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AUTOMODEL BEHAVIOUR OF THE FIXED ANGLE SCATTERING AMPLITUDE IN THE FRAMEWORK OF THE JOST-LEHMANN-DYSON REPRESENTATION

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Автомодельные асимптотики амплитуды рассеяния на фиксированные углы в рамках представления Дайсона-Йоста-Лемана

В работе установлены необходимые и достаточные условия существования степенных автомодельных асимптотик амплитуды рассеяния на большие углы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Automodel Behaviour of the Fixed Angle Scattering Amplitude in the Framework of the Jost-Lehmann-Dyson Representation

Conditions for asymptotic power behaviour of the scattering amplitude at fixed angles and some properties of the limiting expressions have been established. There is a connection between short distance behaviour of the product of currents and the asymptotics of a particular off-shell amplitude at fixed scattering angles.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research
Dubna 1975

1. INTRODUCTION

In the last time the principle of automodelity /1/ has been applied successfully to high-energy scattering at fixed angles. If supplemented by assumptions about the quark structure of hadrons, automodelity allows one to understand the well known power behaviour

$$\frac{d\sigma}{dt} \approx \frac{1}{s^m} f\left(\frac{t}{s}\right) ; \quad s, |t| \to \infty , \quad \frac{s}{t} \text{ fixed} \qquad (1)$$

observed in elastic hadron scattering /2/.

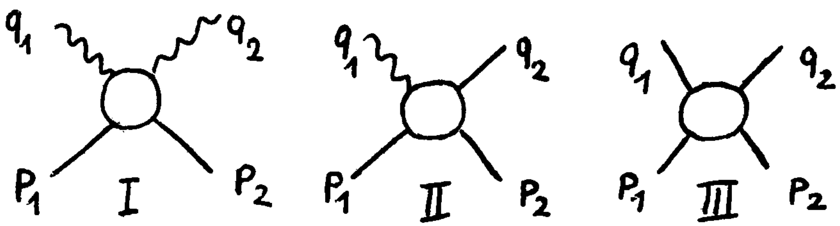
It is desirable however, to give parallel to this intuitive argumentation an investigation on the basis of general QFT, i.e., in the framework of the DJL representation. For deep inelastic electron-nucleon scattering such an investigation has been performed in the basic paper /3/ and continued in a series of subsequent ones /4, 5/. The DJL representation has also been applied to the study of inclusive scattering amplitudes /6/.

The questions we are concerned with in the present note are similar to those studied in connection with deep inelastic scattering. So conditions in terms of DJL spectral functioms are formulated which are sufficient (and necessary, if a 3-dimensional DJL representation can be applied) for automodel behaviour at fixed scattering angles. We compare on-mass-shell and off-mass-shell amplitudes with respect to asymptotic behaviour and establish some properties of the corresponding automodel or scaling functions. Finally, because scattering at fixed angles like deep inelastic scattering involves large momentum transfers one could ask if there are, also in this case, properties in x-space uniquely related to the automodel behaviour.

Whereas particular off-shell processes are found to be correlated with specific small distance behaviour of the product of current operators, the question must be answered negative for on-shell processes.

2. *KINEMATICAL CONDITIONS FOR FIXED ANGLE SCATTERING*

It is useful to consider simultaneously the following three processes as $s \to \infty$, $t \to -\infty$



Wavy lines denote virtual particles with masses tending to infinity in a particular manner. We choose the Breit system to write

$$P = \frac{1}{2}(p_1 + p_2) = (E, \vec{0}), \qquad \Delta = (q_2 - q_1) = (0, 2\vec{p}),$$

$$Q = \frac{1}{2}(q_1 + q_2), \qquad t = \Delta^2 = -4(E_p^2 - m^2),$$

$$s = (P+Q)^2, \qquad q_{1,2}^2 = (Q \mp \frac{1}{2}\Delta)^2, \tag{2}$$

$$p_{1,2}^2 = m^2, \qquad E_p = \sqrt{\vec{p}^{\,2} + m^2}.$$

In every case scattering at fixed angles means $P_\mu \to \infty$, $\Delta_\mu \to \infty$ and $Q_\mu \to \infty$ so that a convenient parametrization is given by

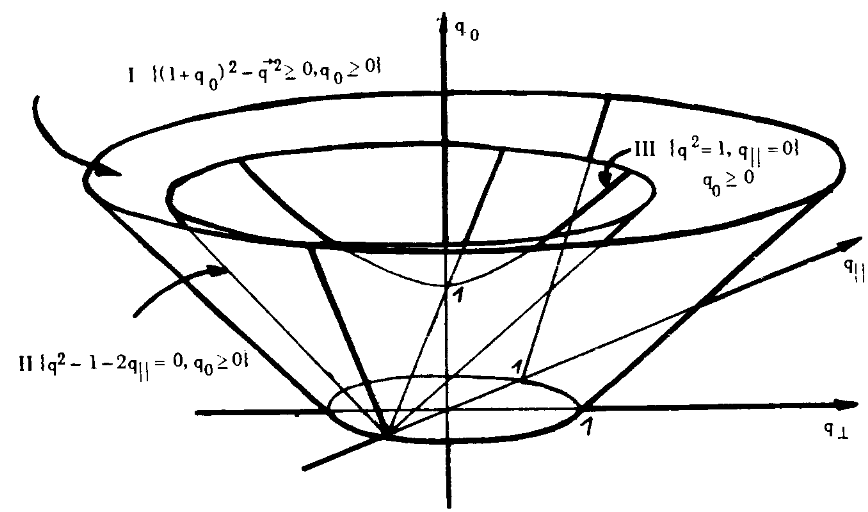
$$Q_\mu = E_p q_\mu, \quad q_\mu \quad \text{fixed}. \tag{3}$$

In other words, the four components of the dimensionless fixed vector q_μ play the role of scaling parameters. The physical range for q_μ is determined from $t \le 0$, $Q \ge 0$, spectrum conditions of the absorptive part ($(P \pm Q)^2 \ge m^2$, $P_0 \pm Q_0 \ge m$) and mass shell conditions ($q_1^2 = m_1^2$ and $q_2^2 = m_2^2$).

Process	Physical range for q_μ	
I	$(1 + q_0)^2 - \vec{q}^{\,2} > 0$	$q_0 \ge 0$
II	$q^2 - 1 = 2q_{\|\|} = 0$	$q_0 \ge 0$
III	$q^2 = 1 \quad q_{\|\|} = 0$	$q_0 \ge 0$

(4)

($q_{||}$, $q_\perp$ denote the components of $\vec{q}$ with respect to $\vec{p}$)
For one-shell process III one free variable remains, say q_0, which is related to $\frac{s}{t}$ by $\frac{s}{t} = -\frac{1}{2}(1 + q_0)$



Physical regions for the processes I, II, III.

Let us note that the endpoints of the physical range $q_0 = 1$ (case III) and $q_0 \to \infty$ belong to the scattering at fixed u and $\frac{t}{s} \to 0$ respectively.

Finally we mention the inequality for the absorptive parts

$$|F(Q;\vec{p})|^2 \leq F(Q_+;0)\, F(Q_-;0)\,,$$
$$P + Q = p_1 + Q_+ = p_2 + Q_-\,, \qquad (5)$$

which relates non-forward scattering to forward scattering.

For on-shell processes at high energy s this inequality is in accordance with the well known fact that fixed angle scattering decreases more strongly than forward scattering. In the off-shell case we obtain the same relation between the process I and the corresponding forward process in the Bjorken region (which would measure the light cone singularities of hadronic currents).

3. *ASYMPTOTIC POWER BEHAVIOUR AND CONDITIONS ON THE DJL SPECTRAL FUNCTIONS*

A general framework for a discussion of asymptotic power behaviour at fixed angles for on- and off-shell processes is the DJL representation. On this basis it is also possible to study the x-space properties (properties of the product of current operators) which could be determined from fixed angle scattering. The general problem with nonidentical currents demands the investigation of the four-dimensional DJL representation. For simplicity we restrict ourselves to the case of identical currents where a three-dimensional representation can be applied.

Let us begin with the absorptive part* $F(Q;\vec{p})$ given by

$$F(Q;\vec{p}) = \epsilon(Q)\int d\vec{u}\int d\lambda^2\, \delta(Q^2-(\vec{Q}-\vec{u})^2-\lambda^2)\,\Psi(\vec{u},\lambda^2;\vec{p})\,,$$
$$|\vec{u}| \leq E_p\,, \qquad (6)$$
$$\lambda \geq \max\{0,\, m-\sqrt{E_p^2-\vec{u}^{\,2}}\}.$$

We introduce the scaling variables $Q_\mu = E_p\, q_\mu$, $\vec{p} = p\vec{e}$, $\vec{e}^{\,2} = 1$ and write

$$\vec{u} = E_p \vec{\mu}\,,$$
$$\lambda^2 = E_p^2 \tau^2, \qquad (7)$$
$$\Psi(\vec{u},\lambda^2;\vec{p}) = E_p^{-3}\,\Phi(E_p\vec{\mu},\, E^2\tau^2;\, p\vec{e})\,.$$

Then

$$F(E_p q;\, p\vec{e}) = \epsilon(q_0)\int d\vec{\mu}\, d\tau^2\, \delta(q_0^2-(\vec{q}-\vec{\mu})^2-\tau^2)\,\Phi(E_p\vec{\mu}, E_p^2\tau^2;\, p\vec{e})\,. \qquad (8)$$

It is important to have a precise definition for the fixed angle limit $E_p \to \infty$. The basic fact that $F(Q;\vec{p})$ is a generalized function with respect to Q_μ indicates that one has to deal with a limit of functionals. We apply a quasi-limit which is a generalization of that used in /4/. In this sense asymptotic power behaviour

$$F(E_p q;\, p\vec{e}) \underset{E_r \to \infty}{\approx} E_p^{\alpha} F_0(q;\vec{e}) \qquad (9)$$

will be understood as

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*Here only the first term of the Jost-Lehmann representation is investigated.
$F(Q,\vec{p}) = \epsilon(Q_0)\int d\vec{u}\int d\lambda^2\, \delta(Q_0^2-(Q-\vec{u})^2-\lambda^2)\,(\Psi(\vec{u},\lambda^2,\vec{p}) + Q^0\Psi_1(\vec{u},\lambda^2,\vec{p}))$,
The second term can be treated analogously.

$$\lim_{E_p\to\infty} \frac{1}{E_p^{\alpha}}\int d^4q\, F(E_p q, p\vec{e})\, f(q) = \int d^4q\, F_0(q;\vec{e})\, f(q)\,, \quad f(q)\in S_4. \tag{10}$$

By Fourier transformation this equation can be rewritten in x-space as

$$\lim_{E_p\to\infty} \frac{1}{E_p^{\alpha+4}}\int d^4x\, \tilde{\tilde{F}}\left(\frac{x}{E_p};p\vec{e}\right) \tilde{\tilde{f}}(x) = \int d^4x\, \tilde{\tilde{F}}_0(x;\vec{e})\, \tilde{\tilde{f}}(x)\,, \tag{11a}$$

where $\tilde{\tilde{F}}(x;p\vec{e})$ and $\tilde{\tilde{F}}_0(x;\vec{e})$ are the Fourier transforms of the absorptive part $F(Q;\vec{p})$ and the scaling function $F_0(q;\vec{e})$ respectively:

$$\tilde{F}(x;p\vec{e}) = \frac{1}{(2\pi)^4}\int d^4Q\, e^{-iQx} F(Q;p\vec{e})\,,$$

$$\tilde{F}_0(x;\vec{e}) = \frac{1}{(2\pi)^4}\int d^4q\, e^{-iqx} F_0(q;\vec{e})\,.$$

Formula (11a) may be used to derive a condition for the spectral function Φ. Therefore we apply the Jost-Lehmann formula determining the spectral function Φ from $\tilde{\tilde{F}}$

$$\begin{aligned}\Psi(\vec{u},\lambda^2;\vec{p}) &= E_p^{-3}\Phi(E_p\vec{\mu}, E_p^2\tau; p\vec{e}) \\ &= \frac{i}{2\pi}\frac{\partial}{\partial\lambda^2}\{\theta(\lambda^2)\int_0^\infty dx^2 J_0(\lambda\sqrt{x^2})\int d^3x\, e^{+iE_p\vec{\mu}\vec{x}}\, F(x,x^2;p)\}, \\ \bar{F}(\vec{x},x^2;\vec{p}) &= \epsilon(x_0)\,\tilde{F}(x;\vec{p})\,,\end{aligned} \tag{12}$$

and study the functional

$$\int d\vec{\mu}\int d\tau^2\, \Phi(E_p\mu, E_p^2\tau^2; p\vec{e})\, f(\vec{\mu},\tau^2)$$

(f: test function). Because of the fact that

$$\bar{f}(\vec{x},x^2) = \int d\vec{\mu}\int_0^\infty d\tau^2 e^{i\vec{\mu}\vec{x}}\frac{\partial}{\partial\tau^2}\{\theta(\tau^2)\, J_0(\tau\sqrt{x^2})\}\, f(\vec{\mu},\tau^2)$$

is a test function with respect to $\vec{x}$ and $x^{2/4/}$ we get the functional

$$\int dx^2\int d\vec{x}\, \bar{F}\left(\frac{\vec{x}}{E_p},\frac{x^2}{E_p^2};p\vec{e}\right)\bar{f}(\vec{x},x^2) =$$

$$= \int d^4x\, \tilde{\tilde{F}}\left(\frac{x}{E_p};p\vec{e}\right) x_0\, \bar{f}(\vec{x},x_0^2-\vec{x}^2)\,.$$

Now it is obvious that equation (11a) is equivalent to the existence of the quasi-limit for the spectral function

$$q\text{-}\lim_{E_p\to\infty}\frac{1}{E_p^{\alpha}}\Phi(E_p\vec{\mu},E_p^2\tau^2;p\vec{e}) = \Phi_0(\vec{\mu},\tau^2;\vec{e})\,. \tag{13}$$

On the other hand, postulating the q-limit (13) for the spectral function, the generalized power behaviour (10) can be derived:

$$q\text{-}\lim_{E_p\to\infty} F(E_p q;p\vec{e}) = E_p^{\alpha}F_0(q;\vec{e})\,. \tag{14a}$$

Notice that the integral

$$\int d^4q\,\epsilon(q_0)\,\delta(q_0^2-(\vec{q}-\vec{\mu})^2-\tau^2)\, f(q)\,, \quad f(q)\in S_4,$$

defines a test function. The automodel or scaling function $F_0(q;\vec{e})$ fulfills the representation

$$F_0(q;\vec{e}) = \epsilon(q_0)\int d\vec{\mu}\, d\tau^2\delta(q^2-(\vec{q}-\vec{\mu})^2-\tau^2)\,\Phi_0(\vec{\mu},\tau^2;\vec{e})\,, \quad |\vec{\mu}|\le 1,\ \tau^2\ge 0. \tag{14b}$$

For illustration we investigate the Born terms:

$$E(Q;\vec{p}) = \epsilon(Q_0+P_0)\,\delta((Q+P)^2-m^2)+\epsilon(Q_0-P_0)\,\delta((Q-P)^2-m^2),$$

$$\lim_{E_p\to\infty} E_p^2F(E_pq;p\vec{e}) = \epsilon(1+q_0)\,\delta((1+q_0)^2-\vec{q}^2)+\epsilon(q_0-1)\,\delta((1-q_0)^2-\vec{q}^2)\,,$$

$$\lim_{E_p\to\infty}\bar{F}\left(\frac{x}{E_p};E_p\vec{e}\right)E_p^{-2} = (2\pi)^{-2}\,\epsilon(x_0)\cos x_0\,\delta(x^2)\,,$$

$$\Psi(\vec{u},\lambda^2;\vec{p}) = \frac{2}{\pi}E_p\{\theta(E^2-\vec{u}^2)\,\delta'(E_p^2-\vec{u}^2-1)\,\delta(\lambda^2)-$$

$$-\theta(\lambda^2)\frac{\partial^2}{\partial(u^2)^2}[\theta(E_p^2-\vec{u}^2)\frac{\partial}{\partial\lambda^2}\chi(\vec{u},\lambda^2)]\},$$

$$\lim_{E_p\to\infty} E_p^5\Psi(E_p\vec{\mu},E_p^2\tau^2;p\vec{e}) = \frac{2}{\pi}\theta(1-\mu^2)\,\delta'(1-\vec{\mu}^2)\,\delta(\tau^2)\,.$$

Let us finally discuss the behaviour of the total amplitude (retarded commutator). As usual, one has to be more careful with the investigation of the R-product (problem of subtractions with respect to λ^2). In addition to condition (13) we must assume that some finite number of subtractions is sufficient for the convergence of the DJL representation for all values of $t \leq 0$. In other words we assume

$$N = \sup_{-\infty < t \leq 0} n(t) < \infty ,$$

where $n_{(t)}$ denotes the number of subtractions for fixed t. Note that equation (10) gives no restrictions on the large λ^2 behaviour of $\Psi(\vec{u},\lambda^2;\vec{p})$ at t fix. If furthermore

$$\Phi(\vec{\mu},\tau^2;\vec{e}) \underset{\tau^2 \to \infty}{\approx} (\tau^2)^k ,$$

we choose

$$n = \max\{N, [k+1]\},$$

so that a representation with n subtractions

$$T(Q;\vec{p}) = -\frac{1}{\pi}\int d\vec{u}[Q_0^2-(\vec{Q}-\vec{u})^2+F_p^2]^n \times$$
$$\times \int \frac{d\lambda^2}{(\lambda^2+E_p^2)^n} \frac{\Psi(\vec{u},\lambda^2;\vec{p})}{Q_0^2-(\vec{Q}-\vec{u})^2-\lambda^2+i0\cdot\epsilon(Q_0)} + P_n(Q;\vec{p}) \qquad (15)$$

is valid for all values of t.

The coefficients of the polynomial P_n are unknown functions of the variable $\vec{p}$.

If we want to understand the limit $E_p \to \infty$ in the sense of a functional

$$\int d^4q\, T(E_p q; p\vec{e})\, f(q) ,$$

then the representation (15) leads to the functional

$$\int d\vec{\mu}\int d\tau^2 \Phi(E_p\vec{\mu}, E_p^2\tau^2; p\vec{e})\, g(\vec{\mu},\tau^2) ,$$

where $g(\vec{\mu},\tau^2)$ is infinitely differentiable and vanishes as some inverse power for $\tau^2 \to +\infty$. This is sufficient for the existence of the limit for $T(E_p q; p\vec{e})$:

$$\lim_{E_p\to\infty} T(E_q q; p\vec{e}) = E_p^{\alpha} T_0(q;\vec{e}) + P_n(E_p q; E_p\vec{e}) , \qquad (16)$$

$$T_0(q;\vec{e}) = -\frac{1}{\pi}\int d\vec{\mu}[q_0^2-(\vec{q}-\vec{\mu})^2+1]^n \int_0^\infty \frac{d\tau^2}{(\tau^2+1)^n} \frac{\Phi_0(\vec{\mu},\tau^2,\vec{e})}{q_0^2-(\vec{q}-\vec{\mu})^2-\tau^2+i0\cdot\epsilon(q_0)} .$$

In any case the subtraction term becomes polynomial in q multiplied, however, by unknown powers of E_p.

Let us summarize the essential properties of the limiting procedure for the off-shell processes:

1. Condition (13) is obviously a necessary and sufficient condition for the generalized power behaviour of process I.

2. The limit (14) of the absorptive part itself fulfills a DJL representation where, however, the spectrum of the intermediate states contains massless states.

3. $F_0(q;\vec{e})$ is a generalized function with respect to q. Therefore the asymptotic behaviour (14) must not necessarily be valid on the restricted regions corresponding to on-shell processes II or III.

4. The limit in equation (11) is the natural generalization of the classical limit $x_\mu \to 0$ for $F(x;\vec{p})$ (whereby the relative momentum p tends to infinity too). Therefore this relation constitutes a connection between the asymptotic fixed angle (off-shell) amplitude and the matrix element of the current commutator near $x_\mu = 0$

$$\lim_{E_p\to\infty} \frac{1}{E_p^{\alpha+4}} \langle p_1 | [j(\frac{x}{2E_p}), j(-\frac{x}{2E_p})] | p \rangle = \frac{1}{(2\pi)^4}\int dq e^{-iqx} F(q;\vec{e}). \qquad (11b)$$

4. *ASYMPTOTIC BEHAVIOUR FOR THE ON-SHELL PROCESSES II AND III (ABSORPTIVE PART)*

A separate investigation of on-shell scattering is needed because we have to expect different asymptotic power behaviour for on-shell and off-shell processes.

The following considerations deal with one-variable processes including case III and special cases II (see eqs. (4)) which are determined by

$$q^2 = -k, \quad 2q_{||} = -(1+k), \quad k \text{ - constant} \tag{17}$$

This means $s \to \infty, \; t \to -\infty, \; q_1^2 \to -\infty,$

$$\frac{s}{t}, \quad \frac{q_1^2}{t} = \frac{1}{2}(1+k), \quad q_2^2 \text{ fixed.}$$

Let us choose q_0 as an independent variable.

As usual, we assume that $F(E_p q_0; \vec{p})$ is a generalized function with respect to q_0. This postulate is the starting point to obtain another condition on the spectral functions guaranteeing power behaviour for process (17). Applying an odd test function $f(q_0) = q_0 g(q_0^2)$ we evaluate

$$\int dq_0 f(q_0) F(E_p q_0) = \int d\vec{\mu}_\perp \int d\mu_{||}\, d\tau^2 \Phi(E_p \vec{\mu}, E_p^2 \tau^2; pe) \int dq_0^2 g(q_0) \times$$
$$\times \delta(\tau^2 - q^2 + \mu_{||}^2 + \mu_\perp^2 - 2q_{||}\mu_{||} - 2q_\perp \mu_\perp \cos\gamma) = \tag{18}$$
$$= 2\int d\mu_\perp \mu_\perp d\mu_{||}\, d\tau^2 \Phi(E_p \vec{\mu}, E_p^2 \tau^2; pe) \int dq_0^2 g(q_0^2) \frac{\theta(q_\perp^2 - y^2)}{\sqrt{q_\perp^2 - y^2}},$$

with

$$y = \frac{\tau^2 + \mu_\perp^2 + (\mu_{||}+1)^2 + (k-1)(1+\mu_{||})}{2\mu_\perp}, \tag{19}$$

and

$$q_\perp^2 = q_0^2 + k - \frac{(1+k)^2}{4},$$

where the restriction (17) has been taken into account.

We note that

$$h(y^2) = \int dq_0^2 g(q_0^2)(q_\perp^2 - y^2)_+^{-1/2} = \int_0^\infty \frac{d\alpha}{\sqrt{\alpha}}\, g\left(\alpha + y^2 + \frac{(1+k)^2}{4} - k\right) \tag{20}$$

defines a test function $h(y^2)$.

Therefore we can write

$$\int dq_0 f(q_0) F(E_p q_0; p\vec{e}) = \int dq_0^2 g(q_0^2)[\epsilon(q_0) F(E_p q_0; p\vec{e})]$$
$$= \int dy\, \chi(E_p y; p\vec{e})\, h(y^2), \tag{21}$$

where

$$\chi(E_p y; p\vec{e}) = \int d\mu_\perp^2 \int d\mu_{||}\, \theta\left(y - \frac{\mu_\perp^2 + (\mu_{||}+1)^2 + (k-1)(1+\mu_{||})}{2\mu_\perp}\right) \Phi(E_p \vec{\mu}, E_p^2 \tau^2; p\vec{e}). \tag{22}$$

From equation (21) we conclude:

1. The absorptive part $F(E_p q_0; p\vec{e})$ has an integral representation in terms of the symmetric part

$$\chi_s = \frac{1}{2}\{\chi(E_p y; \vec{p}) + \chi(-E_p y; \vec{p})\}$$

$$F(E_p q_0; p\vec{e}) = 2\epsilon(q_0) \int_0^{q_\perp} dy \frac{\chi_s(E_p y; p\vec{e})}{\sqrt{q_\perp^2 - y^2}}. \tag{23}$$

2. A sufficient condition for asymptotic power behaviour at fixed angles, i.e., for $F(E_p q_0; p\vec{e}) \approx E_p^\beta F_0(q_0; \vec{e})$ is

$$\lim_{E_p \to \infty} E^{-\beta} \int dy\, \chi_s(E_p y; p\vec{e})\, h(y^2) = \int dy\, \chi_{s0}(y; \vec{e})\, h(y^2). \tag{24}$$

Condition (24) should be understood not as an alternative to formula (13) but as a supplement. In the former case we had a quasi-limit with respect to $\vec{\mu}$ and τ^2, whereas equation (24) is a one-dimensional quasi-limit applied to the reduced spectral function. Indeed, there exist spectral functions fulfilling both conditions (13) and (24) with $\alpha \neq \beta$, e.g., (in case $k = -1$)

$$\Phi(E_p \vec{\mu}, E_p^2 \tau^2; E_p \vec{e}) = \delta(\mu_{||}) \cdot \frac{\theta(\tau^2 - E_p^{-2})}{[E_p^2(\tau^2 - 1 + \mu_\perp^2)] + 4E_p^2 \mu_\perp^2} \cdot \frac{E_p^2 \mu_\perp^2}{(E_p^2 \mu_\perp^2 + 1)^2}$$

$$\Phi \underset{E_p \to \infty}{\approx} E_p^{-4} \delta(\vec{\mu}_\perp)\, \delta(\mu_{||})\, \delta(\tau^2 + \mu^2 - 1)$$

as a functional in $\vec{\mu}$ and τ^2,

$$\chi \underset{E_p \to \infty}{\approx} E_p^{-3}\,\delta(y) \qquad \text{as a functional in } y .$$

This has as consequence for the absorptive part that its on-shell and off-shell asymptotical behaviour are different.

As a result of the condition (23) we obtain for the absorptive part

$$F(E_p q_0; p\vec{e}) \approx E_p^{\beta} F_0(q_0; \vec{e})$$

with the scaling function

$$F_0(q;\vec{e}) = 2\epsilon(q_0)\int_0^{q_\perp} dy \frac{\chi(y;\vec{e})}{\sqrt{q_\perp^2 - y^2}}, \tag{25}$$

$$q_\perp^2 = q_0^2 + k - \frac{(k+1)^2}{4} .$$

It is important to note that condition (24) is a necessary and sufficient one. This follows from the fact that equation (20) defines an one-to-one mapping from S onto S. This guarantees the existence of the limit of the right-hand side of formula (21) for all $h(y^2) \in S$ provided the q-limit for $F(E_p q_0; p\vec{e})$ exists. For completeness we list the inverse transformation to equation (20) and (23):

$$g(q_0^2) = \ell(q_\perp^2) = -[\Gamma(\tfrac{1}{2})]^{-2}\frac{\partial}{\partial q_\perp^2}\int dy^2 h(y^2)(y^2 - q_\perp^2)^{-1/2}; \tag{26}$$

$$\frac{1}{\sqrt{y^2}}\chi_s(E_p y; p\vec{e}) = [\Gamma(\tfrac{1}{2})\Gamma(-\tfrac{1}{2})]^{-1}\int dq_\perp^2\,\epsilon(q_0)\,F(E_p q_0^2; p\vec{e})(y^2 - q_\perp^2)_+^{-3/2}. \tag{27}$$

It is remarkable that there is a one-to-one correspondence between the absorptive part and some integral of the DJL spectral function (even before taking the limit $E_p \to \infty$) in this special kinematical case (17).

5. *ASYMPTOTIC BEHAVIOUR FOR THE ON-SHELL PROCESSES II AND III (DISPERSIVE PART)*

The foregoing considerations will now be extended to the total amplitude (R-product). For simplicity it is assumed, that its DJL representation is free of subtractions

$$T(Q;\vec{p}) = -\frac{1}{\pi}\int d\vec{\mu}\int d\tau^2 \frac{\Phi(E_p\vec{\mu}, E_p^2\tau^2; p\vec{e})}{(q_0 + i0)^2 - (\vec{q} - \vec{\mu})^2 - \tau^2} . \tag{28}$$

In the special region (17) this takes the form

$$T(E_p q_0; p\vec{e}) = \frac{1}{\pi}\int dy \int d\mu_{||}\, d\mu_\perp^2\, \Phi(E_p\vec{\mu}, E_p^2\tau^2; p\vec{e})\int_0^{\pi} d\gamma \frac{1}{y - i0\epsilon(q_0) - q_\perp \cos\gamma} \tag{29}$$

and finally

$$T(E_0 q_0; p\vec{e}) = \int dy \frac{\chi(E_p y; p\vec{e})}{\sqrt{(y - i0\epsilon(q_0))^2 - q_\perp^2}} , \tag{30}$$

where χ has been defined in equation (22).

A careful handling of the cut properties of this square-root leads to

$$T(E_p q_0; pe) = \int_{q_\perp}^{\infty} dy \frac{1}{\sqrt{y^2 - q^2}}\{\chi(E_p y; pe) - \chi(-E_p y; pe)\} +$$

$$+ \int_0^{q_\perp} dy \left\{ \frac{\chi(E_p y; p\vec{e})}{\sqrt{(y - i0\epsilon(q_0))^2 - q_\perp^2}} - \frac{\chi(-E_p y; p\vec{e})}{\sqrt{(y + i0\epsilon(q_0))^2 - q_\perp^2}} \right\},$$

or

$$T(E_p q_0; p\vec{e}) = 2\int_{q_\perp}^{\infty} dy \frac{\chi_a(E_p y; p\vec{e})}{\sqrt{y^2 - q^2}} + 2i\epsilon(q_0)\int_0^{q_\perp} \frac{\chi_s(E_p y; p\vec{e})}{\sqrt{q^2 - y^2}} . \tag{31}$$

Comparing formula (31) with the well known relation $T = D + iF$ we conclude that the dispersive part D is related to the antisymmetric part of χ. By the way, we note that there exists an inverse transformation similar to equation (26).

To obtain the asymptotic power behaviour for the dispersive part, too, condition (24) is not sufficient. Therefore we demand a similar condition for the antisymmetric part χ_a. In principle χ_a could show a power behaviour different from that of χ_s. Consequently in this case absorptive and dispersive parts would behave differently in the asymptotic region of fixed angle scattering. This is a sharp contrast to the situation in off-shell fixed angle scattering (process I considered in section 3).

For the special off-shell process II with $k \geq 1$, however, we have according to equation (22), $\chi(E_p y; p\vec{e}) = 0$ for $y < 0$, so that

$$\chi_a(E_p y; p\vec{e}) = \chi_s(E_p y; p\vec{e}) = \frac{1}{2}\chi(E_p y; p\vec{e}) \quad \text{for} \quad y > 0 .$$

Thus absorptive and dispersive parts have the same asymptotic power behaviour. This must be not the case for the on-shell process III. Here $\mathrm{Re}\,T$ and $\mathrm{Im}\,T$ (which can be identified with D and F in the case of identical particles) may behave differently.

Let us assume for simplicity that both parts χ_a and χ_s show the same power behaviour, e.g.,

$$\chi(E_p y; p\vec{e}) \approx E_p^{\beta} \chi_0(y; \vec{e}), \tag{32}$$

so that

$$T(E_p q_0; p\vec{e}) \approx E_p^{\beta} T_0(q_0; \vec{e}),$$

$$T_0(q_0; \vec{e}) = 2\int_{q_\perp}^{\infty} dy \frac{\chi_{a0}(y; \vec{e})}{\sqrt{y^2 - q_\perp^2}} + 2i\epsilon(q_0)\int_0^{q_\perp} dy \frac{\chi_{s0}(y; \vec{e})}{\sqrt{q_\perp^2 - y^2}} \tag{33}$$

(compare argumentation leading to formula (16)).

It is interesting to discuss formula (33) at the

boundaries of the physical region (compare eq. (17))

$$q_\perp \to \infty \quad \to \quad \frac{s}{t} \to \infty$$

$$q_\perp = 0 \quad \to \quad u \ \text{fix} .$$

In the first case $(q_\perp \to \infty)$ $T_0(q; \vec{e})$ becomes purely absorptive

$$T_0(q_0; \vec{e}) \approx i\epsilon(q_0)\left[\frac{1}{\sqrt{y^2}}\chi_{s0}(y; \vec{e})\right] * (y^2)_+^{-1/2} \Big|_{y^2 = q_\perp^2 \to \infty} \tag{34}$$

depending on the properties of χ_{s0} for large y.

In the second case $(q_\perp \to 0)$ we get

$$T_0(q_0; \vec{e}) = \lim \int^{\infty} dy \frac{2\chi_{a0}(y; \vec{e})}{\sqrt{y^2 - q_\perp^2}} + i\epsilon(q_0)\left[\frac{1}{\sqrt{y^2}}\chi_{s0}(y; \vec{e})\right] * (y^2)_+^{-1/2} \Big|_{y^2 = q_\perp^2 \to 0} \tag{35}$$

which reflects the properties of χ_a and χ_s at $y = 0$. For suitable spectral functions this expression may also become pure absorptive (case of $t-u$ crossing symmetry).

There remain a discussion of the analytic properties of $T_0(q_0; \vec{e})$ which can be studied at the representation

$$T_0(q_0; \vec{e}) = \int dy \frac{\chi_0(y; \vec{e})}{\sqrt{(y - i0\epsilon(q_0))^2 - q_\perp^2}} ; \quad q_\perp^2 = q_0^2 + k - \frac{1}{4}(1+k)^2 . \tag{36}$$

It is crucial to note that the imaginary part $i0\epsilon(q_0)y$ of y can be understood as an imaginary part of q_0 if the integration extends over positive values of y only. This takes place if we choose the region (17) with $k \geq 1$ (compare the definition (22) for χ). In this case $T_0(q, \vec{e})$ is an analytic function in the q_0-plane with two cuts starting from $\pm\sqrt{\frac{1}{4}(1+k)^2 - k}$. Physically the restriction (17) corresponds to processes with masses q_2^2 on shell and $q_1^2 \to -\infty$, where $q_{1/2}^2 = \frac{1}{2}(1+k)$. Thus analytic properties can be proven for semi off-shell ($k \geq 1$) processes but not for the on-shell process ($k = -1$).

It is quite reasonable that in cases of analyticity the dispersive and absorptive parts show equal asymptotic behaviour (see considerations following equation (31)).

Finally a technical remark should be added. It concerns the problem of subtractions. Here we have to use the subtracted representation (15). A straight-forward calculation analogous to that leading from equation (32) to equation (33) gives (remember $T = D + iF$)

$$D_0(q;\vec{e}) = 2\int_{q_\perp}^{\infty} dy \left\{ \frac{\chi_{a0}(y,\vec{e})}{\sqrt{y^2 - y_\perp^2}} - \chi_0'(y,\vec{e}) \right\}.$$

In the case of one subtraction χ_0' is determined by

$$\chi_0'(y;\vec{e}) = \lim_{E_p \to \infty} \frac{1}{E_p^{\beta}} \Big[\int d\mu_{||} d\mu^2 \frac{2\mu_\perp \big(\big[y - \frac{1}{2\mu_\perp}(\mu_\perp^2 + (\mu_{||}+1)^2 + (k-1)(\mu_{||}+1))\big)\big]}{2\mu_\perp y - \mu_\perp^2 - (1+\mu_{||})^2 - (k-1)(\mu_{||}+1) + 1} \times$$

$$\times \Phi(E_p\vec{\mu}, E_p^2 \tau^2; p\vec{e}) \Big]_{\text{antisym. part}}.$$

For higher subtractions there is an explicit polynomial dependence on $q_\perp$ in χ_0'.

Also in this case the dominance of the absorptive part over the dispersive part will be recognized, if we do not take care of the unknown polynomials with respect to $q_\perp$.

6. CONCLUSIONS

The DJL representation allows the investigation of the amplitude for on- and off-shell scattering at fixed angles. A connection between the asymptotic behaviour of the matrix elements of the current commutator at $x_\mu \to 0$, $|\vec{p}| \to \infty$ has been established (see formula (11b).).

On- and off-shell amplitudes can show a different asymptotic behaviour because of the distribution character of the functions involved. There is a mathematical possibility that dispersive and absorptive parts of the on-shell amplitudes have different limits for fixed angle scattering. This is not allowed in the case of off-shell amplitudes where analytic properties can be proved.

For an one-parametric manifold of processes including the on-shell scattering the absorptive part dominates over the dispersive part as $s \to \infty$, $t \to -\infty$, $\frac{t}{s} \to 0$ if we do not take care of subtraction polynomials.

We are deeply indebted to N.N.Bogolubov, A.A.Logunov and M.A.Mestvirishvili for useful discussions.

APPENDIX

Spectral Functions of Born Terms

Here we consider the spectral function of the Born term

$$T = \frac{1}{(P+Q)^2 - m^2} + \frac{1}{(P-Q)^2 - m^2}$$

with the imaginary part

$$\mathrm{Im}\, T = \epsilon(P_0 + Q_0)\,\delta((P+Q)^2 - m^2) + \epsilon(Q_0 - P_0)\,\delta((P-Q)^2 - m^2).$$

In the Breit system $P = (E_\alpha, 0, 0, 0)$ and choosing $m^2 = 1$ we can write

$$\mathrm{Im}\, T = \epsilon(Q_0 + E_p)\,\delta((E_p + Q_0)^2 - \vec{Q}^2 - 1) + \epsilon(Q_0 - E_p)\,\delta((Q_0 - E_p)^2 - \vec{Q}^2 - 1).$$

Following [3] we have to calculate

$$\Psi(\vec{u}, \lambda^2) = \frac{i}{2\pi} \frac{\partial}{\partial \lambda^2} \Big[\theta(\lambda^2) \int_0^{\infty} dk^2 J_0(k\lambda) \int d\eta\, e^{-i\vec{\eta}\vec{u}}\, \Phi(\vec{\eta}, k^2) \Big],$$

$$F(\vec{\eta}, k^2) = \bar{F}(k^2 + \vec{\eta}^2, \vec{\eta}^2), \qquad \bar{F}(x^2, \vec{x}) = \epsilon(x_0)\, \mathrm{Im} T.$$

In the same manner as in [3] we get

$$\bar{F}(x) = -\frac{i}{\pi}\cos E_p x_0 D(x,1),$$

$$\Phi(\rho,k^2) = -i(2\pi)^{-2}\cos(E\sqrt{k^2+\rho^2})\frac{\partial}{\partial k^2}[\theta(k^2)J_0(k)],$$

$$\Psi(\vec{u},\lambda^2) = \frac{1}{\pi^2 u}\frac{\partial}{\partial\lambda^2}\{\theta(\lambda^2)\int_0^\infty dk^2 J_0(k\lambda)\frac{\partial}{\partial k^2}[\theta(k^2)J_0(k)]\times$$

$$\times\int_0^\infty d\rho\,\rho\sin(u\rho)\cos(E_p\sqrt{k^2+\rho^2}),\quad u\equiv|\vec{u}|$$

and also

$$\Psi(u,\lambda^2) = \frac{2}{\pi}\sqrt{E_p^2}\frac{\partial^2}{\partial(u^2)^2}\{\theta(E^2-u^2)[\delta(\lambda^2)-$$

$$-\delta(\lambda^2)\int_0^\infty dk J_0(k\sqrt{E_p^2-u^2})J_1(k)-$$

$$-\theta(\lambda^2)\frac{\partial}{\partial\lambda^2}\int_0^\infty dk J_0(k\lambda)J_0(k\sqrt{E_p^2-u^2})J_1(k)]\}.$$

For the investigation of the first term we use (see /7/ 6.512.3)

$$\int_0^\infty J_0(k\sqrt{E_p^2-u^2})J_1(k)\,dk = \theta(1-E_p^2+u^2)$$

so that

$$\delta(\lambda^2)\frac{\partial^2}{\partial(u^2)^2}\{\theta(E_p^2-u^2)(1-\theta(1-E_p^2+u^2)\} =$$

$$=\delta(\lambda^2)\frac{\partial^2}{\partial(u^2)^2}\{\theta(E_p^2-u^2)\theta(E_p^2-u^2-1)\} =$$

$$=\delta(\lambda^2)\delta'(E_p^2-u^2-1)\theta(E_p^2-u^2).$$

For the spectral function we then write

$$\Psi(u,\lambda^2) = \frac{2}{\pi}\sqrt{E_p^2}\theta(E_p^2-u^2)\delta'(E_p^2-u^2-1)\delta(\lambda^2) - \frac{2}{\pi}\sqrt{E_p^2}\theta(\lambda^2)\times$$

$$\times\frac{\partial^2}{\partial(u^2)^2}[\theta(E_p^2-u^2)\frac{\partial}{\partial\lambda^2}\chi(u^2,\lambda^2)],$$

where

$$\chi(u^2,\lambda^2) = \int_0^\infty J_0(k\sqrt{E_p^2-u^2})J_0(k\lambda)J_1(k)\,dk.$$

Again using /1/ 6.578.3;-4;-8 we obtain

$$\gamma(u^2,\alpha^2) = \begin{cases} 0 & \text{for } 1+\lambda<\sqrt{E_p^2-u^2} \\ 0 & \lambda-1>\sqrt{E_p^2-u^2} \\ 1 & 0<\lambda<1-\sqrt{E_p^2-u^2} \\ \frac{v}{\pi} & |1-\lambda|<\sqrt{E_p^2-u^2}<1+\lambda \begin{cases} \lambda>1 & \lambda<\sqrt{E_p^2-u^2}+1 \\ & \lambda>\sqrt{E_p^2-u^2}-1 \\ \lambda<1 & \lambda>1-\sqrt{E_p^2-u^2} \\ & \lambda>\sqrt{E_p^2-u^2}-1 \end{cases} \end{cases}$$

$$v = \text{acrcos}\,\frac{\lambda^2+E_p^2-u^2-1}{2\lambda\sqrt{E_p^2-u^2}}$$

From general principles we expect as physical range

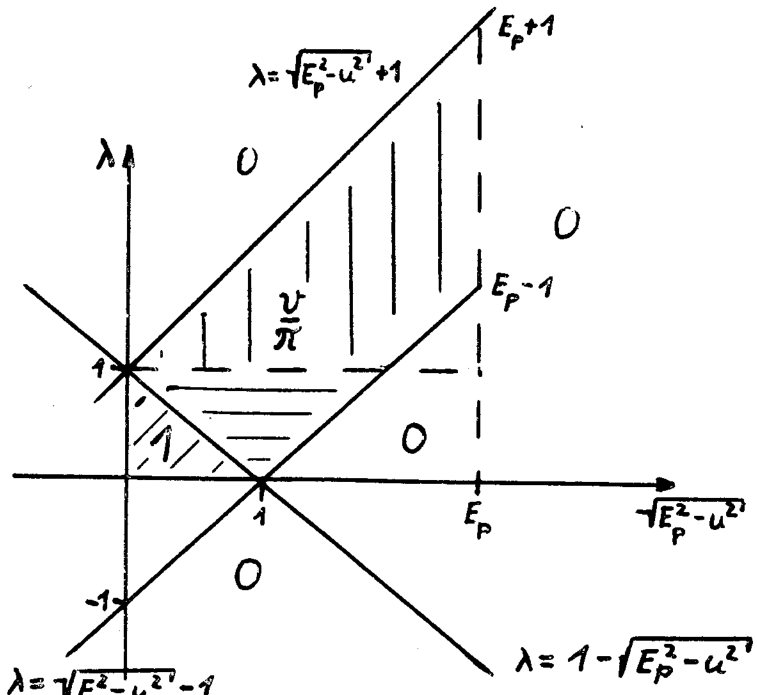
$$\alpha \geq \max\{0,\ 1-\sqrt{E^2-u^2}\}.$$

This means that

$$\chi(u,\lambda^2) = \frac{v}{\pi}\qquad |1-\lambda|<\sqrt{E^2-u^2}<1+\lambda$$

(taking into account the derivative) covers the physical range. Remark

$$\arccos\frac{\lambda^2+E_p^2-u^2-1}{2\lambda\sqrt{E^2-u^2}} = \begin{cases} \arccos 1 = 0 & \text{for } \lambda=\sqrt{E_p^2-u^2}-1, \\ \arccos 1 = 0 & \lambda=\sqrt{E_p^2-u^2}+1, \\ \arccos(-1)=\pi & \lambda=1-\sqrt{E_p^2-u^2}, \\ \arccos(-1)=\pi & \lambda=-1-\sqrt{E_p^2-u^2}, \end{cases}$$



This means there is no discontinuity along the straight lines $\lambda=\sqrt{E_p^2-u^2}\pm 1$ and $\lambda=1-\sqrt{E_p^2-u^2}$ in the allowed range. In the interior of this domain the argument of arc cos is smaller than 1. Now we will perform the quasi-limit of $\Phi(E_p\mu, E_p^2\tau^2; p\vec{e})$:

$$\Phi(E_p\mu, E_p^2\tau^2; p\vec{e}) = E_p^3\Psi(E_p\mu, E_p^2\tau^2; p\vec{e}) ,$$

$$\lim_{E_p\to\infty} E_p^2\Phi(E_p\mu, E_p^2\tau^2; p\vec{e}) =$$

$$= \lim_{E_p\to\infty} \Big\{ \frac{2}{\pi}E_p^6\theta(E_p^2-E_p^2\mu^2)\,\delta'(E_p^2-E_p^2\mu^2-1)\,\delta(E_p^2\tau^2) -$$

$$- \frac{2}{\pi}E_p^6\,\theta(E_p^2\tau^2)\,\frac{1}{E_p^4}\,\frac{\partial^2}{\partial(\mu^2)^2}\Big[\theta(E_p^2-E_p^2\mu^2)\,\frac{\partial}{\partial(E_p^2\tau^2)}\chi(E_p^2\mu^2, E_p^2\tau^2)\Big\} =$$

$$= \frac{2}{\pi}\theta(1-\mu^2)\,\delta'(1-\mu^2)\,\delta(\tau^2) .$$

The second term gives no contribution. To see this, we write the full expression with test functions $\phi(\tau^2)\,g(\mu^2)$:

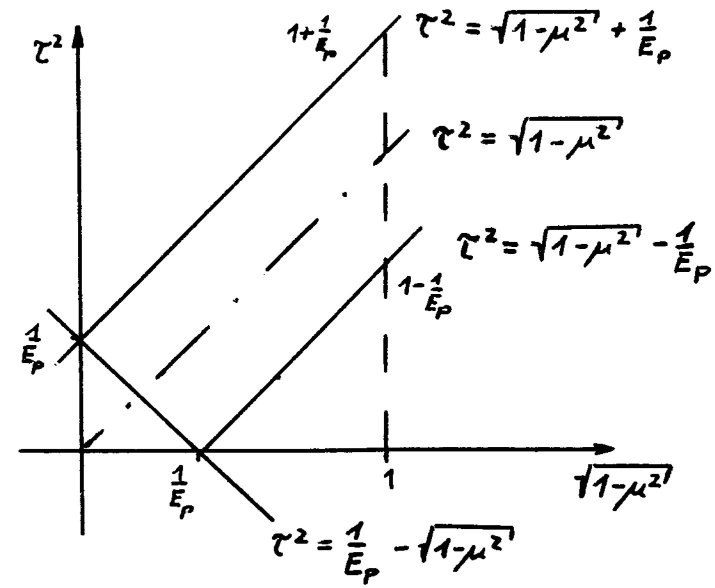
$$\lim_{E_p\to\infty} \int d\mu\,\mu^2 g(\mu^2)\int d\tau^2\phi(\tau^2)\,\theta(\tau^2)\,\frac{\partial^2}{\partial(\mu^2)^2}\Big[\theta(1-\mu^2)\frac{\partial}{\partial\tau^2}\chi(E^2\mu^2, E^2\tau^2)\Big].$$

For the quasi-limit differentiation and limiting procedure can be exchanged.

So we can consider the limit $\lim_{E_p\to\infty}\chi(E_p^2\mu^2, E_p^2\tau^2)$.

We remark: 1. $|\chi| \leq 1$

2. The support contracts to a line



Therefore this term gives no contribution.

REFERENCES

1. *В.А.Матвеев, Р.М.Мурадян, А.Н.Тавхелидзе. Проблемы физики элементарных частиц и атомного ядра, ЭЧАЯ, т. 2, вып. 1, с. 2 /1970/.*
2. *V.A.Matveev, R.M.Muradyan, A.N.Tavkhelidze. Lett. Nuovo Cim., 7, 719 (1973).*
   S.J.Brodsky, G.F.Farrar. Phys.Rev.Lett., 31, 1153 (1973).
3. *Н.Н.Боголюбов, А.Н.Тавхелидзе, В.С.Владимиров. ТМФ, 12, 305 /1972/.*
4. *Б.И.Завьялов. ТМФ, 17, 178 /1973/.*
5. *Э.Вицорек, В.А.Матвеев, Д.Робашик, А.Н.Тавхелидзе. ТМФ, 16, 315 /1973/.*
   Э.Вицорек, В.А.Матвеев, Д.Робашик. ТМФ, 19, 14 /1974/.
6. *А.А.Логунов, М.А.Мествиришвили, В.А.Петров. Препринт ИФВЭ, СТФ 74-76, Серпухов /1974/.*
7. *И.С.Градштейн, И.М.Рыжик. Таблицы интегралов, сумм, рядов и произведений. Москва, 1963.*

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