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SUPERFIELDS WITH ANY SPIN. SPECTRAL REPRESENTATION OF THE DILATATIONAL INVARIANT TWO-POINT FUNCTION

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1. According to Salam and Strathdee [1] the scalar superfield is written as a polynomial in the Majorana anticommuting spinor θ, i.e.,

$$\Phi(x;\theta) = \Phi(x) + \Phi^{a}(x)\,\theta_a + \Phi^{[a,\beta]}(x)\,\theta_a\theta_\beta +$$

$$+\Phi^{[a,\beta,\gamma]}(x)\,\theta_a\theta_\beta\theta_\gamma + \Phi^{[a,\beta,\gamma,\delta]}(x)\,\theta_a\theta_\beta\theta_\gamma\theta_\delta\,. \qquad (1.1)$$

$$(a,\beta,\ldots = 1,\ldots,4)\,.$$

The fields with arbitrary spin may be written in a similar way [2], i.e., as one homogeneous function of the two-component complex (commuting) spinor

$$\Psi(x;\lambda z) = \lambda^{\nu_1}\bar{\lambda}^{\nu_2}\,\Psi(x;z)\,, \qquad (1.2)$$

where $\chi = [\nu_1, \nu_2]$ give irreducible representations of $SL(2,C)$ [2].

We introduce the superfields with arbitrary spin in the following way

$$\Psi(x;\lambda z,\theta) = \lambda^{\nu_1}\bar{\lambda}^{\nu_2}\,\Phi(x,z,\theta)\,. \qquad (1.3)$$

With respect to θ, the fields $\Psi(x;z,\theta)$ are polynomials of degree 2. Here z and θ are two-component complex spinors and θ are elements of the Grassman algebra, i.e.,

$$[\theta_a, \theta_b]_+ = [\theta_a, z_b]_- = [z_a, z_b]_- = 0,\ (a,b = 1,2)\,.$$

The transformation laws of superfield (1.3) are:

$$U(a,\Lambda)\,\Psi(x;z,\theta)\,U^{-1}(a,\Lambda) = \Psi(\Lambda x + a;\, zA^{-1},\, \theta A^{-1}), \qquad (1.4)$$

$$U(\alpha)\,\Psi(x;z,\theta)\,U^{-1}(\alpha) = \Psi[\,x_\mu + \frac{1}{2}(\alpha\sigma_\mu\bar{\theta} - \theta\sigma_\mu\bar{\alpha});\, z, \theta+\alpha\,],$$

where $\Lambda \in SO(3,1)$, $A \in SL(2,C)$, $a \in T_4$ and $\alpha \in T_4$ are two-component complex anticommuting spinor parameters of supertransformations. In the basis in which γ_5 is diagonal we have correspondence between (1.1) and (1.3) for the scalar case.

For the tensor fields $(\nu_1 = \nu_2)$ the supertransformation law (1.4) may be generalized in the following way

$$U(\alpha)\Phi(x_\mu, \xi_\mu, \theta)\,U(\alpha)^{-1} =$$

$$=\Phi[\,x_\mu + \frac{ia}{2}(\alpha\sigma_\mu\bar{\theta} - \theta\sigma_\mu\bar{\alpha}),\, \xi_\mu + \frac{ib}{2}(\alpha\sigma_\mu\bar{\theta} - \theta\sigma_\mu\bar{\alpha}),\, \theta+\alpha\,], \qquad (1.5)$$

where a and b are two arbitary parameters. In the infinitesimal form (1.4) and (1.5) are given by

$$[\,G^c, \Phi] = \{\frac{\partial}{\partial\theta_{\dot{c}}} + \frac{ia}{2}\frac{\partial^+}{\partial x_\mu}(\sigma_\mu\bar{\theta})^c + \frac{ib}{2}\frac{\partial}{\partial\xi^\mu}(\sigma^\mu\bar{\theta})^c\}\Phi,$$

$$(1.6)$$

$$[\,\bar{G}^{\dot{c}}, \Phi] = \{\frac{\partial}{\partial\bar{\theta}_{\dot{c}}} + \frac{ia}{2}\frac{\partial}{\partial x^\mu}(\theta\sigma^\mu)^{\dot{c}} + \frac{ib}{2}\frac{\partial}{\partial\xi^\mu}(\theta\sigma^\mu)^{\dot{c}}\}\Phi.$$

We have the transformation law (1.4) if $a = 1$ and $b = 0$.

2. Consider the two-point function

$$F(x_1, z_1, \theta_1; x_2, z_2, \theta_2) = \langle 0|\Psi_1(x_1, z_1, \theta_1)\,\Psi_2(x_2, z_2, \theta_2|0\rangle, \qquad (2.1)$$

where the fields $\Psi_a(x_a; z_a)$ are transformed according to some irreducible representations $\chi = [\nu_1, \nu_2]$ of

SL(2,C). The superinvariance conditions for the two-point function (2.1) are

$$F(\Lambda x_b + a, z^b A^{-1}, \theta^b A^{-1}) = F(x_b, z^b, \theta^b),$$

$$F[x_\mu^b + \frac{i}{2}(\alpha\sigma_\mu \bar{\theta}^b - \theta^b \sigma_\mu \bar{\alpha}), z^b, \theta^b + \alpha] = F(x_\mu^b, z^b, \theta^b). \qquad (2.3)$$

To satisfy (2.2) and (2.3) it is convenient to pass to the momentum space. Taking into account the translational invariance and spectrality, we write down

$$F(x_1 - x_2; z^a, \theta^a) = \int d^4p\, \Theta(p)\, e^{-ip(x_1-x_2)} \tilde{\tilde{F}}(p; z^a, \theta^a), \qquad (2.4)$$

where $\Theta(p) = \Theta(p^0)\Theta(p^2)$ is the characteristic function of the future cone.

Consider first the relativistic invariance condition (2.2). This condition is satisfied if the kernel $\tilde{\tilde{F}}(p; z^a, \theta^a)$ is the function of the relativistic invariants which may be constructed out of 4-vector p and spinors z^a and θ^a. Out of the spinors z^a and θ^a we may construct the following 4-vectors

$$\xi_\mu^{ab} = z^a \sigma_\mu \bar{z}^b, \qquad \eta_\mu^{ab} = \theta^a \sigma_\mu \bar{\theta}^b,$$

$$\zeta_\mu^{ab} = z^a \sigma_\mu \bar{\theta}^b, \qquad \bar{\zeta}_\mu^{ab} = \theta^a \sigma_\mu \bar{z}^b. \qquad (2.5)$$

Using the identity

$$g^{\mu\nu} (\sigma_\mu)^{a\dot{b}} (\sigma_\nu)^{c\dot{d}} = 2\epsilon^{ac}\epsilon^{\dot{b}\dot{d}},$$

one may prove the following identities

$$X_\mu^{11} Y_\nu^{22} + X_\mu^{12} Y_\nu^{21} = \pm 2 g_{\mu\nu} x^1 \epsilon y^2 \bar{x}^1 \epsilon \bar{y}^2,$$

$$(\xi^{ab})^2 = (\zeta^{ab})^2 = (\bar{\zeta}^{ab})^2 = 0,$$

$$(\eta^{ab})^2 = -2\theta^a \epsilon \theta^a \bar{\theta}^b \epsilon \bar{\theta}^b, \qquad (a, b = 1, 2), \qquad (2.6)$$

where $\epsilon = i\sigma_2$ and the sign $+1$ is for the case $[x,y]_- = 0$ and sign -1 when $[x,y]_+ = 0$.

Following paper /2/, one can prove that the kernel $\tilde{F}$ is a function of the following 14 independent relativistical invariants

$$\tilde{F}(p;z^a,\theta^a) = \tilde{F}(p^2, z^1\epsilon z^2, \bar{z}^1 \cdot \bar{z}^2, z^1 \tilde{p}\bar{z}^2, z^1\tilde{p}\bar{z}^1, z^2\tilde{p}\bar{z}^2,$$
$$\theta^1\epsilon\theta^2, \theta^a\tilde{p}\bar{\theta}^b, \theta^1\epsilon z^1, \bar{\theta}^1\epsilon\bar{z}^1, \theta^1\tilde{p}\bar{z}^1) . \qquad (2.7)$$

The irreducibility condition (1.3) gives /2/

$$\tilde{F}(p;z^a,\theta^a) = \sum_{k,m,n=0}^{2} (z^1\epsilon z^2)^{\ell_0+\ell_0^1+\frac{k-m-n}{2}} (z^1\tilde{p}\bar{z}^2)^{\ell_0-\ell_0^1+\frac{k-m-n}{2}} \times$$

$$\times (z^1\tilde{p}\bar{z}^1)^{\ell_1-\ell_0-k-1} (z^2\tilde{p}\bar{z}^2)^{\ell_1^1-\ell_0+\frac{m+n-k}{2}-1} (z^1 p\,\bar{\theta}^1)^m (z^1\epsilon\theta^1)^n (\bar{z}^1\epsilon\bar{\theta}^1)^k \times$$

$$\times H_{k\ell m}\left(p^2, w = 1 - \frac{p^2 (z^1\sigma_\mu\bar{z}^1)(z^2\sigma^\mu\bar{z}^2)}{(z^1\tilde{p}\bar{z}^1)(z^2\tilde{p}\bar{z}^2)}, \theta^1\epsilon\theta^2, U^{ab}\right), \qquad (2.8)$$

where $\nu_1 = \ell_1 + \ell_0 - 1$, $\nu_2 = \ell_1 - \ell_0 - 1$. $H_{k\ell m}$ are arbitrary functions of p^2 and w and which are polynomials in θ^1 and θ^2 of degree 2.

Equation (2.8) gives the general form of relativistically invariant two-point kernel for the superfields with any spin. The superinvariance condition (2.3) in the infinitesimal form is written as follows

$$G^a\tilde{F}(p;z^b,\theta^b) = 0,$$
$$\bar{G}^{\dot{a}}\tilde{F}(p;z^b,\theta^b) = 0, \qquad (a,\dot{a} = 1,2), \qquad (2.9)$$

where G and $\bar{G}$ are the generators of the supertransformations acting on the kernel $\tilde{F}$. From (1.10), (1.11) (for $a = 1$ and $b = 0$) and (2.4) we have

$$G^a = \frac{\partial}{\partial\theta^1_a} + \frac{\partial}{\partial\theta^2_a} + \frac{1}{2}(p\bar{\theta}^1)^a + \frac{1}{2}(p\bar{\theta}^2)^a \qquad (2.10)$$

and

$$\bar{G}^{\dot{a}} = \frac{\partial}{\partial\bar{\theta}^1_{\dot{a}}} + \frac{\partial}{\partial\bar{\theta}^2_{\dot{a}}} + \frac{1}{2}(\theta^1 p)^{\dot{a}} - \frac{1}{2}(\theta^2 p)^{\dot{a}} .$$

Substituting (2.10) in eq. (2.9) we have

$$(\theta^1\epsilon + \epsilon\theta^2)^a \frac{\partial\tilde{F}}{\partial\theta^1\epsilon\theta^2} + (\underset{\sim}{p}\bar{\theta}^1)^a \left[\frac{\partial\tilde{F}}{\partial u^{11}} + \frac{\partial\tilde{F}}{\partial u^{21}} + \frac{1}{2}\tilde{F}\right] +$$

$$+ (p\bar{\theta}^2)^a \left[\frac{\partial\tilde{F}}{\partial u^{12}} + \frac{\partial\tilde{F}}{\partial u^{22}} - \frac{1}{2}\tilde{F}\right] + (z^1\epsilon)^a \frac{\partial\tilde{F}}{\partial z^1\epsilon\theta^1} = 0, \qquad (2.11)$$

$$(\theta^1\underset{\sim}{p})^{\dot{a}}\left[-\frac{\partial\tilde{F}}{\partial u^{11}} - \frac{\partial\tilde{F}}{\partial u^{12}} + \frac{1}{2}\tilde{F}\right] - (\theta^2 p)^{\dot{a}}\left[\frac{\partial F}{\partial u^{21}} + \frac{\partial F}{\partial u^{22}} + \frac{1}{2}F\right] +$$

$$+ (z^1\underset{\sim}{p})^{\dot{a}} \frac{\partial\tilde{F}}{\partial z^1 p\bar{\theta}^1} + (\bar{z}^1\epsilon)^{\dot{a}} \frac{\partial F}{\partial\bar{z}^1\epsilon\bar{\theta}^1} = 0. \qquad (2.12)$$

The solution of this system is given by

$$\tilde{F}(p, z^a, \theta^a) = (z^1\epsilon z^2)^{\ell_0+\ell_0^1} (z^1\underset{\sim}{p}z^2)^{\ell_0-\ell_0^1} (z^1\underset{\sim}{p}z^1)^{\ell_1-\ell_0-1} (z^2\underset{\sim}{p}z^2)^{\ell_1^1-\ell_0-1} \times$$

$$\times \int d\kappa\, f(p^2,\kappa)\, e^{\kappa(u_{11}+u_{22}) + (\frac{1}{2}-\kappa)u_{12} - (\frac{1}{2}+\kappa)u_{21}} , \qquad (2.13)$$

where $f(p^2,\kappa)$ is an arbitrary function of p^2 and κ. There is a second solution given by

$$F^1 = \delta(\theta_1 - \theta_2)(z^1 \epsilon z^2)^{\ell_0 + \ell_0^1} (z^1 p \tilde{z}^2)^{\ell_0 - \ell_0^1} (z^1 p \tilde{z}^1)^{\ell_1 - \ell_0 - 1} \times$$
$$\times (\tilde{z}^2 p z^2)^{\ell_1^1 - \ell_0 - 1} g(p^2, \theta^1 \epsilon \theta^1), \qquad (2.14)$$

where $g(p^2, \theta\epsilon\theta)$ is an arbitrary function.

The function (2.13) is not invariant under the space reflections. In the case $\ell_0 = \ell_0^1 = 0$ the function (2.13) possesses such an invariance if /3/

$$f(p^2, \kappa) = \sigma(p^2)\delta(\kappa),$$

when $\ell_0, \ell_0^1 \neq 0$ the invariance with respect to space reflections takes place provided the field $\Psi(x, z)$ is transformed according to representations $[\ell_0, \ell_1] \oplus [-\ell_0, \ell_1]$ of $SL(2,C)$.

3. The two-point function (2.13) describes the propagation of particles with spin $\max(0, |\ell_0| - 1) \leq s \leq \ell_1$ for finite-component fields and $\max(0, |\ell_0| - 1) \leq s \leq \infty$ when both the fields are infinite-component /4/.

Let us decompose the superinvariant kernel (2.13) into the sum over the spin variables, i.e., over the eigenvalues of the second Casimir operators of the Poincare subgroup

$$\tilde{\tilde{F}}(p, z^a, \theta^a) = \sum_s F_s(p; z^a, \theta^a). \qquad (3.1)$$

Here $\tilde{\tilde{F}}_s$ satisfy the following eq.

$$[\hat{S}^2 - s(s+1)]\tilde{\tilde{F}}_s(p; z^a, \theta^a) = 0, \qquad (3.2)$$

where

$$\hat{S}^2 = \frac{1}{2}\Sigma_{\mu\nu}\Sigma^{\mu\nu} - \frac{1}{p^2}\Sigma_{\mu\lambda}\Sigma^{\nu\lambda} p^\mu p_\nu, \qquad (3.3)$$

is the second Casimir operator of the Poincare group. Consider first scalar superfields (these fields contain spins $s = 0, 1/2, 1$). In this case the generators of the Lorentz group are given by

$$\Sigma_j = \frac{1}{2}\epsilon_{jk\ell}\,\Sigma_{k\ell} = \frac{1}{2}(\theta\sigma_j\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\bar\theta}\sigma_j\bar\theta),$$

$$\Sigma_{0j} = \frac{i}{2}(\theta\sigma_j\frac{\partial}{\partial\theta} - \frac{\partial}{\partial\bar\theta}\sigma_j\bar\theta). \quad (3.4)$$

Substituting (3.4) in (3.3) we write eq. (3.2) in the relativistically invariant variables

$$\{\frac{3}{4}(u_{12}\frac{\partial}{\partial u_{12}} + u_{21}\frac{\partial}{\partial u_{21}} - u_{12}^2\frac{\partial^2}{\partial u_{12}^2} - u_{21}^2\frac{\partial^2}{\partial u_{21}^2}) +$$

$$+ (u_{11}u_{22} + 2u_{12}u_{21})\frac{\partial^2}{\partial u_{12}\,\partial u_{21}} - s(s+1)\}\bar{\bar F}_s(p;\theta^1,\theta^2) = 0. \quad (3.5)$$

The solutions of this eq. are given by

$$\bar{\bar F}_s(p;\theta^1,\theta^2) = f(p^2)\sum_m C_s^m t_s^m(u), \quad (3.6)$$

where $f(p^2)$ is an arbitrary function, t_s^m are given by

$$t_0^1 = 1,\ t_0^2 = u_{11}u_{22},\ t_0^3 = u_{12}^2,\ t_0^4 = u_{21}^2,\ t_0^5 = u_{12}^2u_{21}^2,$$

$$t_{1/2}^1 = u_{12},\ t_{1/2}^2 = u_{21},\ t_{1/2}^3 = u_{12}u_{21}^2,\ t_{1/2}^4 = u_{21}u_{12}^2,$$

$$t_1^1 = u_{11}u_{22} + 2u_{12}u_{21}. \quad (3.7)$$

The coefficients C_s^m may be found from the power decomposition of (2.13). From (2.13) and (3.6) and (3.7) we have

$$C^1_0 = 8C^2_0 = 8C^3_0 = 8C^4_0 = 64C^5_0 =$$

$$= 2C^1_{1/2} = -2C^2_{1/2} = 16C^3_{1/2} = -16C^4_{1/2} = -8C^1_1 = 1. \qquad (3.8)$$

In the case of superfields with spin the decomposition of the kernel (2.14) is more complicated. From (2.15) it follows that we may decompose the kernel into the sum over the variables θ and z separately. The decomposition with respect to the commuting variables z is given in paper /2/. Combining these decompositions we have

$$\tilde{F}(p;z^a,\theta^a) = (z^1 \epsilon z^2)^{\ell_0+\ell^1_0} (z^1 \tilde{p} \bar{z}^1)^{\ell_0-\ell^1_0} (z^1 \tilde{p} \bar{z}^1)^{\ell_1-\ell_0-1} \times$$

$$\times (z^2 p \bar{z}^2)^{\ell_1-\ell_0-1} \sum_{s_1, m} C^m_{s_1} t^m_{s_1} (u_{12}, u_{21}, u_{11}, u_{22}) \times$$

$$\times \sum_{s_2=|\ell_0|} \sigma_{s_2}(p^2)\, P^{(\ell_0-\ell^1_0,\, \ell_0+\ell^1_0)}_{s_2-|\ell_0|}(w), \qquad (3.9)$$

where σ_s is an arbitrary function, C^m_s are given by (3.8) and $P^{\alpha\beta}_n(x)$ are the Jacoby polynomials. Any term in (3.9) describes propagation of particles with spin $|s_1-s_2| \leq s \leq s_1+s_2$.

4. The kernel (3.8) is local for the finite-component fields /2/.

The kernel (2.15) is invariant with respect to dilatations if it satisfies the following eq. /5/.

$$D\tilde{\tilde{F}}(p, \theta^1, z^1, \theta^2, z^2) = 0, \qquad (4.1)$$

where D is the generator of dilatations, which is the sum of corresponding operator acting on the fields, given by

$$[D, \Phi(x;z,\theta)] = i[d + \frac{1}{2}(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}) + \frac{1}{2}(\theta\frac{\partial}{\partial \theta} + \bar{\theta}\frac{\partial}{\partial \bar{\theta}})]\Phi(x;z,\theta). \qquad (4.2)$$

This operator obeys the following commutation relations with generators of the supersymmetric algebra

$$[D, M_{\mu\nu}] = 0\,, \qquad [D, P_\mu] = -iP_\mu\,,$$

$$[D, G^a] = \frac{i}{2} G^a, \;\; [D, \bar{G}^{\dot{a}}] = \frac{i}{2} \bar{G}^{\dot{a}}. \tag{4.3}$$

Substituting (3.2) into (3.1) we obtain

$$\{ d_1 + d_2 - 4 - 2p^2 \frac{\partial}{\partial p^2} + z^1 \epsilon\, z^2 \frac{\partial}{\partial z^1 \epsilon\, z^2} + \theta^1 \epsilon\, \theta^2 \frac{\partial}{\partial \theta^1 \epsilon\, \theta^2} \} \tilde{F}(p; z^a, \theta^a) = 0. \tag{4.4}$$

From (2.13) and (4.4) we have that the general form of dilatationally invariant supersymmetric two-point kernel is given by

$$\tilde{F}(p; z^1, \theta^1, z^2, \theta^2) = N (p^2)^{\frac{d_1 + d_2 + \ell_0 + \ell_0^1}{2} - 2} (z^1 \epsilon\, z^2)^{\ell_0 + \ell_0} e^{\frac{1}{2}(u_{12} - u_{21})} \times$$

$$\times (\underset{\sim}{z}^1 p \bar{z}^2)^{\ell_0 - \ell_0^1} (z^1 \underset{\sim}{p} \bar{z}^2)^{\ell_1 - \ell_0 - 1} (z^2 p \bar{z}^2)^{\ell_1^1 - \ell_0 - 1} \sum_s N_s P_{s - |\ell_0|}^{(\ell_0 - \ell_0^1,\, \ell_0 + \ell_0^1)}(w),$$

where N_s are arbitrary constants.

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