объединенный
институт
ядерных
исследований
дубна

E2-92-582

V.N.Pervushin, G.S.Pogosyan*, A.N.Sissakian, S.I.Vinitsky

EQUATION FOR QUASIRADIAL FUNCTIONS IN MOMENTUM REPRESENTATION ON A THREE-DIMENSIONAL SPHERE

Submitted to «Soviet Journal of Nuclear Physics»

*G.S.Pogosyan E-mail: POGOS@THEOR.JINRC.DUBNA.SU

1992

Первушин В.Н. и др. E2-92-582

Уравнение для квазирадиальных функций в импульсном представлении на трехмерной сфере

В работе получено радиальное уравнение Шредингера в дискретном импульсном представлении для центральных потенциалов на трехмерной сфере в виде системы однородных алгебраических уравнений. В пределе плоского пространства эта система соответствует известному интегральному уравнению Шредингера для радиальных функций в импульсном представлении. Явно вычислены ядра этого уравнения для ряда потенциалов, имеющих геометрический смысл и встречающихся в приложениях. Предложен метод вычисления квазирадиальных решений на основе чебышевской процедуры построения системы ортогональных полиномов дискретной переменной.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1992

Pervushin V.N. et al. E2-92-582

Equation for Quasiradial Functions in Momentum Representation on a Three-Dimensional Sphere

The radial Schrödinger equation for the wave functions in discrete the momentum representation for central potentials on a three-dimensional sphere are obtained in the form of a system of homogeneous algebraic equations. This system corresponds to the ordinary integral Schrödinger equation for radial wave functions in the momentum representation in the limit of a flat Euclidean space. The kernels of this equation are calculated explicitly for a class of central potentials having a geometrical sense and appearing in applications. The numerical method of calculation of quasiradial solutions and spectrum is proposed on the basis of the Chebyshev procedure of constructing a suitable system of orthogonal polynomials of a discrete variable.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1992

In memory of Prof. Ya.A.Smorodinsky

1 Introduction

The quantum mechanics on the Riemannian manifolds attracts the attention of many investigators during a long period of time, see, for example, refs. in [1,2]. Starting from the papers by Schroedinger, Infeld and Stivenson [3-5] the Hydrogen atom problem with the harmonic potential in the coordinate representation on a three-dimensional sphere, $S^3(\chi, \theta, \phi)$, with radius, R, has been studied by many authors [6-10]. However in the momentum representation this problem has been not considered. Nowadays the investigation of the above problem in the momentum representation and development of methods of its solution are of interest due to possible applications in more complicate three-body systems [11] and in quarkonium physics [12]. It also has a direct relation to the QCD with a global topological variable and with a self-dual condensate like a "bag" in which quasiparticle excitation states of quarks and gluons are described by the Gegenbauer polynomials [13].

In the present paper the Schroedinger equation for quasiradial wave functions with a central potential in the discrete momentum representation on a three-dimensional sphere are constructed in the form of an infinite system of homogeneous algebraic equations which in the limit of $R \to \infty$ transforms into the known integral Schroedinger equation for radial wave function in the usual momentum representation [14]. For the kernel of this equation, explicit formulae are found for a certain class of central potentials having geometrical interpretation. First, it is concerns the harmonic potential

$$V_S^\alpha(\chi, R) = -\frac{\alpha}{R} ctg\chi$$

defined as a solution of the Laplace equation on the three- dimensional sphere and has the same sense as the conventional Coulomb potential. The corresponding quasiradial solutions in the discrete momentum representation have been obtained in our previous paper [15] through the general hypergeometric function of a unit argument with the help of a direct calculation of the overlap integral between the known quasiradial function in the coordinate representation and the Gegenbauer polynomials. These functions in the limit of $R \to \infty$ transform into the known radial wave functions of the Hydrogen atom in the usual momentum representation [14]. This result can be used as a test for any numerical solution of the above-mentioned problem if it has no exact solution in the coordinate representation.

For solving this problem we propose a procedure based on a suitable orthogonal transformation realizing the change of the initial system of equations to an equivalent system of algebraic equations with a more simple structure of the kernel. The corresponding transformation matrix is given by the generalized Chebyshev procedure of constructing a system of orthogonal polynomials of a discrete variable, that produces the finest approximation in the transition to the integral equation in the limit of $R \to \infty$.

2 Solutions of free Schroedinger equation on a three-dimensional sphere

A three-dimensional space with constant curvature can be realized on a three-dimensional sphere S^3_R with a radius $0 < R < \infty$ embedded in the four-dimensional Euclidian space M_4 defined in a proper way. The coordinates $\vec{x} = \{x_1, x_2, x_3, x_4\}$ of the four-dimensional space M_4 are connected with the spherical coordinate $\omega = \{\chi, \vartheta, \varphi\}$ describing the motion on the three-dimensional sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$$

by the equations

$$x_1 = R\sin\chi\sin\vartheta\cos\varphi \ , \ x_2 = R\sin\chi\sin\vartheta\sin\varphi,$$

$$x_3 = R\sin\chi\cos\vartheta \ , \ x_4 = R\cos\chi, \tag{1}$$

$$0 \le \chi \le \pi \ , \ 0 \le \vartheta \le \pi \ , \ 0 \le \varphi \le 2\pi.$$

The spherical coordinate system (1) is more appropriate because the variable χ can be treated both as the spherical angle and as the length of geodesic line on the three-dimensional sphere (with the radius R =1). The angular part of the Laplace operator $\widehat{\Box}$ and elements of volume $d\widehat{M}_4$ and length dS^2 in the $\widehat{M}_4$ have the form

$$\frac{1}{R^2}\widehat{\Box} = \frac{1}{R^2}\left[\frac{1}{\sin^2\chi}\frac{\partial}{\partial\chi}\sin^2\chi\frac{\partial}{\partial\chi} - \frac{l^2}{\sin^2\chi}\right],$$

$$d\widehat{M}_4 = R^3\sin^2\chi d\chi d\widehat{M}_3 \ ,$$

$$dS^2 = R^2 d\chi^2 + R^2 sin^2\chi(d\theta^2 + sin^2\theta d\phi^2).$$

Here

$$l^2 = -\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\sin\vartheta\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right],$$

$$d\widehat{M}_3 = \sin\vartheta d\vartheta d\varphi$$

is the operator of the of angular momentum squared at $\widehat{M}_3 \sim S^2(\vartheta, \varphi)$.

The Schroedinger equation describing the particle motion in a potential field $V^\alpha_S(\Omega, R)$ on S^3_R at any value of R can be written in the following form ($\hbar = \mu = 1$):

$$\left[-\frac{1}{2R^2}\widehat{\Box} + V^\alpha_S(\Omega, R)\right]\Psi^\alpha_S(\Omega, R) = E^\alpha(R)\Psi^\alpha_S(\Omega, R). \tag{2}$$

For free particle motion on the sphere S^3_R, i.e. in the case of $V^\alpha_S \equiv 0$, the solution of the Schrödinger equation in the spherical coordinate system (1) has the form

$$\Psi^{\alpha=0}_{nlm}(\chi, \vartheta, \varphi, R) = S_{nl}(\chi, R)Y_{lm}(\vartheta, \varphi).$$

The energy spectrum of this dynamical system is defined by formula

$$E^{\alpha=0}_n(R) = \frac{(n^2-1)}{2R^2}, \quad n = 1, 2, \ldots$$

The spherical functions $Y_{lm}(\vartheta, \varphi)$ obey to the orthogonality and completeness conditions

$$\sum_{lm} Y_{lm}(\vartheta, \varphi)Y_{lm}(\vartheta', \varphi') = \delta(\cos\vartheta - \cos\vartheta')\delta(\varphi - \varphi'),$$

$$\int\int Y_{lm}(\vartheta, \varphi)Y_{l'm'}(\vartheta, \varphi)\sin\vartheta d\vartheta d\varphi = \delta_{ll'}\delta_{mm'}.$$

The quasiradial functions $S_{nl}(\chi, R)$ are linked with the Gegenbauer polynomials by

$$S_{nl}(\chi, R) = \frac{2^{l+1}l!}{R}\sqrt{\frac{n(n-l-1)!}{2\pi R(n+l)!}}(\sin\chi)^l C^{l+1}_{n-l-1}(\cos\chi), \tag{3}$$

where the orbital quantum number l at a fixed quantum number n takes the values $l = 0, 1, \cdots, n-1$. These functions can be represented through the elementary functions

$$S_{nl}(0, R) = \sqrt{\frac{2n^2}{\pi R^3}}\delta_{l0}, \quad S_{n0}(\chi, R) = \sqrt{\frac{2}{\pi R^3}}\frac{\sin n\chi}{\sin\chi},$$

$$S_{nl}(\chi, R) = \frac{(\sin\chi)^l}{\sqrt{(n^2-1)\cdots(n^2-l^2)}}\frac{d^l}{(d\cos\chi)^l}S_{n0}(\chi, R).$$

Using the condition of orthogonality and completeness for the Gegenbauer polynomials one can verify that the functions $S_{nl}(\chi, R)$ satisfy the following relations:

$$R^3\sin\chi\sin\chi'\sum_n S_{nl}(\chi, R)S_{nl}(\chi', R) = \delta(\chi - \chi'), \tag{4}$$

$$R^3\int_0^\pi S_{nl}(\chi, R)S_{n'l}(\chi, R)\sin^2\chi d\chi = \delta_{nn'}. \tag{5}$$

The functions $S_{nl}(\chi, R)$ depending on two discrete variables l, n, and the continuous variable χ satisfy the three-term recurrence relations

$$\frac{\sqrt{n^2-(l+1)^2}}{2l+1}S_{n,l+1}(\chi, R) + \frac{\sqrt{n^2-l^2}}{2l+1}S_{n,l-1}(\chi, R) = ctg\chi S_{nl}(\chi, R),$$

$$\sqrt{\frac{(n-l)(n+l+1)}{4n(n+1)}}S_{n+1,l}(\chi, R) + \sqrt{\frac{(n+l)(n-l-1)}{4n(n-1)}}S_{n-1,l}(\chi, R) = \cos\chi S_{nl}(\chi, R)$$

and the differential equation

$$\left\{\frac{d^2}{d\chi^2} + 2ctg\chi\frac{d}{d\chi} - \frac{l(l+1)}{\sin^2\chi}\right\}S_{nl}(\chi, R) + (n^2-1)S_{nl}(\chi, R) = 0. \tag{6}$$

In the limit of $R \to \infty$ one can realize the transformation from the space S_R^3 to the flat space M_3, so that this implies the following approximation:

$$\chi \ll 1,\ n \gg 1;\quad n/R = const.\ \ \chi \cdot R = const,$$

and besides the following relations $\chi \cdot R \to r$, and $n/R \to p$, take place where r is the radius vector and p is the momentum of a free particle in the flat space M_3. It is not hard to see also that in the limit $R \to \infty$ the wave functions $S_{nl}(\chi, R)$ convert to the spherical Bessel functions

$$\lim_{\substack{R\to\infty \\ n/R\to p}} \sqrt{R} \cdot S_{nl}(\chi, R) = \sqrt{\frac{2p^2}{\pi}} j_l(pr),$$

i.e. to the radial functions of the free motion in M_3.

3 Equation for quasiradial momentum functions on three-dimensional sphere

For any central potential $V(\chi, R)$ the Schroedinger equation (2) admits the separation of variables in the spherical coordinate system and the corresponding wave function may be presented in the conventional form

$$\Psi^{\alpha}_{Elm}(\chi, \vartheta, \varphi; R) = R_{El}(\chi, R) Y_{lm}(\vartheta, \varphi) \tag{7}$$

where quasiradial function $R_{El}(\chi, R)$ satisfies to equation

$$\left\{\frac{d^2}{d\chi^2} + 2ctg\chi\frac{d}{d\chi} - \frac{l(l+1)}{sin^2\chi}\right\} R_{El}(\chi, R) + 2R^2\left\{E - V^{\alpha}(\chi, R)\right\} R_{El}(\chi, R) = 0. \tag{8}$$

We will suppose that the Sturm-Liouville problem for the equation (8) in a finite interval $[0, \pi]$ has the pure discrete spectrum $E \equiv E_N$, $N = 1, 2, \ldots$ at all values $R \in (0, \infty)$. Then we can expand the solution $S_{El}(\chi, R)$ of eq.(8) into a series over the complete basis of free functions $S_{nl}(\chi, R)$

$$R_{Nl}(\chi, R) = \sum_{n=l+1}^{\infty} F_{Nl}(n, R) S_{nl}(\chi, R), \tag{9}$$

where the coefficients $F_{Nl}(n; R)$ are defined by the relation

$$F_{Nl}(n, R) = R^3 \int_0^{\pi} S_{nl}(\chi, R) R_{Nl}(\chi, R) \sin^2\chi d\chi. \tag{10}$$

We can consider the last relation as a transformation of the function $R_{Nl}(\chi, R)$ with respect to the continuous variable χ into the function $F_{Nl}(n, R)$ of the discrete variable $n = 1, 2, \ldots$ at fixed values of the energy E_N, the orbital quantum number l, and the parameter R. In this case the free functions $S_{nl}(\chi, R)$ play a role of the transformation matrix from the $R_{Nl}(\chi, R)$ to the $F_{Nl}(n, R)$. Like in the flat case we can

call the $F_{Nl}(n, R)$ the quasiradial momentum functions of the discrete variable n and respectively the transformation (9) and (10) may be called as the Fourier-Gegenbauer transformation.

From the normalization condition for the $R_{Nl}(\chi, R)$

$$R^3 \int_0^{\pi} R_{Nl}(\chi, R) R_{N'l}(\chi, R) \sin^2\chi d\chi = \delta_{NN'}$$

the orthogonality relation follows for the quasiradial momentum wave functions

$$R^3 \sum_{n=l+1}^{\infty} F_{Nl}(n, R) F_{N'l}(n, R) = \delta_{NN'}.$$

If we substitute the expansion (9) over the free functions into the equation (8) for the quasiradial function, use the equation (6), multiply from $R^3 \sin^2\chi S_{n'l}(\chi, R)$ the left and then integrate over angle χ in the interval $[0, \pi]$ we arrive at an infinite system of homogeneous algebraic equations for the quasiradial momentum functions $F_{Nl}(n, R)$

$$\sum_{n=l+1}^{\infty} \left\{\left(\frac{n^2-1}{2R^2} - E_N\right)\delta_{n,n'} + W_l(n, n')\right\} F_{Nl}(n, R) = 0, \tag{11}$$

where the matrix elements of the potential energy have the form

$$W_l(n, n') = R^3 \int_0^{\pi} S_{nl}(\chi, R) V^{\alpha}(\chi, R) S_{nl}(\chi, R) \sin^2\chi d\chi. \tag{12}$$

As it follows from the definition (12) at any values of the orbital momentum l the kernel $F_l(n, n')$ is a symmetric matrix with respect to n and n' and depends on the parameter R and on the behavior of the central potential $V^{\alpha}(\chi, R)$. The eigenvalues of the energy E_N of the dynamical system with potential $V^{\alpha}(\chi, R)$ are calculated from the zero determinant of eq.(11). It is obvious that as $R \to \infty$ the homogeneous system of equations (11) transforms into the conventional integral equation for the radial momentum functions in the flat space [14].

From the practical point of view the following potentials are of greate importance: the harmonic potential $V = -(\alpha/R)\text{ctg}\chi$ defined as the solution of the Laplace equation on the three-dimensional sphere [3] and the potentials $V^{\alpha}(\chi, R) = \alpha(R\sin\chi)^{p-2}$, where $p = 1, 2 \ldots$

3.1 Calculation of the kernel $W_l(n, n')$ for the harmonic potential

For the harmonic potential $V = -(\alpha/R)\text{ctg}\chi$ the kernel $W_l(n, n')$ has the following form

$$W_l(n, n') = -\frac{\alpha}{R}\int_0^{\pi} S_{nl}(\chi, R) ctg\chi S_{n'l}(\chi, R) R^3 \sin^2\chi d\chi. \tag{13}$$

As the $ctg\chi$ is an odd function then $W_l(n, n')$ is not zero if quantum numbers n и n' are of different parity.

For calculation of the integral we make use of the known expansion for the Gegenbauer polynomials in the Fourier series:

$$C_n^{\nu}(\cos\varphi) = \frac{\Gamma(n+\nu)}{n!\Gamma(\nu)} \sum_{s=0}^{n} \frac{(-n)_s(\nu)_s}{s!(1-\nu-n)_s} e^{-i(n-2s)\varphi}, \qquad (14)$$

where the $(\alpha)_n = \Gamma(\alpha+n)/\Gamma(\alpha)$ is the Pochhammer symbol. Substitution of the equation (14) into the integral representation (13) for $W_l(n,n')$ results in

$$W_l(n,n') = -\frac{\alpha}{\pi R} \frac{2^{2l+1}\sqrt{nn'}(n-1)!(n'-1)!}{\sqrt{(n-l-1)!(n'-l-1)!(n+l)!(n'+l)!}},$$

$$\sum_{s=0}^{n-l-1} \frac{(-n+l+1)_s(l+1)_s}{(-n+1)_s s!} \sum_{t=0}^{n'-l-1} \frac{(-n'+l+1)_t(l+1)_t}{(-n'+1)_t t!} A_{st}^l(n,n'), \qquad (15)$$

where

$$A_{st}^l(n,n') = \int_0^{\pi} expi\chi\{2(s+t+l+1) - (n+n')\}\cos\chi(\sin\chi)^{2l+1}d\chi,$$

Now using the formula

$$\int_0^{\pi} (sin\chi)^{\alpha} e^{i\beta\chi} d\chi = \frac{\pi}{2^{\alpha}} \frac{e^{i\frac{\pi}{2}\beta}\Gamma(\alpha+1)}{\Gamma(1+\frac{\alpha-\beta}{2})\Gamma(1+\frac{\alpha+\beta}{2})},$$

one can easily see that

$$A_{st}^l(n,n') = (-1)^{l+1}\frac{(2l+1)!}{2^{2l+2}} \left\{ \frac{\Gamma(1+s+t-\frac{n+n'}{2})}{\Gamma(2l+3+s+t-\frac{n+n'}{2})} + \frac{\Gamma(s+t-\frac{n+n'}{2})}{\Gamma(2l+2+s+t-\frac{n+n'}{2})} \right\}.$$

Substituting the formula obtained for the $A_{st}^l(n,n')$ into expansion (15) and collecting one of the sets into the general hypergeometric function

$${}_{p+1}F_p \left\{ \begin{matrix} \alpha_0,\ \alpha_1,\ \ldots,\ \alpha_p \\ \beta_1,\ \beta_2,\ldots,\ \beta_p \end{matrix} \middle| 1 \right\} = \sum_{s=0}^{\infty} \frac{(\alpha_0)_s(\alpha_1)_s\ldots(\alpha_p)_s}{(\beta_1)_s\ldots(\beta_p)_s(s)!}$$

with respect to the unit argument, we immediately get

$$W_l(n,n') = \frac{\alpha}{2\pi R} \frac{(-1)^l\sqrt{nn'}(2l+1)!(n-1)!(n'-1)!}{\sqrt{(n-l-1)!(n'-l-1)!(n+l)!(n'+l)!}} \sum_{s=0}^{n-l-1} \frac{(-n+l+1)_s(l+1)_s}{(-n+1)_s s!}$$

$$\left\{ \frac{\Gamma(s-\frac{n+n'}{2})}{\Gamma(2l+2+s-\frac{n+n'}{2})} {}_3F_2 \left\{ \begin{matrix} -n'+l+1,\ l+1,\ s-\frac{n+n'}{2} \\ -n'+1,\ 2l+s+2-\frac{n+n'}{2} \end{matrix} \middle| 1 \right\} + \right.$$

$$\left. \frac{\Gamma(s+1-\frac{n+n'}{2})}{\Gamma(2l+3+s-\frac{n+n'}{2})} {}_3F_2 \left\{ \begin{matrix} -n'+l+1,\ l+1,\ s+1-\frac{n+n'}{2} \\ -n'+1,\ 2l+s+3-\frac{n+n'}{2} \end{matrix} \middle| 1 \right\} \right\}. \qquad (16)$$

The functions ${}_3F_2(1)$ in the relation (16) are given by finite expansions and may be summarized in according to the Saalschutz theorem [17]

$${}_3F_2 \left\{ \begin{matrix} -n,\ a,\ b \\ c,\ 1+a+b-c-n \end{matrix} \middle| 1 \right\} = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

The last relation admits us to write the formula for $W_l(n,n')$ in terms of the sum of the two generalized hypergeometrical functions ${}_4F_3$ with respect to unit argumetns

$$W_l(n,n') = \frac{\alpha}{2\pi R} \frac{(-1)^l l!(n-1)!\sqrt{nn'(n'+l)!}}{\sqrt{(n-l-1)!(n'-l-1)!(n+l)!}},$$

$$\left\{ \frac{\Gamma(-\frac{n+n'}{2})\Gamma(\frac{n'-n}{2})}{\Gamma(l+1-\frac{n+n'}{2})\Gamma(l+1+\frac{n'-n}{2})} {}_4F_3 \left\{ \begin{matrix} -n+l+1,\ l+1,\ -\frac{n+n'}{2},\ \frac{n'-n}{2} \\ -n+1,\ l+1-\frac{n+n'}{2},\ l+1+\frac{n'-n}{2} \end{matrix} \middle| 1 \right\} + \right.$$

$$\left. + \frac{\Gamma(1-\frac{n+n'}{2})\Gamma(1+\frac{n'-n}{2})}{\Gamma(l+2-\frac{n+n'}{2})\Gamma(l+2+\frac{n'-n}{2})} {}_4F_3 \left\{ \begin{matrix} -n+l+1,\ l+1,\ 1-\frac{n+n'}{2},\ 1+\frac{n'-n}{2} \\ -n+1,\ l+2-\frac{n+n'}{2},\ l+2+\frac{n'-n}{2} \end{matrix} \middle| 1 \right\} \right\}.$$

Taking into account the well-known symmetry relation for the series like to the ${}_4F_3(1)$ of the Saalschutz type [18]

$${}_4F_3 \left\{ \begin{matrix} -n,\ b,\ c,\ d \\ e,\ f,\ g \end{matrix} \middle| 1 \right\} = \frac{(f-b)_n(g-b)_n}{(f)_n(g)_n} {}_4F_3 \left\{ \begin{matrix} -n,\ b,\ e-c,\ e-d \\ e,\ b-f-n+1,\ b-g-n+1 \end{matrix} \middle| 1 \right\}, \qquad (17)$$

$$-n+b+c+d = 1+e+f+g,$$

we can easily verify that

$${}_4F_3 \left\{ \begin{matrix} -n+l+1,\ l+1,\ -\frac{n+n'}{2},\ -\frac{n-n'}{2} \\ -n+1,\ l+1-\frac{n+n'}{2},\ l+1-\frac{n-n'}{2} \end{matrix} \middle| 1 \right\} = \frac{l!(n+l)!\Gamma(l+1+\frac{n'-n}{2})}{(2l+1)!(n-1)!\Gamma(\frac{n'-n}{2})}.$$

$$\frac{\Gamma(\frac{n+n'}{2}-l-1)}{\Gamma(\frac{n+n'}{2})} {}_4F_3 \left\{ \begin{matrix} -n+l+1,\ -n'+l+1,\ l+1,\ l+1 \\ 2l+2,\ l+2-\frac{n+n'}{2},\ l+1-\frac{n+n'}{2} \end{matrix} \middle| 1 \right\}$$

$${}_4F_3 \left\{ \begin{matrix} -n+l+1,\ l+1,\ 1-\frac{n+n'}{2},\ 1-\frac{n-n'}{2} \\ -n+1,\ l+2-\frac{n+n'}{2},\ l+2-\frac{n-n'}{2} \end{matrix} \middle| 1 \right\} = \frac{l!(n+l)!\Gamma(l+2+\frac{n'-n}{2})}{(2l+1)!(n-1)!\Gamma(1+\frac{n'-n}{2})}.$$

$$\frac{\Gamma(\frac{n+n'}{2}-l)}{\Gamma(1+\frac{n+n'}{2})} {}_4F_3 \left\{ \begin{matrix} -n+l+1,\ -n'+l+1,\ l+1,\ l+1 \\ 2l+2,\ l+2-\frac{n+n'}{2},\ l+1-\frac{n+n'}{2} \end{matrix} \middle| 1 \right\}.$$

In this way after the obvious transformations

$$\frac{\Gamma(-\frac{n+n'}{2})\Gamma(\frac{n+n'}{2}-l-1)}{\Gamma(l+1-\frac{n+n'}{2})\Gamma(\frac{n+n'}{2})} = \frac{\Gamma(1+\frac{n+n'}{2})\Gamma(\frac{n+n'}{2}-l)}{\Gamma(l+2-\frac{n+n'}{2})\Gamma(1-\frac{n+n'}{2})} =$$

$$= (-1)^{l+1}\frac{\Gamma(\frac{n+n'}{2}-l)\Gamma(\frac{n+n'}{2}-l-1)}{\Gamma(\frac{n+n'}{2})\Gamma(\frac{n+n'}{2}+1)}$$

we finally obtain the following formula for the kernel $W_l(n,n')$:

$$W_l(n,n') = -\frac{\alpha}{\pi R}\frac{\sqrt{nn'}(l!)^2}{(2l+1)!}\frac{\Gamma(\frac{n+n'}{2}-l-1)\Gamma(\frac{n+n'}{2}-l)}{\Gamma(\frac{n+n'}{2}+1)\Gamma(\frac{n+n'}{2})}\sqrt{\frac{(n+l)!(n'+l)!}{(n-l-1)!(n'-l-1)!}}$$

$$_4F_3\left\{\begin{matrix}-n+l+1, \ -n'+l+1, \ l+1, \ l+1\\ 2l+2, \ l+2-\frac{n+n'}{2}, \ l+1-\frac{n+n'}{2}\end{matrix}\middle| 1\right\}. \qquad (18)$$

Note, this form of the $W_l(n,n')$ is explicitly symmetric in n и n', which ensures the spectrum of the corresponding algebraic eigenvalue problem for the equation (11) being real.

Let us investigate now the transformation of this kernel in the limit of $R\to\infty$. As $R\to\infty$ and $n/R\to p$, $n'/R\to p'$ we have the following correspondence rule $n/R\to p$, $n'/R\to p'$

$$\frac{\Gamma(\frac{n+n'}{2}-l-1)\Gamma(\frac{n+n'}{2}-l)}{\Gamma(\frac{n+n'}{2}+1)\Gamma(\frac{n+n'}{2})}\sqrt{\frac{nn'(n+l)!(n'+l)!}{(n-l-1)!(n'-l-1)!}} \Rightarrow \left\{\frac{(4pp')}{(p+p')^2}\right\}^{l+1}$$

$$_4F_3\left\{\begin{matrix}-n+1, \ -n'+1, \ 1, \ 1\\ 2, \ 2-\frac{n+n'}{2}, \ 1-\frac{n+n'}{2}\end{matrix}\middle| 1\right\} \Rightarrow {}_2F_1\left\{\begin{matrix}l+1, \ l+1\\ 2l+2\end{matrix}\middle| \frac{4pp'}{(p+p')^2}\right\}$$

and consequently

$$\lim_{R\to\infty} R\cdot W_l(n,n') = -\frac{\alpha}{\pi}\frac{(l!)^2}{(2l+1)!}\left\{\frac{(4pp')}{(p+p')^2}\right\}^{l+1} {}_2F_1\left\{\begin{matrix}l+1, \ l+1\\ 2l+2\end{matrix}\middle| \frac{4pp'}{(p+p')^2}\right\}.$$

Comparing with the representation of the Legendre function of the second kind Q_μ through the hypergeometrical sets [17]

$$Q_\mu = \frac{2^\mu}{(z+1)^{\mu+1}}\frac{[\Gamma(1+\mu)]^2}{\Gamma(2+2\mu)}{}_2F_1\left\{\begin{matrix}1+\mu, \ 1+\mu\\ 2+2\mu\end{matrix}\middle| \frac{2}{1+z}\right\},$$

in the case of $z=(p^2+p'^2)/2pp'$ we obtain that

$$\lim_{R\to\infty} R\cdot W_l(n,n') = -\frac{2\alpha}{\pi}Q_l\left(\frac{p^2+p'^2}{2pp'}\right)$$

and thus we get the well- known Schroedinger integral equation of the hydrogen atom in the flat momentum space [14]

$$(p^2-2E)F(p) = \frac{2\alpha}{\pi p}\int_0^\infty Q_l\left(\frac{p^2+p'^2}{2pp'}\right)F(p')p'dp'.$$

In the case of $l=0$ the form of the kernel essentially simplifies. Indeed, we have

$$W_0(n,n') = -\frac{\alpha}{\pi R}\frac{4nn'}{(n+n')(n+n'-2)}{}_4F_3\left\{\begin{matrix}-n+1, \ -n'+1, \ 1, \ 1\\ 2, \ 2-\frac{n+n'}{2}, \ 1-\frac{n+n'}{2}\end{matrix}\middle| 1\right\}.$$

Now using the formula [19]

$$_4F_3\left\{\begin{matrix}-N, \ a, \ 1, \ 1\\ 2, \ b, \ 1+a-b-N\end{matrix}\middle| 1\right\} =$$

$$= \frac{(b-1)(a-b-N)}{(N+1)(a-1)}\left\{\Psi(N+b)+\Psi(1+a-b)-\Psi(b-1)-\Psi(a-b-N)\right\},$$

where $\Psi(n)$ is the logarithmic derivative of the gamma function, one can easily see that

$$W_0(n,n') = -\frac{\alpha}{\pi R}\left\{\Psi(\frac{n+n'}{2})+\Psi(-\frac{n+n'}{2})-\Psi(\frac{n-n'}{2})-\Psi(\frac{n'-n}{2})\right\}.$$

At large numbers n and n' the kernel has the asymptotic form

$$W_l(n,n') \sim -\frac{\alpha}{R}\frac{2l!}{(2l+1)!!}\left(\frac{2}{n'}\right)^{l+1}\sqrt{\frac{n(n+l)!}{(n-l-1)!}}$$

The last relation admits us to examine the Kantorovich reduction theorem and obtain explicit estimations of the convergence rate of the solution with respect to large numbers n, n'.

3.2 The kernel $W_l(n,n')$ for the potentials $V^\alpha(\chi,R)=\alpha(R\sin\chi)^{p-2}$

Let us write the integral representation for the kernel $W_l(n,n')$ with the potentials $V^\alpha(\chi,R)=\alpha(R\sin\chi)^{p-2}$

$$W_l^{p-2}(n,n') = \frac{\alpha 2^{2l+1}(l!)^2}{\pi(R)^{2-p}}\sqrt{\frac{n\cdot n'\Gamma(n-l)\Gamma(n'-l)}{\Gamma(n+l+1)\Gamma(n'+l+1)}}$$

$$\int_{-1}^{1}(1-x^2)^{l+p-1/2}C^{l+1}_{n-l-1}(x)C^{l+1}_{n'-l-1}(x)dx. \qquad (19)$$

As the integrand contains an odd function, $W_l^{p-2}(n,n')$, is not zero if the quantum numbers n и n' have the same parity. The integral (19) has been calculated in ref. [20] and in our notation it reads as follows:

If $(n-l-1)$, $(n'-l-1)$ are even, then

$$W_l^{p-2}(n,n') = (-1)^{\frac{n'-l-1}{2}}\frac{\alpha\sqrt{nn'}(R)^{p-2}\Gamma(l+\frac{p+1}{2})\Gamma(\frac{p}{2})\Gamma(\frac{n'+l+1}{2})}{\Gamma(l+3/2)\Gamma(\frac{n'-l+1}{2})\Gamma(\frac{p}{2}+\frac{n'+l+1}{2})\Gamma(\frac{p}{2}-\frac{n'-l-1}{2})}$$

$$\sqrt{\frac{\Gamma(n+l+1)\Gamma(n'-l)}{\Gamma(n'+l+1)\Gamma(n-l)}}{}_4F_3\left\{\begin{matrix}-\frac{n-l-1}{2}, \ \frac{n+l+1}{2}, \ l+\frac{p+1}{2}, \ \frac{p}{2}\\ l+\frac{3}{2}, \ \frac{p}{2}+\frac{n'+l+1}{2}, \ \frac{p}{2}-\frac{n'-l-1}{2}\end{matrix}\middle| 1\right\}. \qquad (20)$$

If $(n-l-1)$, $(n'-l-1)$ are odd, then

$$W_l^{p-2}(n,n') = (-1)^{\frac{n'-l}{2}-1}\frac{\alpha\sqrt{nn'}(R)^{p-2}\Gamma(l+\frac{p+1}{2})\Gamma(\frac{p}{2})\Gamma(\frac{n'+l}{2}+1)}{\Gamma(l+3/2)\Gamma(\frac{n'-l}{2})\Gamma(1+\frac{p}{2}+\frac{n'+l}{2})\Gamma(1+\frac{p}{2}-\frac{n'-l}{2})}.$$

$$\sqrt{\frac{\Gamma(n+l+1)\Gamma(n'-l)}{\Gamma(n'+l+1)\Gamma(n-l)}}\,{}_4F_3\left\{\begin{matrix} -\frac{n-l}{2}+1,\ \frac{n+l}{2}+1,\ l+\frac{p+1}{2},\ \frac{p}{2} \\ l+\frac{3}{2},\ 1+\frac{p}{2}+\frac{n'+l}{2},\ 1+\frac{p}{2}-\frac{n'-l}{2} \end{matrix}\middle|\, 1\right\}. \tag{21}$$

The formulae (20) and (21) may be combined into one formula if we make use of the transformation (17) for ${}_4F_3(1)$. As a result the kernel takes in the following form

$$W_l^{p-2}(n,n') = (-1)^{\frac{n-n'}{2}} \frac{\alpha\sqrt{nn'}(R)^{p-2}\Gamma(l+\frac{p+1}{2})\Gamma(\frac{p}{2})\Gamma(\frac{n'+n}{2})}{\Gamma(l+3/2)\Gamma(1+\frac{n'-n}{2})\Gamma(\frac{p}{2}+\frac{n-n'}{2})\Gamma(\frac{p}{2}+\frac{n+n'}{2})}$$

$$\sqrt{\frac{\Gamma(n+l+1)\Gamma(n'-l)}{\Gamma(n'+l+1)\Gamma(n-l)}}\,{}_4F_3\left\{\begin{matrix} -\frac{n-l-1}{2},\ -\frac{n-l-2}{2},\ 1-\frac{p}{2},\ \frac{p}{2} \\ l+\frac{3}{2},\ 1-\frac{n+n'}{2},\ 1+\frac{n'-n}{2} \end{matrix}\middle|\, 1\right\}. \tag{22}$$

The last formula is not symmetric in the quantum numbers n and n', however, it does not depend on the parity $(n-l-1)$ (or $(n'-l-1)$).

In the case of $p = 2M+2, M = 0,1,2,\dots$ only the following matrix elements

$$W_l^M(n,n),\ \ W_l^M(n,n\pm 2),\ \ \dots,\ \ W_l^M(n,n\pm 2M).$$

do not vanish. As a result the equation (11) is replaced by the $(2M+1)$-term recurrence relation for quasiradial momemtum functions $F_{Nl}(n,R)$:

$$\sum_{k=0}^{M} \delta_{n',n\pm 2k} W_l^M(n,n') F_{Nl}(n',R) - \Big(E_N - W_l^M(n,n) - \frac{n^2-1}{2R^2}\Big) F_{Nl}(n,R) = 0,$$

$$F_{Nl}(0,R) = F_{Nl}(1,R) = \dots = F_{Nl}(l,R) = 0.$$

As an example, we report certain simplest formulae of the nonvanishing coefficients $W_l^M(n,n')$ at $M = 1$:

$$W_l^2(n,n) = \frac{(\alpha R)^2}{2}\frac{\{(n^2-1)+l(l+1)\}}{(n^2-1)},$$

$$W_l^2(n,n+2) = -\frac{(\alpha R)^2}{4}\sqrt{\frac{(n-l)(n-l+1)(n+l+1)(n+l+2)}{n(n+1)^2(n+2)}},$$

$$W_l^2(n,n-2) = -\frac{(\alpha R)^2}{4}\sqrt{\frac{(n-l-2)(n-l-1)(n+l-1)(n+l)}{(n-2)(n-1)^2 n}}.$$

In the case of $p = 2M+1$, $M = 0,1,2,\dots$, the algebraic equation (11) is not reduced to the system of simple recurrence relations for $F_{NL}(n,R)$.

When $M = 0$ the kernel $W_l^p(n,n')$ of equation (11) can be expressed through $6j$ - symbols or Racah coefficients for quarter-multiple moments of the group $SU(1,1)$. Comparing the expression for $W_l^p(n,n')$ with the representation for $6j$ - symbols in terms of the hypergeometric functions ${}_4F_3(1)$ of unit argument [21]:

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = (-1)^{a+b+d+e}\cdot$$

$$\frac{\Delta(abc)\Delta(cde)\Delta(aef)\Delta(bdf)}{\Gamma(2f+2)\Gamma(a+b-c+1)\Gamma(a+c-b+1)\Gamma(d+e-c+1)}\cdot$$

$$\frac{\Gamma(a+f+e+2)\Gamma(b+f+d+2)\Gamma(a+c+d-f+1)}{\Gamma(c+d-c+1)\Gamma(a+e-f+1)\Gamma(b+d-f+1)\Gamma(c-a-d+f+1)}\cdot$$

$${}_4F_3\left\{\begin{matrix} -a-c+f,\ -b-d+f,\ -a+e+f+1,\ b-d+f+1 \\ -a-c-d+f,\ -a+c-d+f+1,\ 2f+2 \end{matrix}\middle|\, 1\right\}$$

where $\Delta(abc)$

$$\Delta(abc) = \sqrt{\frac{\Gamma(a+b-c+1)\Gamma(a-b+c+1)\Gamma(b+c-a+1)}{\Gamma(a+b+c+2)}}$$

we obtain the required formula:

$$W_l^{-1}(n,n') = (-1)^{\frac{n'+l}{2}-3/2}\sqrt{nn'}\begin{pmatrix} \frac{n-1}{2} & -\frac{1}{2} & \frac{n'-1}{2} \\ \frac{l-1/2}{2} & -\frac{1}{4} & \frac{l-1/2}{2} \end{pmatrix}.$$

4 The calculation procedure for eigenfunctions in the momentum representation

To solve the eigenvalue problem for the homogeneous system of equations (11) obtained in the previous section in a wide range of the quantum numbers n, l and the parameter R the application of special calculation methods is needed. In practice one usually make use of some recurrence relations, admitting to find out normal and nonsingular solution of the evolution equation for $F(R)$:

$$\frac{\partial F(R)}{\partial R} = AF(R), \tag{23}$$

$$F(R_0) = 1, \tag{24}$$

with the matrix of coefficients A formally defined by the equation

$$A(R) = \Big(\frac{\partial F(R)}{\partial R}\Big)F(R)^{-1}. \tag{25}$$

The matrix A corresponds to the connection operator in the Hilbert fibre bundle $\mathcal{H}(\mathcal{F}_R,\pi,B)$ with a typical layer $\mathcal{F}_R \simeq L^2(M_4, dM_4(R))$ and projection π: $\mathcal{H} \to B$ on the base $B = R_+^1$ or C, associated with the solutions of the spectral problem (8), (11) which acquire the meaning of the local cross sections in the $\mathcal{H}$. Then the standard gauge transformation

$$F(R) = \Lambda(R)U(R) \tag{26}$$

induces a new connection operator

$$A'(R) = \Lambda(R)^{-1} A \Lambda(R) - \Lambda(R)^{-1} \frac{\partial \Lambda(R)}{\partial R}, \tag{27}$$

and for the matrix of the coefficients $U(R)$ which plays the role of the transport operator of the frame of reference $\hat{e} \in \mathcal{F}_{\hat{R}}$ over the base B, U: $\mathcal{F}_{\hat{R}} \to \mathcal{F}_R$, or an element of the holonomy group, G, we have an equivalent evolution equation:

$$\frac{\partial U(R)}{\partial R} = AU(R), \tag{28}$$

$$U(0) = 1. \tag{29}$$

Its solutions are represented in the form of P-exponential

$$U(R) = Pexp \int_0^R A'(R')dR'. \tag{30}$$

Note that an analogous construction for the matrices of finite rank realized in practice is well known from investigations of the Lypunov stability of systems of linear differential equations with variable coefficients [22].

So, the standard way of obtaining the recurrence relations for the coefficients F consists in averaging of the equation (8) over some local frame of reference $\hat{e} \in \mathcal{F}_{\hat{R}}$, $\hat{R} \in \hat{U} \in B$, for example, in the vicinity $\hat{U} = (0 < R < \epsilon)$ at $\epsilon > O$. Really, this reduces to averaging with the basis of the free quasiradial solutions (3) and obtaining a finite system with the matrices of finite rank $n_{max} - l$ instead of the infinite system of equations (11) :

$$\sum_{n'=l+1}^{n_{max}} \left\{ [E_{n'}^{\alpha=0}(R) - E_N^{\alpha}(R)]\delta_{n,n'} + W_l(n,n';R) \right\} F_{Nl}(n,R) = 0. \tag{31}$$

Another possibility consists in obtaining the approximate recurrence relations at $l+1 \leq n \leq n_{max}$ for the gauge-equivalent transformation (26). It may be constructed by means of a suitable orthogonal transformation:

$$\sum_{k=1}^{n_{max}-l} T_{nk} T_{n'k} = \delta_{nn'},$$
$$\sum_{n=l+1}^{n_{max}} T_{nk} T_{nk'} = \delta_{kk'} \tag{32}$$

of the initial frame of reference (3) in the expansion (9) of the qusiradial solutions

$$R_{Nl}(\chi, R) = \sum_{k=1}^{n_{max}-l} P_{kl}(\chi, R) U_{kl}^N(R) \tag{33}$$

where

$$P_{kl}(\chi, R) = \sum_{n=l+1}^{n_{max}} S_{nl}(\chi, R) T_{nk}, \quad U_{kl}^N(R) = \sum_{n=l+1}^{n_{max}} T_{nk} F_{nl}^N. \tag{34}$$

The corresponding recurrence relations for the new coefficients (34) take the following form

$$\sum_{k'=1}^{n_{max}-l} \left\{ E_{kk'}^{\alpha=0}(R) + V_{kk'}^{\alpha}(R) - E_N^{\alpha}(R)\delta_{kk'} \right\} U_{k'l}^N(R) = 0. \tag{35}$$

Here the kinetic and potential energies are defined as

$$E_{kk'}^{\alpha=0}(R) = \sum_{n=l+1}^{n_{max}} T_{nk} E_n^{\alpha=0}(R) T_{nk'}, \tag{36}$$

$$V_{kk'}^{\alpha}(R) = \sum_{n=l+1}^{n_{max}} \sum_{n'=l+1}^{n_{max}} T_{nk} W_l(n,n';R) T_{n'k}, \tag{37}$$

and the matrix elements $W_l(n,n';R)$ are defined by eq.(12).

As coefficients T_{nk} one can make use of the complete and orthogonal with the weight w_k generalized Chebyshev system of polynomials of the discrete variable $x_k = \cos \chi_k$

$$T_{nk} = w_k^{1/2} S_{nl}(x_k) \tag{38}$$

on the grid $\omega_k = \{x_k \in (-1,1), k = 1, n_{max} - l\}$, composed of the nodes of the polynomials $S_{n_{max}l}(x_k) = 0$ of the continuous variable $x_k \in (-1,1)$. Then the integral (12) may be calculated explicitly via the Gauss-Jacobi quadrature:

$$W_l(n,n';R) = \sum_{k=1}^{n_{max}-l} w_k S_{nl}(x_k) V^{\alpha}(x_k;R) S_{n'l}(x_k).$$

Thus, in the new representation (33) the kinetic energy operator on the sphere is given by the nondiogonal matrix (36) of the rank $n_{max} - l$, whereas the operator of the potential energy (37), because of the completeness and orthogonality relation (32) at the meshpoints of the grid ω_k, is defined by the diagonal matrix of the same rank $n_{max} - l$

$$V_{kk'}^{\alpha}(R) = \sum_{k''=1}^{n_{max}-l} \sum_{n=l+1}^{n_{max}} T_{nk} T_{nk''} V_{kk'}^{\alpha}(x_{k''};R) \sum_{n'=l+1}^{n_{max}} T_{n'k''} T_{n'k'} = V^{\alpha}(x_k;R)\delta_{kk'}. \tag{39}$$

These elements are defined by the original values of the potential energy $V^{\alpha}(x_k;R)$ on the discrete grid of meshpoints corresponding to the nodes $x_k \in \omega_k$, which are as usual well tabulated.

As a result the system of recurrence relations (35) takes the form

$$\sum_{k'=l}^{n_{max}-l} \left\{ \sum_{n=l+1}^{n_{max}} T_{nk} \frac{n^2-1}{2R^2} T_{nk'} + \left(V^{\alpha}(x_k;R) - E_N^{\alpha}(R) \right) \delta_{kk'} \right\} U_{k'l}^N(R) = 0. \tag{40}$$

This system is useful for the investigations of the asymptotics of the solutions(11) with respect to the number $n_{max} \to \infty$ and the different limiting relations, because the nodes x_k in this case are defined by means of the corresponding nodes of the Bessel functions [17]. Such investigations are required for constructing the asymptotics of matrix elements of the connection operator $A(R)$ necessary for understanding peculiarities of the structure of the Hilbert fibre bundle $\mathcal{H}$ defining the geometry and dynamics of more complicated three-body systems.

It is to be noted that for low-lying states one should immediately solve the system of equations (31) with nuclei calculated in section 3. For highly excited states it is preferable to use the system of equations (40).

5 Conclusion

We should like to note that for the relativistic generalisation of the Schrödinger equation on the sphere S^3_R the obtained form of the equation is more preferable since single-particle energies are solutions of the Schwinger-Dyson equations which, as a rule, depend in a complicated way on momentum. As has been shown in ref.[12], the nonrelativistic Schwinger equation on the three-dimensional sphere for the harmonic potential well describes the spectrum of heavy quarkonia. In this connection, it is interesting to derive the system of Schwinger-Dyson and Bethe-Salpeter equations in the three-dimensional space of constant positive curvature for describing light quarkonia as Goldstone modes of these equations [23]. The solution of eq.(11) for the central potentials and the relativistic generalisation of this equation are the goal of our further investigations.

We would like to thank A.A.Izmest'ev, Yu.L.Kalinovsky, V.Kallis, I.V.Puzynin and V.M.Ter-Antonyan for fruitful disscussions.

Литература

[1] A.P.Kobushkin, N.Ogava, K.Fudzii, N.M.Chepilko. Yad.Fiz. 1990, v.52, p.772; N.M.Chepilko. Yad.Fiz. 1992, v.55, p. 273.

[2] C.Grosche. J.Math.Phys. 1991, 32, 1984.

[3] E.Shrödinger. Proc.Irish.Acad. 1940, v.A46, 9; 1941, v.A46, 183; v.A47, 53.

[4] L.Infeld. Phys.Rev., 1941, v.59, 737.

[5] A.F.Stivenson. Phys.Rev. 1941, v.59, 842.

[6] A.A.Bogush, V.S.Otchik, V.M.Red'kov. Vestnik AN BSSR, 1983, v.3, 56; V.S.Otchik, V.M.Red'kov. Preprint Nu. 298, IN AN BSSR, 1983.

[7] A.O.Barut and R.Wilson. Phys.Lett.A, 1985, v.110, 351.
A.O.Barut, A.Inomata and G.Junker. J.Phys. A20, 1987, 6271; 1990, A23, 1179.

[8] L.G.Mardoyan, A.N.Sissakian. Yad.Fiz. 1992, v.55

[9] P.W.Higgs. J.Phys A12, 1979, 309.

[10] H.I.Leemon. J.Phys A12, 1979, 489.

[11] S.I.Vinitsky, B.L.Markovski, A.A.Suzko. Yad.Fiz. 1992, v.55, 669.
S.I.Vinitsky, M.B.Kadomtsev, A.A.Suzko. Yad.Fiz. 1990, v.51, 952.

[12] A.A.Izmest'ev. Yad.Fiz. 1990, v.52, 1697; 1991, v.53, 1402.

[13] V.N.Pervuchin. Theor.Math.Fiz. 1980, v.45, 394; Yad.Fiz. 1982, v.36, 262.

[14] H.Bethe and E.Salpeter."Quantum Mechanics of one- and two- electron atoms". Springer-Verlag, 1957.

[15] S.I.Vinitsky, L.G.Mardoyan, G.S.Pogosyan, A.N.Sissakian T.A.Strizh. Yad.Fiz.1993, v.56. 69.

[16] L.D.Landau and E.M.Lifshiz. "Quntum Mechanics" Nauka,M., 1963.(in Russian)

[17] H.Batemannand A.Erdelye."Higher transcendental functions." New York, Mc Graw-Hill,1953.

[18] W.N.Bailey. Generalized hypergeometric series, Cambridge Tracts, No 32, Cambridge, 1935.

[19] A.P.Prudnikov, Yu.A.Bruchkov, O.I.Marichev. "Integral and expansion. Additional topics". Nauka, M.,1986 (in Russian)

[20] M.S.Kul'dushov. Yad.Fiz. 1972, v.15, 197.

[21] J.A.Wilson. SIAM J.Math.Anal. 1980, v.11, 690.

[22] F.R. Gantmacher."Theory of Matricies". Nauka, M., 1988 (in Russian)

[23] V.N.Pervushin et al., Fortschr.Phys.1989,38,323.
Yu.L.Kalinovsky et al., Sov.J.Nucl.Phys.1989,49,1059.
Yu.L.Kalinovsky, L.Kaschluhn and V.N.Pervushin. Fortschr.Phys. 1989, 38, 353; Phys.Lett. B231, 1989, 288.

Received by Publishing Department
on December 30, 1992