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R.P.Zaikov, V.Tcholakov

INFINITE-COMPONENT CONFORMAL FIELDS. SPECTRAL REPRESENTATION OF THE TWO-POINT FUNCTION

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* Present address: Higher Pedagogical Institute, Schumen, Bulgaria.

1. *Introduction*

As a rule, one considers the finite-component conformal fields [1-3]. These fields are transformed according to representations of class Ia or Ib given by Mack and Salam [1], i.e., when the generators of special conformal transformations acting on the spin variables are represented by nilpotent operators or trivially. In those cases the stability sybgroup of the conformal group has finite-dimensional representations.

In the cases of conformal invariant operator product expansion, or of the conformal invariant partial wave expansions of Green functions [5] we deal with infinite dimensional representations of the conformal group (with respect to its stability subgroup).

In this paper we consider the fields which are transformed according to the representations of class II [1], i.e., when the generators of the special conformal group acting on the spin variables are represented in a nontrivial way. In that case the representations of stability subgroup are essentially infinite-dimensional. To specify the irreducible representations of the conformal group $SO(4,2)$ we use its Casimir operators [6-9]. The unitary irreducible representations of the conformal group are given in papers [6-8].

The conformal invariant spectral representation of the two-point function for the fields with arbitrary integer spin, which are transformed according to any irreducible representations of $SO(4,2)$ group, is obtained. The case of half integer spin may be considered analogously.

2. *Irreducible Representations of Conformal Group*

Consider the fields $\Phi(x;\xi)$ which have the following transformation properties with respect to the conformal group

$$U(\Lambda)\,\Phi(x;\xi)\,U(\Lambda)^{-1} = \Phi(\Lambda x;\Lambda\xi), \qquad (2.1.)$$

where $\Lambda \in SO(4,2)$ and $\xi \in C^{+}$ here $C^{+} = \{\xi_\mu \mid \xi^2 = 0,\ \xi^0 > 0\}$ is the future light cone.

In the infinitesimal form the transformation law (2.1) has the form:

$$[P_\mu, \Phi(x;\xi)] = i\frac{\partial}{\partial x^\mu}\Phi(x;\xi),$$

$$[M_{\mu\nu}, \Phi(x;\xi)] = \{-i(x_\mu\frac{\partial}{\partial x^\nu} - x_\nu\frac{\partial}{\partial x^\mu}) + \Sigma_{\mu\nu}\}\Phi(x;\xi),$$

$$[D, \Phi(x;\xi)] = (ix^\nu\frac{\partial}{\partial x^\nu} + \Lambda)\Phi(x;\xi), \qquad (2.2)$$

$$[K_\mu, \Phi(x;\xi)] = \{i[2x_\mu x^\nu\frac{\partial}{\partial x^\nu} - x^2\frac{\partial}{\partial x^\mu}] - 2ix^\nu(g_{\mu\nu}\Lambda + \Sigma_{\mu\nu}) +$$

$$+ \epsilon k_\mu\}\Phi(x;\xi),$$

where $\epsilon = 0$ for the "fundamental" tensor fields and $\epsilon = 1$ for any other fields. The operators

$$\Sigma_{\mu\nu} = -i(\xi_\mu\frac{\partial}{\partial\xi^\nu} - \xi_\nu\frac{\partial}{\partial\xi^\mu}),$$

$$\Lambda = i(d + \xi^\nu\frac{\partial}{\partial\xi^\nu}), \qquad (2.3)$$

$$k_\mu = 2i\xi_\mu(d_1 + \xi^\nu\frac{\partial}{\partial\xi^\nu})$$

are generators of the stability subgroup of the conformal group, i.e., the subgroup which leaves $x = 0$.

It is well known that the conformal group has three independent Casimir operators

$$\hat{C}_{II} = \frac{1}{2} J_{AB} J^{AB} ,$$

$$\hat{C}_{III} = \frac{1}{48} E_{ABCDEF} \, J^{AB} J^{CD} J^{EF} , \qquad (2.4)$$

$$\hat{C}_{IV} = J_{AB} J^{BC} J_{CD} J^{CA} ,$$

where $(A, B, \ldots = 0,1,2,3,5,6)$ and

$$J_{\mu\nu} = M_{\mu\nu} , \quad J_{5\mu} = \frac{1}{2} (P_{\mu} - K_{\mu}) , \qquad (2.5)$$

$$J_{6\mu} = \frac{1}{2} (P_{\mu} + K_{\mu}) , \quad J_{65} = D .$$

The fields which transform according to arbitrary irreducible representations of the conformal group are given as a solution of the following system of eqs.

$$[\hat{C}_{k} , \Phi(x;\xi)] = c_{k} \Phi(x;\xi) \quad (k = II, III, IV) , \qquad (2.6)$$

where c_k are the eigenvalues of the corresponding Casimir operators. For the "tensor" representations $c_{III} = 0$, i.e., in our case any unitary irreducible representation is labelled by the pair of real parameters $\chi = \{ c_{II} , c_{IV} \}$. Substituting (2.2) and (2.3) into (2.6), we have

$$[\hat{C}_{II} , \Phi(x;\xi)] = \{ d(d-4) - 2(d-1) \xi^{\mu} \frac{\partial}{\partial x^{\mu}} + 2(d - \xi^{\mu} \frac{\partial}{\partial x^{\mu}}) \xi^{\nu} \frac{\partial}{\partial \xi^{\nu}} + 2 \xi^{\mu} \xi^{\nu} \frac{\partial^{2}}{\partial \xi^{\mu} \partial \xi^{\nu}} \} \Phi(x;\xi) , \qquad (2.7)$$

$$[\hat{C}_{III} , \Phi(x;\xi)] = 0, \qquad (2.8)$$

$$[C_{IV}, \Phi(x;\xi)] = [d(d-4)(d-2)^2 + 4(d-2)(d-4)[-\epsilon d_1 \xi^{\mu} \frac{\partial}{\partial x^{\mu}} + (d-\epsilon \xi^{\mu} \frac{\partial}{\partial x^{\mu}}) \xi^{\nu} \frac{\partial}{\partial \xi^{\nu}} + \xi^{\mu} \xi^{\nu} \frac{\partial^2}{\partial \xi^{\mu} \partial \xi^{\nu}}]] \Phi(x;\xi), \quad (2.9)$$

where $\epsilon = 0$ for the "fundamental" fields and $\epsilon = 1$ for any other fields. In our case k_{μ} are not nilpotent operators and, consequently, the corresponding representations of the stability subgroup are infinite-dimensional.

3. *Teo-Point Function for the Irreducible Fields*

Consider the two-point function

$$F^{[\chi_1,\chi_2]}(x_1,\xi_1;x_2,\xi_2) = \langle 0|\Phi(x_1;\xi_1,\chi_1)\,\Phi(x_2,\xi_2,\chi_2)|0\rangle, \quad (3.1)$$

where the fields $\Phi(x,\xi,\chi)$ transform according to an arbitrary "tensor" representation of the conformal group $\chi = [c_{II}, c_{IV}]$. We consider as well, the conventional "fundamental" tensor fields ($\epsilon = 0$) and any "nonfundamental" fields. The conformal invariance for the two-point function (3.1) is

$$F^{[\chi_1,\chi_2]}(\Lambda x_1, \Lambda\xi_1; \Lambda x_2, \Lambda\xi_2) = F^{[\chi_1,\chi_2]}(x_1,\xi_1;x_2,\xi_2), \quad (3.2)$$

where $\Lambda \in SO(4,2)$.

From (2.7) and (2.9) it follows that it is convenient to pass to the momentum space. Taking into account the translational invariance and spectrum condition, we have

$$F^{[\chi_1,\chi_2]}(x_1 - x_2;\xi_1,\xi_2) = \int d^4p\, \Theta(p)\, e^{-ip(x_1-x_2)} \times \tilde{\tilde{F}}^{[\chi_1,\chi_2]}(p;\xi_1,\xi_2), \quad (3.3)$$

where $\Theta(p) = \theta(p^0)\,\theta(p^2)$ is the characteristic funstion of the future cone, and $F^{\{\chi_1,\chi_2\}}(p;\xi_1,\xi_2)$ is the kernel of the two-point function.

From the irreducibility conditions for the fields (2.6) there follow the corresponding conditions for the two-point function and consequently for its kernel $F^{\{\chi_1,\chi_2\}}(p;\xi_1,\xi_2)$, i.e., one has

$$(\hat{C}^a_k - c^k_a)\,\tilde{F}^{\{\chi_1,\chi_2\}}(p;\xi_1,\xi_2) = 0, \qquad \begin{matrix}(a=1,2),\\ (k = II, IV).\end{matrix} \tag{3.4}$$

From (2.7), (2.8), (2.9), (3.3) and (3.4) we have the following system of partial differential equations

$$\{d^a(d^a-4) + 2i\epsilon(3/2-a)\,d^a_1\,p\xi^a + 2[d^a + i\epsilon(3/2-a)\,p\xi^a] \times$$

$$\times\, \xi^\nu_a \frac{\partial}{\partial \xi^\nu_a} + 2\xi^\mu_a \xi^\nu_a \frac{\partial^2}{\partial \xi^\mu_a \partial \xi^\nu_a} - c^a_{II}\}F^{\{\chi_1,\chi_2\}}(p;\xi_1,\xi_2) = 0, \tag{3.5}$$

$$\{d^a(d^a-4)(d^a-2)^2 + 4(d^a-4)(d^a-2)[i\epsilon(3/2-a)\,p\xi^a +$$

$$+ (d^a_1 - i\epsilon(3/2-a)\,p\xi^a)\,\xi^\nu_a \frac{\partial}{\partial \xi^\nu_a} + \xi^\mu_a \xi^\nu_a \frac{\partial^2}{\partial \xi^\mu_a \partial \xi^\nu_a}] -$$

$$- c^a_{IV}\}F^{\{\chi_1,\chi_2\}}(p;\xi_1,\xi_2) = 0 \qquad (a=1,2). \tag{3.6}$$

where $\epsilon(a) = 0$ for the "fundamental" fields and $\epsilon(a) = \Theta(a) - \Theta(-a)$ for any other fields.

Let us write down eqs. (3.5) and (3.6) in terms of relativistically invariant variables $\sqrt{p^2}$, $p\xi^a = z^a$ and $w = 1 - \frac{p^2(\xi^1\xi^2)}{(p\xi^1)(p\xi^2)}$. Then we have

$$\Big\{ d^a(d^a-4) + 2i\epsilon(3/2-a)\, d_1^a z_a + 2\,[\, d^a + i\epsilon\,(3/2-a) z_a]\, z_a \frac{\partial}{\partial z_a} + 2 z_a^2 \frac{\partial^2}{\partial z_a^2} - c_{II}^a \Big\}\, F^{[\lambda_1,\lambda_2]}(p^2, z_a, w) = 0, \tag{3.7}$$

$$\Big\{ d^a(d^a-4)(d^a-2)^2 + 4(d^a-4)(d^a-2)\,\Big[\, i\epsilon(3/2-a)\, z_a + (d^a + i\epsilon(3/2-a)\, z_a)\, z_a \frac{\partial}{\partial z_a} + z_a^2 \frac{\partial^2}{\partial z_a^2} \Big] - c_{IV}^a \Big\}\, F(p^2, z_a, w) = 0 \tag{3.8}$$

The solutions of the system of partial differential eqs. (3.7) and (3.8) may be written down in the form

$$F^{[\lambda_1,\lambda_2]}(p^2, z_a, w) = f^{[\lambda_1,\lambda_2]}(p^2, w)\, z_1^{\sigma_1} z_2^{\sigma_2} \cdot t_{\sigma_1}^{\lambda_1}(z_1)\, t_{\sigma_2}^{\lambda_2}(z_2), \tag{3.9}$$

where $f^{[\lambda_1,\lambda_2]}(p^2, w)$ are arbitrary functions (which will be determined from the dilatation and special conformal invariance) and $t_{\sigma}^{\lambda}(z)$ are solutions of the following differential equation

$$\Big\{ z_a \frac{d^2}{dz_a^2} + [\, d^a + 2\sigma_a + i\epsilon(3/2-a)\, z_a] \frac{d}{dz_a} + i\epsilon\,(3/2-a) \cdot (d_1^a + \sigma_a) \Big\}\, t_{\sigma_a}^{\lambda_a} = 0. \tag{3.10}$$

Consider, first, the case of the "fundamental" fields. In that case the solution of eq. (3.10) $(\sigma = 0)$ is given by

$$\varphi_\ell^\lambda(z) = c_1 + c_2 z^{1-d-2\ell} \quad . \tag{3.11}$$

where c_1 and c_2 are arbitrary constants.

For "nonfundamental" fields ($\epsilon \neq 0$) the solutions of eq. (3.10) are the degenerate hypergeometrical functions. If $d^a + 2\ell^a$ is noninteger, the general solution of (3.10) is

$$\varphi^{\lambda_a}_{\ell_a}(z_a) = A_a \Phi(d_1^a + \ell_a, d^a + 2\ell^a; i\epsilon(3/2-a) z_a) + B_a [i\epsilon(3/2-a) z_a]^{1-d_a-2\ell_a} \Phi(d_1^a - d^a - \ell^a, 2-d^a-2\ell^a; i\epsilon(3/2-a) z_a) \quad (a = 1, 2). \tag{3.12}$$

For $d^a + 2\ell^a$ integer the general solution of eq. (3.10) has the form

$$\varphi^{\lambda_a}_{\ell_a}(z_a) = A'_a \Psi(d_1^a + \ell^a, d^a + 2\ell^a, i\epsilon(3/2-a) z_a) + B'_a e^{i\epsilon(3/2-a) z_a} \Psi(d^a - d_1^a - \ell_a, d^a + 2\ell^a, -i\epsilon(3/2-a) z_a), \tag{3.13}$$

where Ψ are the Tricomi functions.

The functions $\Phi(a,c,x)$ and $x^{1-c}\Phi(a-c, 2-c, x)$ as well as $\Psi(a,c,x)$ and $e^x \Psi(c-a, c, -x)$ give the nonequivalent representations labeling with the same numbers $\lambda = [c_{II}, c_{IV}]$.

The functions Ψ are connected with Φ by the following relations

$$\Psi(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(a,c;x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} \Phi(a-c+1, 2-c; x).$$

The functions Φ and Ψ have the following integral representations

$$\Phi(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 du\, e^{ux}\, u^{a-1}(1-u)^{c-a-1},$$

$$\mathrm{Re}\, c > \mathrm{Re}\, a > 0,$$

and

$$\Psi(a,c;x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(a+s)\Gamma(-s)\Gamma(1-c-s)}{\Gamma(a)\Gamma(a-c+1)}\, x^{s} ds,$$

where $-\mathrm{Re}\, a < \gamma < \min(0, 1-\mathrm{Re}\, c)$.

Consider now in more detail the case when $a = -n$ with n integer. In this case $d_1^a + r_a = -n_a$ the degenerate hypergeometrical function Φ transforms into the Lagere polynomials, i.e., in this case the solutions of eq. (3.10) are

$$f_{r_a}^{\lambda_a} = A_a'' L_n^{d_1^a+2r_a}(i\epsilon(3/2-a)z_a) + B_a''\,[i\epsilon(3/2-a)z_a]^{1-d_1^a-2r_a} \times$$

$$\times\Phi(-n_a-d_1^a-2r_a, 2-d_a-2r_a, i\epsilon(3/2-a)z_a). \tag{3.14}$$

The Lagere polynomials satisfy the following differential equations

$$\left[x\frac{d^2}{dx^2} + (\alpha - x + 1)\frac{d}{dx} + n\right] L_n^{\alpha}(x) = 0 \tag{3.15}$$

and are connected with the functions Φ as follows

$$L_n^{\alpha}(x) = \frac{\Gamma(n+\alpha+1)}{n!\,\Gamma(\alpha+1)}\Phi(-n, \alpha+1, x).$$

The numbers ρ are related to the eigenvalues of the Casimir operators c_{II} and c_{IV} via the following relations

$$c_{II} = (\rho_1 + 2)^2 + (\rho_2 + 1)^2 - 5, \tag{3.16}$$

$$c_{IV} = (\rho_1 + 2)^2 [(\rho_1 + 2)^2 - 2(\rho_2 + 1)^2 + 1] +$$
$$+ (\rho_2 + 1)^2 [(\rho_2 + 1)^2 + 1], \tag{3.17}$$

where we use the following labelling:

$$\rho_1 = d + l - 4, \quad \rho_2 = l. \tag{3.18}$$

For the unitary representations of the conformal group, following , we have the following cases

$$\begin{aligned} &1)\ \rho_1 = -2 + i\sigma_1, \quad \rho_2 = -1 + i\sigma_2 \\ &\quad 0 \le \sigma_1 < \infty \qquad 0 \le \sigma_2 < \infty \\ &2)\ -1 < \rho_2 + 1 < 0 \\ &3)\ \mathrm{Im}\,\rho_1 = \mathrm{Im}\,\rho_2 = 0 . \end{aligned} \tag{3.19}$$

From (3.9) and (3.16) it follows that in any case (3.19) there are both "fundamental" and "nonfundamental" fields transforming according to some irreducible representations which are specified by the same pair of numbers $\chi = [\rho_1, \rho_2]$. In such a manner we have generalized the theorem of Gatto et al. [11] to the case of any representations of conformal group.

4. *Conformal Invariant Two-Point Kernel*

The kernel (3.9) will be conformally invariant if the following equations [3]

$$\begin{aligned} &D\tilde{F}(p^2, z_a, w) = 0, \\ &K_\mu \tilde{F}(p^2, z_a, w) = 0, \end{aligned} \tag{4.1}$$

are satisfied, where D and K_μ are the generators of dilatations and special conformal transformations acting on the two-point function. Following paper [3], we have that the solution of eqs. (4.1) exists only for

$$\lambda_1 \quad \lambda_2 \tag{4.2}$$

and they are given by

$$F^{|\lambda_1 - \lambda_2|}(p^2, z_a, w) \quad N^{\lambda}(p^2)^{d-2} \; z_1^{\ell} z_2^{\ell} \; t^{\lambda}(z_1)\, t^{\lambda}(z_2)$$

$$\sum_{s=0} \frac{(2s+1)\,\Gamma(\ell+s+2)\,\Gamma(d+s-1)\,\Gamma(d+s-2)}{\Gamma(\ell-s+1)} P_s(w) \; .$$

where $P_s(w)$ is the Legendre polynomials.
From (4.3) it follows that the two-point function exists when either both the fields are "fundamental" or one is "fundamental", or both the fields are "nonfundamental".

In any case of unitary representations of SO(1,2) the kernel (4.3) is positively defined, but is local only when $\ell - n$ is positive integer number.

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