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A.M.Chervyakov, V.V.Nesterenko

# PARABOLIC APPROXIMATION FOR THE TRANSVERSE VIBRATIONS OF BEAMS AND RODS

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# 1 Introduction

So far, the basic tool for calculating the transverse vibrations of beams and rods is the classical Bernoulli–Euler equation ( B–E equation ) [1]. Only two generalizations of this theory can be mentioned here. The effect of the gyration inertia of beam elements under vibrations was taken into account by Rayleigh [2]. At the beginning of this century Timoshenko proposed a more realistic model [3] involving both the bending and shear deformations of rods under transverse vibrations. However, all attempts to obtain more general approximations in this problem lead as a rule to very complicated constructions (see, for example, [4]).

It is widely believed that the Timoshenko model is successful because it is a hyperbolic approximation to exact equations in the theory of elasticity [4]. However, in the present paper we would like to show that the same corrections to the B–E frequencies as in the Timoshenko theory can be obtained remaining inside the scope of the parabolic approximation that is dealing with a parabolic equation for flexural vibrations of the beam.

The layout of the paper is as follows. In sect. 2 the essentials of the Timoshenko model are presented. In sect. 3 the same description of transverse vibrations of a beam but without of an auxiliary nonphysical mode of oscillations is proposed by making use of the parabolic type equation. In Conclusion (sect. 4) it is shown that the energy in the proposed model is positive definite.

## 2 Timoshenko model

We begin with a short summary of the Timoshenko model [1, 2, 3]. Let $y_1(t,x)$ and $y_2(t,x)$ be lateral deflections of the beam due to the bending and shear, respectively. The Timoshenko model is defined by the following Lagrange function

$$L = T - V, \tag{1}$$

$$T = \frac{\rho F}{2}\int_0^l (\dot{y}_1 + \dot{y}_2)^2\, dx + \frac{\rho I}{2}\int_0^l (\dot{y}_1')^2\, dx\,; \tag{2}$$

$$V = \frac{EI}{2}\int_0^l (y_1'')^2 dx + \frac{kFG}{2}\int_0^l (y_2')^2 dx. \tag{3}$$

Here $E$ is the Young modulus of the beam material, $G$ is the shear modulus, $\rho$ is the mass density, $I$ is the moment of inertia of a cross section around principal axis normal to the plane of motion, $F$ is the cross section area, $l$ is the beam length, and $k$ is the shear coefficient. The dot and prime stand for differentiation with respect to $t$ and $x$, respectively.

The Lagrange function (1) – (3) leads to equations of motion for $y = y_1 + y_2$ and $\psi = y' + y_2'$

$$\ddot{y} - k\frac{G}{\rho}(y'' - \psi') = 0\,,$$

$$\psi'' - \frac{\rho}{E}\ddot{\psi} + k\frac{GF}{EI}(y' - \psi) = 0 \tag{4}$$

and to the boundary conditions which in the case of hinged beam ends have the form

$$y(t,0) = y(t,l) = 0 \quad y''(t,0) = y''(t,l) = 0\,,$$

$$\psi'(t,o) = \psi'(t,l) = 0. \tag{5}$$



Let us substitute into equations (4) the normal forms of the beam vibrations

$$y(t,x) = A_n \sin(\omega_n t + \epsilon_n)\sin(\lambda_n x)\,,$$

$$\psi(t,x) = B_n \sin(\omega_n t + \epsilon_n)\cos(\lambda_n x)\,, \quad \lambda_n = n\pi/l\,. \tag{6}$$

corresponding to the boundary conditions (5). As a result, we obtain the following equation for eigenfrequencies

$$a^2\lambda_n^4 - \omega^2 - \left(1 + \frac{E}{kG}\right)r^2\lambda_n^2\omega^2 + r^2\frac{\rho}{kG}\omega^4 = 0\,, \tag{7}$$

where $a^2 = IF/\rho F$ and $r$ is the radius of inertia of the beam cross section, $r^2 = I/F$. Besides, there arises the relation between the amplitudes of oscillations $A_n$ and $B_n$

$$\frac{B_n}{A_n} = \lambda_n\left[1 - \frac{E}{kG}\frac{\omega_n^2}{\Omega_n^2}r^2\lambda_n^2\right]. \tag{8}$$

Here $\Omega_n$ are the frequencies of beam oscillations in the classical Bernoulli–Euler theory [1], which can be obtained by taking into account only the first two terms in eq. (7)

$$\Omega_n = a\,\lambda_n^2. \tag{9}$$

Equation (7) defines two series of eigenfrequencies or two modes of oscillations but frequencies of lower mode alone turn smoothly into the Bernoulli–Euler frequencies (9) when the small parameter $r^2\lambda_n^2$ in the Timoshenko theory tends to zero [5]. Timoshenko himself [1, 3] has considered only the basic low-frequency mode iterating eq. (7) with respect to the small parameter $r^2\lambda_n^2$. Moreover, the last term in eq. (7) corresponding to the kinetic energy of shear deformations was neglected as a quantity of higher order in $r^2\lambda_n^2$ [6]. The effect of shear deformations under flexural vibrations was taken into account by the term in (7) proportional to $E/kG$. In this way the following formula for eigenfrequencies was obtained [1]:

$$\omega_n \simeq a\lambda_n^2\left[1 - \frac{1}{2}\left(1 + \frac{E}{kG}\right)r^2\lambda_n^2\right]. \tag{10}$$

As was shown in [5], the second higher-frequency mode of oscillations in the Timoshenko theory plays an auxiliary role. The vibrations with such frequencies are not excited practically but their incorporation by the frequency equation improves the frequencies of the basic oscillatory mode.

## 3 Parabolic equation for flexural vibrations of beams

We shall show here that the same corrections to the B–E frequencies as in (10) can be obtained by making use of the parabolic equation describing flexural vibrations of the Timoshenko beam. It is appealing that this new equation does not entail the nonphysical duplication of frequencies. The basic idea of our approach is to remove from eqs. (1) – (3) the variable $y_2$ describing the shear deformations but at the same time to take into account the effect of shear upon the bending vibrations.

For the kinetic energy of beam (2) the bending contribution to the first term comes from the bending variables $\dot{y}_1$ and $\dot{y}_2$ being only the correction to it. Indeed, using the explicit solutions, as those in the case of hinged beam ends (6), for example, and the relation between amplitudes (8) one finds

$$\frac{\dot{y}_2}{\dot{y}_1} = \frac{r^2\lambda_n^2 \frac{E}{kG}\left(\frac{\omega_n}{\Omega_n}\right)^2}{1 - r^2\lambda_n^2\frac{E}{kG}\left(\frac{\omega_n}{\Omega_n}\right)^2}. \qquad (11)$$

For sufficiently long beams ($h/l$ is small, where $h$ is the beam height) $\omega_n^2/\Omega_n^2 \sim 1$ (see, for example, [5] ). Then formula (11) gives

$$\frac{\dot{y}_2}{\dot{y}_1} \simeq r^2\lambda_n^2 \ll 1. \qquad (12)$$

It should be noted that the estimation (12) is true only for vibrations of the basic mode in the Timoshenko theory. Hence, the variable $\dot{y}_2$ can be dropped out of formula (1).



In (3) the second term represents the elastic energy of shear deformations. The variable $y_2'$ here can be expressed through $y_1$. Actually, according to the Hooke law for shear deformations we have

$$y_2' = \frac{Q}{kGF}$$

where $Q$ is the transverse force arising under flexural vibrations. From the elementary theory of beams [1] it follows that

$$Q = -EI\frac{d^3y_1}{dx^3}.$$

and consequently,

$$y_2' = -\frac{EI}{kFG\,y_1'''}. \qquad (13)$$

To construct a correct Lagrange function describing bending vibrations we must insert (13) into (3) and change to the opposite sign of the second term. Thereby, it will be taken into account that the appropriate portion of the energy of bending vibrations turns into the potential energy of shear deformations.

Finally, we obtain the following Lagrange function specifying our model

$$L_1 = \frac{\rho F}{2}\int_0^l \dot{y}^2 dx + \frac{\rho I}{2}\int_0^l \dot{y}'^2 dx - \frac{EI}{2}\int_0^l y''^2 dx + \frac{E^2I^2}{2kFG}\int_0^l y'''^2 dx. \qquad (14)$$

For simplicity, index 1 of the bending variable $y(t,x)$ is omitted.

Varying (14), one arrives at the parabolic equation for flexural vibrations of a beam that takes into account the effect of shear deformations upon the bending vibrations

$$\ddot{y} + a^2 y_x^{(4)} - r^2\ddot{y}'' + a^2r^2\frac{E}{kG}y_x^{(6)} = 0, \qquad (15)$$

where $y_x^{(n)} \equiv \partial^n y/\partial x^n$. For the hinged beam ends the variation of (14) ought to be performed under the following boundary conditions

$$y(t,0) = y(t,l) = 0, \quad y''(t,0) = y''(t,l) = 0. \qquad (16)$$

Substituting the normal form of oscillations dictated by the boundary conditions (16) into (15) we obtain the equation for eigenfrequencies

$$\omega_n^2\left(1+r^2\lambda_n^2\right)-a^2\lambda_n^4\left(1+\frac{E}{k\,G}r^2\lambda_n^2\right)=0\,. \qquad (17)$$

Solving this equation to the same precision as it was done by Timoshenko for equation (7) we arrive at the same corrections to frequencies defined by (10)

$$\omega_n=a\lambda_n^2\sqrt{\frac{1-\frac{E}{k\,G}r^2\lambda_n^2}{1+r^2\lambda_n^2}}\simeq a\lambda_n^2\left[1-\frac{1}{2}\left(1+\frac{E}{kG}\right)r^2\lambda_n^2\right]. \qquad (18)$$

# 4 Conclusion

Thus, the parabolic equation (15) gives practically the same description of flexural vibrations of beams as the Timoshenko theory. The appealing feature of the proposed approach is the absence of an auxiliary nonphysical mode of oscillation entering into the Timoshenko model.

Using the explicit solution (6) for $y(t,x)$ and the frequency equation (17) one can find the energy corresponding to the Lagrange function (14)

$$E_n=\frac{1}{4}A_n^2\,E\,I\,l\,\lambda_n^4\left(1-\frac{E}{k\,G}r^2\lambda_n^2\right). \qquad (19)$$

It is easy to see that the quantity (19) is positive definite. Indeed, for the prismatic beam $E/kG=3.2$ (see, for example, [1]) and within the range of application of the theory in question $r^2\lambda_n^2\ll 1$ the quantity in parentheses in (18) is wittingly positive. Therefore the Lagrange function (14) containing the shear potential energy with opposite sign, as compared with the Timoshenko model, does not result in the negative total energy.



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