^{\alpha\beta}\delta_{ab},$$

i.e., $J^\alpha = (J^\alpha_{ab})$ give a real matrix representation for the generators J^α of the Clifford algebra for the space R^q with the metric $g_{\alpha\beta} = -\delta_{\alpha\beta}$. For the description of the explicit form of tensors J^α_{ab} see [14].

We shall look for solutions of the YM equations (1) in the form

$$A_a = -J^\alpha_{ac}T_\alpha(\varphi)\partial_c\varphi, \qquad (3)$$

where the real antisymmetric tensors J^α_{ab} satisfy (2); φ is an arbitrary function of coordinates $x_a \in R^n$; $T_1, ..., T_q$ depend only on φ, take values in the Lie algebra $\mathcal{G}$ and satisfy the Rouhani-Ward (RW) equations (see [15, 16, 12-14]):

$$f_{\alpha\beta\gamma}\dot{T}_\gamma + [T_\alpha, T_\beta] = 0. \qquad (4)$$

Here $f_{\alpha\beta\gamma}$ is some totally antisymmetric three-index tensor in R^q satisfying $f_{\alpha\gamma\delta}f_{\beta\gamma\delta} = 2\delta_{\alpha\beta}$ and $\dot{T}_\gamma \equiv dT_\gamma/d\varphi$. If q coincides with the dimension of the simple compact Lie algebra $\mathcal{H}$, then as $f_{\alpha\beta\gamma}$ one may take the structure constants of $\mathcal{H}$.

In [13] and [14], it has been shown that after substituting (3) into (1) and using the identities (2), the YM equations are reduced to the following system of linear equations:

$$f^\alpha_{\beta\gamma}\Sigma^{\beta\gamma}_{ac}\partial_c\partial_b\varphi - 2J^\alpha_{bc}\partial_c\partial_a\varphi + 2f^\alpha_{\beta\gamma}J^\beta_{ac}J^\gamma_{be}\partial_c\partial_e\varphi + J^\alpha_{ab}\Box\varphi = 0, \qquad (5)$$

where $\Box \equiv \partial_c\partial_c$. Equations (4) and (5) have solutions. Their explicit form for different special cases can be found in [7-9, 12-14].

3. Solutions of the massless Klein-Gordon equation

In R^n let us consider the massless scalar field χ with values in the adjoint representation of the Lie algebra $\mathcal{G}$. The Klein-Gordon equation for χ in the external field A_a has the form

$$\Big(\partial_a + [A_a, \ ]\Big)\Big(\partial_a + [A_a, \ ]\Big)\chi = 0, \qquad (6)$$

where $a, ... = 1, ..., n$.

Now, substitute the ansatz (3) for A_a into (6). Suppose that $T_\alpha(\varphi)$ and φ obey the equations (4) and (5). For χ let us consider the following ansatz:

$$\chi = \chi_\alpha T_\alpha(\varphi), \qquad \chi_\alpha = const. \qquad (7)$$

In this case, the Klein-Gordon equation (6) is reduced to the following equation:

$$\chi_\alpha \dot{T}_\alpha \Box\varphi + \chi_\alpha\partial_c\varphi\partial_c\varphi\Big\{\ddot{T}_\alpha - [T_\beta, [T_\alpha, T_\beta]]\Big\} = 0. \qquad (8)$$

Here we have used the identities (2); $\ddot{T}_\alpha \equiv d^2T_\alpha/d\varphi^2$.

Thus, if $T_\alpha(\varphi)$ satisfy the equations

$$\ddot{T}_\alpha - \Big[T_\beta, [T_\alpha, T_\beta]\Big] = 0. \qquad (9)$$

and φ satisfies the Laplace equation

$$\Box\varphi = 0, \qquad (10)$$

then the ansatz (7) gives the solution of the massless Klein-Gordon equation (6).

It is easy to see that each solution of the RW equations (4) satisfies Eqs. (9). Indeed, if one multiplies Eqs. (4) by $f_{\alpha\beta\delta}$ and differentiates these equations once more, then obtains

$$\ddot{T}_\alpha = -f_{\alpha\beta\gamma}[T_\beta, \dot{T}_\gamma].$$

At the same time, from Eqs.(4) it follows that

$$\Big[T_\beta, [T_\alpha, T_\beta]\Big] = -f_{\alpha\beta\gamma}[T_\beta, \dot{T}_\gamma].$$

Therefore, if T_α satisfy Eqs.(4), then T_α satisfy Eqs.(9). Remind that the function φ must satisfy Eqs.(5). Comparing Eqs.(5) with Eq.(10), we obtain the following system of equations:

$$f^\alpha_{\beta\gamma}\Sigma^{\beta\gamma}_{ac}\partial_c\partial_b\varphi - 2J^\alpha_{bc}\partial_c\partial_a\varphi + 2f^\alpha_{\beta\gamma}J^\beta_{ac}J^\gamma_{be}\partial_c\partial_e\varphi = 0. \qquad (11a)$$

$$\Box\varphi = 0. \tag{11b}$$

Equations (11) have solutions. Some of them were written out in [13, 14].

4. Klein-Gordon equation in n=4 and Nahm's equations

Let us consider the case $n = 4$. In R^4, as J^α_{ab} one may take the self-dual 't Hooft tensors η^α_{ab}, where $\eta^\alpha_{\beta\gamma} = \epsilon^\alpha_{\beta\gamma}$; $\eta^\alpha_{a4} = -\eta^\alpha_{4a} = \delta^\alpha_a$, $\alpha, \beta, ... =$ 1,2,3; $a, b, ... =$ 1,...,4. As $f^\alpha_{\beta\gamma}$ one may choose the structure constants $\epsilon^\alpha_{\beta\gamma}$ of the group $SU(2)$. Then, Eqs.(4) coincide with the well-known Nahm equations [15,16,7,13]

$$\epsilon_{\alpha\beta\gamma}\dot{T}_\gamma + [T_\alpha, T_\beta] = 0. \tag{12}$$

Using the following identities [4]:

$$\epsilon^\alpha_{\beta\gamma}\eta^\beta_{ab}\eta^\gamma_{cd} = \delta_{ac}\eta^\alpha_{bd} - \delta_{ad}\eta^\alpha_{bc} - \delta_{bc}\eta^\alpha_{ad} + \delta_{bd}\eta^\alpha_{ac},$$

we obtain that Eqs.(11a) reduce to

$$\eta^\alpha_{ab}\Box\varphi = 0.$$

Therefore, the system (11) converts into the Laplace equation (10). As shown in [6-8], ansatz (3) gives the anti-self-dual solutions of the YM equations in R^4.

It is known that in terms of theta functions one can write a general solution of Nahm's equations (12) for any semisimple Lie algebra $\mathcal{G}$ (see, e.g., [15-17]). All solutions of the Laplace equation (10) are known, too. The explicit form of some solutions of Nahm's equations may be found in [7, 8, 13].

5. Solutions of the massless Dirac equation in n=4

Let us consider the Euclidean four-vector matrices [1]:

$$\alpha_a = (-i\sigma_1, -i\sigma_2, -i\sigma_3, \mathbf{1}_2), \qquad \bar{\alpha}_a = (i\sigma_1, i\sigma_2, i\sigma_3, \mathbf{1}_2),$$

where σ_α are the Pauli matrices and $\mathbf{1}_2$ is a unit 2×2 matrix. These matrices have the following basic properties [1]:

$$\alpha_a\bar{\alpha}_b = \delta_{ab} + \bar{\eta}^\alpha_{ab}(i\sigma_\alpha), \quad \bar{\alpha}_a\alpha_b = \delta_{ab} + \eta^\alpha_{ab}(i\sigma_\alpha), \tag{13}$$

where $\bar{\eta}^\alpha_{\beta\gamma} = \epsilon^\alpha_{\beta\gamma}$, $\bar{\eta}^\alpha_{a4} = -\bar{\eta}^\alpha_{4a} = -\delta^\alpha_a$ are anti-self-dual 't Hooft tensors. The self-dual 't Hooft tensors η^α_{ab} were introduced in Sect.4. Let us define the Euclidean γ-matrices as

$$\gamma_a = \begin{pmatrix} 0 & \alpha_a \\ \bar{\alpha}_a & 0 \end{pmatrix}, \qquad [\gamma_a, \gamma_b] = 2i \begin{pmatrix} \bar{\eta}^\alpha_{ab}\sigma_\alpha & 0 \\ 0 & \eta^\alpha_{ab}\sigma_\alpha \end{pmatrix}.$$

The Dirac equation in n=4 for massless spinor ψ in the adjoint representation of the Lie group G has the form

$$\gamma_a(\partial_a\psi + [A_a, \psi]) = 0. \tag{14}$$

Each component of ψ takes values in the Lie algebra $\mathcal{G}$.

Rewrite the four-component column ψ through two-component spinors ψ_+ and ψ_- : $\psi = \binom{\psi_+}{\psi_-}$. The matrix equation (14) can be rewritten as a system of two equations

$$\alpha_a(\partial_a\psi_- + [A_a, \psi_-]) = 0, \quad \bar{\alpha}_a(\partial_a\psi_+ + [A_a, \psi_+]) = 0. \tag{15}$$

We shall look for solutions in the form

$$\psi_+ = \alpha_a(\partial_a\bar{\chi} + [A_a, \bar{\chi}]), \quad \psi_- = 0, \quad \bar{\chi} = \phi\chi = \phi\chi_\alpha T_\alpha(\varphi), \tag{16}$$

where ϕ is a constant two-component spinor, and χ satisfies the massless Klein-Gordon equation (6).

Substituting (16) into (15) and using (13), we obtain that Eqs.(15) reduce to

$$\phi\Big(\partial_a + [A_a, \;]\Big)\Big(\partial_a + [A_a, \;]\Big)\chi + \frac{i}{2}\sigma_\alpha\phi[\eta^\alpha_{ab}F_{ab}, \chi] = 0. \tag{17}$$

In virtue of the anti-self-duality of the field F_{ab} [8], we have $\eta^\alpha_{ab}F_{ab} = 0$. Since χ satisfies Eq.(6), we obtain that the ansatz (16) gives a solution of the Dirac equation (14).

In [7] and [8], the ansatz

$$A_a = 2\bar{\eta}^\alpha_{ab}x_bT_\alpha(\tau), \quad \tau = x_cx_c, \tag{18}$$

has also been considered. For the ansatz (18) the anti-self-duality equations for A_a reduce to the Nahm equations

$$\epsilon_{\alpha\beta\gamma}\dot{T}_\gamma - [T_\alpha, T_\beta] = 0,$$

obtained from Eqs.(12) by the replacement $T_\alpha \longrightarrow -T_\alpha$. It is easy to see that for the ansatz (18) one can also obtain the solutions of the massless Klein-Gordon and Dirac equations if one replaces everywhere φ by τ.

6. Conclusion

Here we do not write out the explicit form of solutions of Eqs.(4) and (5), because this has been done, for example, in [7, 12-14]. Each solution of Eqs.(4) and Eqs.(5) gives the solution (3) of the YM equations in the d≥4 dimension. The ansätze (7) and (16) permit one to obtain the solutions of the massless Klein-Gordon and Dirac equations in these background YM fields. We hope that some of these solutions may be used in quantum chromodynamics and in the superstring theory.

References

1. E.F.Corrigan, D.B.Fairlie, S.Templeton and P.Goddard, *Nucl.Phys.* **B140**, 31 (1978); H.Osborn, *Nucl.Phys.* **B140**, 45 (1978); E.Corrigan and P.Goddard, *Ann.Phys.* **154**, 253 (1983).
2. R.Rajaraman, *Solitons and instantons in quantum field theory* (Amsterdam, North-Holland, 1982).
3. A.Actor, *Rev.Mod.Phys.* **51**, 461 (1979).
4. M.K.Prasad, *Physica* **D1**, 167 (1980).
5. A.N.Leznov and M.V.Saveliev, *The group methods of the integration of nonlinear equations of dynamical systems* (Moscow, Nauka, 1985, in Russian).
6. P.Minkowsky, *Nucl.Phys.* **B177**, 203 (1981); T.A.Ivanova and A.D.Popov, *Sov.J.Nucl.Phys.* **51**, 163 (1990).
7. T.A.Ivanova, Usp.Matem.Nauk 46, 149 (1991); T.A.Ivanova and A.D.Popov, *Lett.Math.Phys.* **23**, 29 (1991).
8. A.D.Popov, *JETP Lett.* **54**, 69 (1991); *Theor.Math.Phys.* **89**, 1297 (1991).
9. R.S.Ward, *Nucl.Phys.* **B236**, 381 (1984).
10. D.B.Fairlie and J.Nuyts, *J.Phys.* **A17**, 2867 (1984); S.Fubini and H.Nicolai, *Phys.Lett.* **B155**, 369 (1985).
11. E.Corrigan, P.Goddard and A.Kent, *Commun.Math.Phys.* **100**, 1 (1985).
12. A.D.Popov, *Europhys.Lett.* **17**, 23 (1992); T.A.Ivanova and A.D.Popov, *Lett.Math.Phys.* **24**, 85 (1992).
13. A.D.Popov, *Mod.Phys.Lett.* **A7**, 2077 (1992); *Europhys.Lett.* **19**, 465 (1992); *JETP Lett.* **55**, 259 (1992).
14. T.A.Ivanova and A.D.Popov, *J.Math.Phys.* **34**, 674 (1993); *Teor.Mat.Fiz.* **94**, 316 (1993).
15. S.Rouhani, *Phys.Lett.* **A104**, 7 (1984).
16. R.S.Ward, *Phys.Lett.* **A112**, 3 (1985).
17. J.Hartubise, *Comm.Math.Phys.* **100**, 191 (1985); **120**, 613 (1989).

Received by Publishing Department on April 20, 1993.