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THEORY OF FUNDAMENTAL INTERACTIONS

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Теория фундаментальных взаимодействий

Дан последовательный вывод теории фундаментальных взаимодействий из первых принципов. В развитой теории нет разделения между пространством-временем и внутренним калибровочным пространством. Выведены основные уравнения для базисных полей. Показано, что теория удовлетворяет принципу соответствия и приводит к новым представлениям в рассматриваемой области. В частности, сделан вывод о существовании частиц, которые характеризуются не только массой, спином, зарядом, но также моментом инерции. Это вращающиеся частицы, которые на микроскопическом уровне соответствуют понятию твердого тела и дают ключ к пониманию природы сильных взаимодействий. Сформулированы основные понятия и законы, относящиеся к этим частицам. Теория может быть проверена на опыте в ближайшее время.

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Theory of Fundamental Interactions

In the present article the theory of fundamental interactions is derived in a systematic way from the first principles. In the developed theory there is no separation between space-time and internal gauge space. Main equations for basic fields are derived. In is shown that the theory satisfies the correspondence principle and gives rise to new notions in the considered region. In particular, the conclusion is made about the existence of particles which are characterized not only by the mass, spin, charge but also by the moment of inertia. These are rotating particles, the particles which represent the notion of the rigid body on the microscopical level and give the key for understanding strong interactions. The main concepts and dynamical laws for these particles are formulated. The basic principles of the theory may be examined experimentally not in the distant future.

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1 Introduction

Despite great successes achieved in the physics of elementary particles, it is known that the search for a complete and consistent theory of the fundamental interactions is yet far from being complete. It is expected that the final theory will be a result of the theoretical analysis of numerous experiments. Of course, such a permanent interplay of the theory and experiment is the most reliable way of development of the physics as a whole. However, in the case under consideration we see that this approach is followed be great difficulties and may have a very long history. So, only the deductive method alone can give a highly reasonable and simple solution of the problem in question. Now it is interesting to recall the Einstein's opinion on the discussed question (Einstein, 1933). He wrote: "It is my conviction that pure mathematical construction enables us to discover the concepts and the laws connecting them which give us the key to the understanding of the phenomena of Nature. Experience can of course, guide us in our choice of serviceable mathematical concepts; it cannot possibly be the source from which they are derived; experience, of course, remains the sole criterion of the serviceability of a mathematical construction for physics, but the truly creative principle resides in mathematics."

In this paper we develop the theory of fundamental interaction from the very general and most well-established first principles. The starting point of the present research is the well known link between the fundamental symmetry of space-time and the laws of conservation of the energy, momentum and angular momentum. It can be used to formulate the guiding physical principle: Important properties of the space-time are tightly related with the most fundamental laws governing the behavior of matter. In this sense the essential space-time property predetermine the objective physical regularities. Thus,the problem is to express all details of this intimate relationship in the exact mathematical language and thereby to develop a new unified approach to the problem of fundamental interactions.

The article is organized as follows. In section 2 we formulate the general assumption which allows us to establish the structure of the fundamental symmetry groups. Then the proper exposition of the gauge principle is given. The type of the basic fields is indicated and their detailed description is presented from the point of view of symmetry principles. Section 3,4,5 content the derivation of the simplest gauge invariant equations which describe different form of interactions of the basic fields. In accordance with the correspondence principle in section 6 we examine the link with the Dirac theory of the electron.

In sections 4 and 6 one rather general consequence is considered which has to do with the gravitational interactions. It is suggested here to measure the force of gravity on electrons and positrons. The theory predicts that the measurements of the gravitational acceleration of electrons and positrons will have a zero result. As it is known, Fairbank and Witteborn (1967) have measured the gravitational acceleration of electrons in the earth's field. So it is very important to make this for positrons. It must be noted that not so long ago Fairbank and Witteborn (1988) reported that now there are no principal technical difficulties for the propounding of the experiment with positrons.

In section 7 the notion of hyperbolic space-time is introduced that is a key for understanding the nature of the so-called strong interactions. Here and in the preceding section one can find some results on a fundamental role of the Yang-Mills field (1954).

2 Essential Structures

2.1 Two First Principles

1. Space-time is an infinite-differentiable finite-dimensional simply connected and orientable manifold M.

   Comments : It is now well understood that the theory of fundamental interactions represents a highly complex and intricate problem. As is well known, it is just abstraction that allows us to introduce the simplicity into the study of the most knotty questions. Therefore we apply to abstract space-time. All possible properties of space-time that do not enter into its definition are not fixed and represent a kind of parameters of the theory which, in particular, include the dimension of space-time. It is expected that constraints on these "parameters" will naturally appear in the course of development of the theory. The definition of a differentiable manifold is supposed to be known (see Kobayashi and Nomizu, 1963; Schutz, 1982).

2. Essential properties of space-time are closely related to fundamental laws of matter and thus predetermine them.

   In what follows, the latter principle will be of crucial importance as a means for narrowing an infinite set of possible mathematical constructions.

2.2 Symmetry Groups

According to the first principles, the symmetry groups that are intimately connected with the concept of a differentiable manifold represent the important property of space-time. One group of this kind is a group of transformations of the differentiable manifold M itself, and another is a group of transformations acting in tangent vector spaces. Before describing these symmetry groups in detail, we make some comments: the definition of a tangent vector or simply a vector is assumed to be known; a set of tangent vectors at point p from M denoted by $T_p(M)$ is called the tangent space of the manifold M at point p. This concept is extremely clearly expounded in the treatise by Misner, Thorne, Wheeler, (1973). The vector field X on a manifold M is the correspondence of a vector to every point p on M. The set of all vector fields on M is a real vector space $L(M)$. A vector field can be expressed in terms of a local coordinate system $x^0, \dots, x^{n-1}$ as follows: $X = V^i\partial_i$, where functions V^i are defined in a coordinate neighbourhood and are called the components of X with respect to $x^0, \dots, x^{n-1}$.

A diffeomorphism of a manifold M onto itself is a homeomorphism φ such that φ and φ^{-1} are differentiable (Kobayashi and Nomizu, 1963). A diffeomorphism is a transformation on M. Transformations on M form a group denoted $Diff(M)$. The group of diffeomorphisms is often called the group of general transformations of coordinates. Under the transformation φ , a curve γ transforms into a curve $\tilde{\gamma}$ called equivalent to γ. The transformation φ on M induces an automorphism $\tilde{\varphi}$ of the algebra of tensor fields that preserves the type of tensor fields and is transposable with tensor contractions. Let $\tilde{T} = \tilde{\varphi}T$ for any tensor field T; the tensor field $\tilde{T}$ is called equivalent to T with respect to the group $Diff(M)$.

The other group of symmetry can be characterized as follows: Tensor fields of the type (1,1) on M are called affinor fields. Let S be a nondegenerate affinor field on $M, \det(S^i_j) \neq 0$ and $X \in L(M)$. In a coordinate patch U with local coordinates $x^0, \dots, x^{n-1}$ a nondegenerate linear transformation $\bar{X} \rightarrow X = SX$ has the form $\bar{V}^i(x) = S^i_j(x)V^j(x)$, where $\bar{V}^i(x)$,$V^i(x)$, $S^i_j(x)$ are components of $\bar{X}$, X, S, respectively, with respect to $x^0, \dots, x^{n-1}$. A set of nondegenerate affinor fields is a group with an associative binary operation P=ST , where

$$P^i_j(x) = S^i_k(x)T^k_j(x). \tag{1}$$

The relation (1) at every point p from M can be considered as the composition law for parameters of the general linear group $\bar{x}^i = a^i_j x^j$. Consequently, to the coordinates a^i_j on the group $GL(n, R)$ one can put nondegenerate affinor fields on the manifold M into correspondence. So, the second group of symmetry underlying the very notion of differentiable manifold will be called the group $GL(n, R)$ of tangent bundle or simply the gauge group.

2.3 Gauge Principle

Now we are ready to formulate the gauge principle as the very important property of space-time. The tangent space $T_p(M)$ is identified here with the so-called gauge or internal space. The group $GL(n, R)$ of tangent bundle is the gauge-symmetry group. To complete the gauge principle it is to be added with an important concept of a polarized particle.

Any pair (p, X) , where p is a point of space-time M and X is a vector tangent to the manifold M at the point p , will be called the polarized particle. Polarization of a particle is associated with direction of the related vector X.

2.4 Physical Meaning of the Diffeomorphism Group

Here we will expound some known results of General Relativity in an appropriate form.

Let g be a symmetric tensor field of the type (0,2). It is the only field that can obey equations derived from the variational principle and invariant under transformations of the diffeomorphism group, $\tilde{g} = \tilde{\varphi} g$. For simplicity we assume that the transformation φ in M maps the coordinate patch U onto itself. If $x^0, \ldots; x^{n-1}$ is a local coordinate system in U , then the transformation φ can be represented by smooth functions in U

$$\varphi : x^i \Rightarrow \varphi^i(x), \varphi^{-1} : x^i \Rightarrow f^i(x); \varphi^i(f(x)) = x^i, f^i(\varphi(x)) = x^i.$$

The transformation $\tilde{g} = \tilde{\varphi} g$ in terms of the local coordinate system is of the form

$$\tilde{g}_{ij}(x) = g_{kl}(f(x)) f_i^k(x) f_j^l(x), \qquad (2)$$

where $f_i^k(x) = \partial_i f^k(x)$. Equations of motion of a particle defined by the field g, are the Euler-Lagrange equations for extremals of the functional $S = \int_p^q \sqrt{g_{ij}(x)\dot{x}^i\dot{x}^j} dt$. If γ is an extremal of the functional S, a curve $\tilde{\gamma} = \varphi\gamma$ equivalent to it will be an extremal of the functional $\tilde{S} = \int_p^q \sqrt{\tilde{g}_{ij}(x)\dot{x}^i\dot{x}^j} dt$. Thus, it can be said that the tensor field g is unseparable from the diffeomorphism group that is a group of symmetry of gravitational interactions, in accordance with the Einstein theory of gravity. Systematic and deep thorough

account of the questions raised in this section may be found in ref.(Anderson, 1967).

2.5 Equations of motion of polarized particles

To establish the type of fields defined by the gauge principle, we derive equations of motion for polarized particles. Transformations of the gauge group act on tangent vectors and do not act on coordinates. Therefore, it is important to determine the laws of change of polarization. Our aim is then to derive the simplest equations of motion of the polarization vector when a particle is moving along the given curve.

Let us take points p and q on the curve $\gamma(t)$ corresponding to the moments of time t and $\bar{t} = t + dt$. To determine an infinitesimal change in the vector $X(t)$ in time dt, the vector $X(\bar{t})$ at point q should be transported along the curve $\gamma(t)$ to the point p and compared there with the vector $X(t)$ at point p. In the general case the infinitesimal change of the vector field on curve $\gamma(t)$ is given by the expression (Schrödinger, 1950; Anderson, 1967)

$$\delta V^i = dV^i + \Gamma^i_{jk}\dot{x}^j V^k dt,$$

where V^i are components of the vector $X(t)$. To derive equations for $V(t)$, we apply to symmetry considerations. We assume that at every moment of time t an infinitesimal change of a vector along the curve $\gamma(t)$ is equal to an infinitesimal linear transformation of the vector induced by the gauge group. If $S^i_j = \delta^i_j + B^i_j dt$ is an infinitesimal gauge transformation, then we have

$$\delta V^i = B^i_j V^j dt,$$

from which we obtain a system of ordinary linear homogeneous differential equations

$$\frac{dV^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}V^k = B^i_j V^j, \qquad (3)$$

defining the law of change of the polarization vector in the course of motion of a polarized particle along the curve $\gamma(t)$.

A qualitatively new system of equations can be obtained when an infinitesimal change in the polarization vector is set equal to an infinitesimal linear transformation of the vector $\dot{\gamma}$ tangent to the curve $\gamma(t)$. So, we have

$$\delta V^i = F^i_j \dot{x}^j dt$$

and consequently,

$$\frac{dV^i}{dt} + \Gamma^i_{jk}\frac{dx^j}{dt}V^k = F^i_j\frac{dx^j}{dt}. \qquad (4)$$

Functions Γ^i_{jk} in eqs. (3) and (4) are called the Christoffel symbols or components of affine connection Γ .

Let us show that eqs. (3) and (4) allow us to establish the laws of change of fields Γ, B, F under gauge transformations in a natural way. Let vector fields $\bar{X}$ and X are equivalent with respect to the gauge group, then $\bar{V}^i = S^i_j V^j$. If the components V^i obey the equation (3), it is not difficult to verify that the components $\bar{V}^i$ will be a solution of the equation

$$\frac{d\bar{V}^i}{dt} + \bar{\Gamma}^i_{jk}\frac{dx^j}{dt}\bar{V}^k = \bar{B}^i_j\bar{V}^j,$$

where

$$\bar{\Gamma}^i_{jk} = S^i_l\Gamma^l_{jm}T^m_k + S^i_l\partial_j T^l_k,\ \bar{B}^i_j = S^i_k B^k_l T^l_j,$$

and T^i_j are components of the affinor field S^{-1} inverse to S . In the other case, we get

$$\frac{d\bar{V}^i}{dt} + \bar{\Gamma}^i_{jk}\frac{dx^j}{dt}\bar{V}^k = \bar{F}^i_j\frac{dx^j}{dt},$$

where the functions $\bar{\Gamma}^i_{jk}$ are expressed through Γ^i_{jk} and S by the same formulae as early , and

$$\bar{F}^i_j = S^i_k F^k_j.$$

Thus, the transformation laws of fields Γ, B, F under gauge transformations are determined.

2.6 A General Description of Basis Fields

Inasmuch as the fields g, Γ, B, F are directly connected with the most general structure properties of space-time, they are to be treated as primary fields according to the first principles. Let us mention some characteristic properties of the basis fields.

For brevity, we will use the matrix notation

$$B = (B^i_j),\ F = (F^i_j),\ \Gamma_j = (\Gamma^i_{jk}),\ E = (\delta^i_j),\ TrB = B^i_i,\ ST = (S^i_k T^k_j),$$

in which the transformation law of the affine connection Γ is of the form

$$\bar{\Gamma}_i = S\Gamma_i S^{-1} + S\partial_i S^{-1} = \Gamma_i + S\nabla_i S^{-1}, \qquad (5)$$

where ∇_i stands for the covariant derivative with respect to the connection Γ

$$\nabla_i S = \partial_i S + \Gamma_i S - S\Gamma_i = \partial_i S + [\Gamma_i, S].$$

As $S\nabla_i S^{-1}$ is a tensor field of the type (1,2), then $\bar{\Gamma}$ is the affine connection together with Γ . Let

$$(R^k_{ijl}) = R_{ij} = \partial_i\Gamma_j - \partial_j\Gamma_i + [\Gamma_i, \Gamma_j] \qquad (6)$$

components of the Riemann tensor of the affine connection Γ , then from (5) and (6) we obtain

$$\bar{R}_{ij} = SR_{ij}S^{-1}. \qquad (7)$$

From relations (5), (6), and (7) it follows that the affine connection is a gauge field.

The affinor field B transforms under gauge transformations by the following law

$$\bar{B} = SBS^{-1}, \qquad (8)$$

whereas the affinor field F,

$$\bar{F} = SF. \qquad (9)$$

As is seen from these formulae, affinor fields can variously transform under gauge transformations; therefore, they are denoted by different letters in eqs. (3) and (4). A fundamental difference between the gauge transformations (8) and (9) consists in the following: let S be an element of the gauge group; obviously, $(-S)$ also belongs to that group. Further, under the transformations $B \Rightarrow \bar{B} = SBS^{-1}$, the same transformation of the affinor field corresponds to different elements S and $(-S)$ of the gauge group; whereas under the transformations $F \Rightarrow \bar{F} = SF$ this is not the case. For comparison, we remark that two different elements of the group $SU(2)$, x and $(-x)$, $x = (x_1, x_2, x_3, x_4)$, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ give the same transformation in the usual vector space and different transformations in the spinor space. Therefore it is not difficult to suggest that theories of affinor fields B and F will radically differ from each other. We will distinguish affinor fields of the boson and fermion type, which can, in fact, be understood from the adopted symbols B and F.

So, it is clear that an affinor field should be associated with structure elements of matter. An important point is that the correspondence between fields and observable particles is here more implicit unlike the usually assumed direct correspondence.

To complete this section, we express, in terms of the local coordinate system, other required transformation laws of basis fields induced by the diffeomorphism group, $\tilde{B} = \tilde{\varphi}B$, $\tilde{\Gamma} = \tilde{\varphi}\Gamma$. Using (3) and (4) we immediately obtain

$$B^i_j(x) = \varphi^i_k(f(x))B^k_l(f(x))f^l_j(x), \qquad (10)$$

where $\varphi^i_j(x) = \partial_j \varphi^i(x)$,

$$\tilde{\Gamma}^i_{jk}(x) = \varphi^i_l(f(x))\Gamma^l_{mn}(f(x))f^m_j(x))f^n_k(x) + \varphi^i_l(f(x))\partial_j f^l_k(x). \qquad (11)$$

As can be verified by simple algebra, under changes of the variables $\bar{x}^i = \bar{x}^i(x^0, \ldots, x^{n-1})$, $x^i = x^i(\bar{x}^0, \ldots, \bar{x}^{n-1})$ the components $\tilde{g}^{ij}(x)$, $\tilde{B}^i_j(x)$, $\tilde{\Gamma}^i_{jk}(x)$ of fields $\tilde{g}$, $\tilde{B}$, $\tilde{\Gamma}$, are transformed as needed.

3 Affinor fields of the Boson Type

3.1 Formulation of the Problem

As is noted above, the diffeomorphism group is responsible for gravitational interactions, and thus, the gauge group we have defined is a symmetry group of all other interactions. All that is required is to derive all the simplest equations for basis fields invariant under transformations of the gauge group or its subgroup because any of them can exist independently. In this section we derive the simplest gauge-invariant equations of second order describing interactions of an affinor field of the boson type with a gauge field . It is assumed that the field B like g and Γ is real. This assumption means the following: we would like to determine how the transition to complex fields is related to the properties of space-time. We have no such possibility when introduce the complex field ad hoc. This relation is of fundamental importance as it will provide further information on the nature of the electric charge.

3.2 Connection between Symmetry Groups

Here we will carefully analyse the relation between groups of gauge and space-time symmetries and start with a general consideration. Let K is a class of theories whose equations can be written in a functional form $F(Q, B) = 0$, where B is a set of dynamical fields and Q is a set of external fields. Note that almost all known field theories belong to the class K. Let G be the group of internal symmetry of a theory; it induces transformations $B \Rightarrow \bar{B} = SB$ and $F(Q, \bar{B}) = 0$ if $F(Q, B) = 0$. It is also assumed that the group G is the group of invariance of the equations $F(Q, B) = 0$ for all admissible changes of the external fields Q . Besides,let H be the group of covariance of the theory under consideration. The covariance means that the group H induces transformations $Q \Rightarrow \tilde{Q} = AQ$, $B \Rightarrow \tilde{B} = AB$, and $F(\tilde{Q}.\tilde{B}) = 0$, if $F(Q, B) = 0$. If B is a solution of the equation $F(Q, B) = 0$, it can be shown that $B' = (A^{-1}SA)B = S'B$ is also a solution of the same equation. Hence it follows that

the covariance group H is a group of external automorphisms of the group G. In other words, the group H does not break the equivalence determined by the group G. Really, from the relation $\bar{B} = SB$ it follows that the fields $\tilde{\bar{B}} = A\bar{B}$ and $\tilde{B} = AB$ are also equivalent as $\tilde{\bar{B}} = ASB = (ASA^{-1})\tilde{B}$, and the transformation $ASA^{-1} = S'$ is shown to belong to the group G together with S . The symmetry group of the theory under consideration can be extended with the use of its covariant group in two ways. One way is to solve the equation $Q = AQ$. If a set of solutions of this equation is not reduced to the identical transformation of the group H , the external fields will be invariant under transformations of a certain subgroup of the group H. The most interesting are, of course, such external fields for which the invariance group is a maximally large subgroup L of the group H. The other way is to treat external fields on an equal footing with dynamical fields and to construct a theory invariant under transformation of both the groups, G and H, if a theory like that does exist.

It can be verified that the group of diffeomorphisms is a group of external automorphisms of the gauge group, i.e. the gauge group is invariant under transformations of the group $Diff(M)$. It can then be concluded that the gauge group $GL(n, R)$ is to be considered as the group of fundamental symmetry. To answer the question on the diffeomorphism group as a symmetry group, we should first construct a theory invariant under transformations of the gauge group and covariant under transformations of the diffeomorphism group. Thus, we will consider at the first glance the fields B and Γ as dynamical, and the field g as external.

3.3 Gauge Covariant Derivative

To simplify computations and to write equations in a symmetric and manifestly gauge-invariant form, we introduce the gauge covariant derivative. We will say that a tensor field T of the type (m, n) has the gauge type (p, q) if under the transformations of the gauge group there is the correspondence

$$T \Rightarrow \bar{T} = \underbrace{S \cdots S}_{p} T \underbrace{S^{-1} \cdots S^{-1}}_{q},$$

where

$$0 \leq p \leq m \text{ and } 0 \leq q \leq n.$$

The Riemann tensor is a tensor field of the type (1,3) and according to (7) has the gauge type (1,1). From (8) and (9) it follows that the affinor field being a tensor field of the type (1,1) may have the gauge type (1,1) or (1,0).

As follows from the consideration made in sect. 2, the geometrical quantity g being a tensor field of the type (0,2) is to be assigned the gauge type (0,0).

Let now T be components of the tensor field (tensor density) of the gauge type (1,1), then by definition

$$D_i T = \partial_i T + [\Gamma_i, T]$$

is the gauge covariant derivative. For instance, for the Riemann tensor

$$D_i R_{jk} = \partial_i R_{jk} + [\Gamma_i, R_{jk}].$$

For the affinor field of the boson type the gauge covariant derivative coincides with the standard covariant derivative ,

$$D_i B = \partial_i B + [\Gamma_i, B] = \nabla_i B.$$

In the general case the operator D_i is not covariant since $D_i T$ will not always be components of the tensor field together with T . However, the commutator $[D_i, D_j]$ is covariant, since

$$[D_i, D_j]T = [R_{ij}, T].$$

Hence we obtain the important relation for the Riemann tensor

$$[D_i, D_j]R_{kl} = [R_{ij}, R_{kl}]. \quad (12)$$

The basic property of the gauge covariant derivative follows from its definition

$$\bar{D}_i \bar{T} = S(D_i T)S^{-1}, \quad (13)$$

where $\bar{T} = STS^{-1}$, and $\bar{D}_i$ is the gauge covariant derivative with respect to the connection $\bar{\Gamma} = \Gamma + S\nabla S^{-1}$. So, the tensor fields B and $D_i B$ are of the same gauge type.

3.4 Gauge Invariant Equations

Now it's easy to write the simplest Lagrangian of the affinor field which is invariant under the gauge transformations $B \Rightarrow \bar{B} = SBS^{-1}, \Gamma \Rightarrow \bar{\Gamma} = \Gamma + SDS^{-1}, g \Rightarrow \bar{g} = g$. We have:

$$L_B = -\frac{1}{2} Tr(D_i B D^i B - m^2 BB), \quad (14)$$

where m is a constant, $D^i = g^{ij} D_j$. In the last formula g are the components of the tensor field g^{-1} inverse to $g, g_{ij} g^{kj} = \delta_i^k$. Here g is simply the tensor field

on a manifold and nothing else. But we shall use it for raising and lowering indices in the usual way as this contradicts nothing (Anderson, 1967). From (14) by the variation with respect to B we obtain the following equations of the affinor field

$$D_i(\sqrt{|g|} D^i B) + m^2 \sqrt{|g|} B = 0, \quad (15)$$

where $|g|$ is the absolute value of the determinant of the matrix (g_{ij}). When deriving (15) one should take into account that $Tr(D_i B) = \partial_i (Tr B)$. From (7) it follows that the simplest Lagrangian of the gauge field is given by the expression

$$L_\Gamma = -\frac{1}{4} Tr(R_{ij} R^{ij}), \quad (16)$$

where $R^{ij} = g^{ik} g^{jl} R_{kl}$. The total Lagrangian of the interacting fields B and Γ is equal to the sum of the Lagrangians (15) and (16). Varying the Lagrangian $L = L_B + L_\Gamma$ with respect to Γ with the help of the relation $\delta R_{ij} = D_i \delta\Gamma_j - D_j \delta\Gamma_i$, we obtain the following equations of the gauge field

$$D_i(\sqrt{|g|} R^{ij}) = \sqrt{|g|} J^j \quad (17)$$

the right hand side of which contains the tensor field of the third rank

$$J^i = [B, D^i B]. \quad (18)$$

This field obviously has the gauge type (1,1). The tensor current J has to satisfy the equation

$$D_i(\sqrt{|g|} J^i) = 0 \quad (19)$$

as in accordance with (12), $D_i D_j(\sqrt{|g|} R^{ij}) = 0$. From (15),(18) it follows that J really satisfies the equation (19) and thus, the system of the equations (15),(17) is consistent. The tensor character of the equations (15),(17) can be seen from the relations

$$\frac{1}{\sqrt{|g|}} D_i(\sqrt{|g|} D^i B) = (\nabla_i \nabla^i B + \omega_i \nabla^i B),$$

$$\frac{1}{\sqrt{|g|}} D_i(\sqrt{|g|} R^{ij}) = \nabla_i R^{ij} + \omega_i R^{ij} - \frac{1}{2}(\Gamma^j_{ik} - \Gamma^j_{ki}) R^{ik},$$

where $\omega_i = \partial_i ln\sqrt{|g|} - \Gamma^k_{ki}$ are the components of the covector field. Thus, it's shown that the group of diffeomorphisms is the group of covariance of the equations (15),(17).

3.5 The Metric Tensor of Energy-Momentum

Varying the Lagrangian $L = L_B + L_\Gamma$ with respect to g we obtain the so-called metric tensor of energy-momentum of the considered system of the interacting fields

$$T_{ij} = Tr(D_iBD_jB) + Tr(R_{ik}R_j^k) + g_{ij}L, \tag{20}$$

where $R_j^k = g^{kl}R_{jl}$. As is known, the metric tensor of the energy-momentum plays an extremely important role in General Relativity. Let us find the divergence $T^{ij}{}_{;i}$. The semicolon denote the covariant derivative with respect to the Levi-Civita connection, belonging to the field g. The Christoffel symbols of this connection are

$$\{{}^{\,i}_{jk}\} = \frac{1}{2}g^{il}(\partial_j g_{kl} + \partial_k g_{jl} - \partial_l g_{jk}). \tag{21}$$

If the fields B and Γ satisfy the equations (15),(17) we can show that the metric tensor of the energy-momentum satisfies the equation

$$T^{ij}{}_{;i} = 0. \tag{22}$$

When deriving (22) besides (14)-(18) one should use the standard relations of the tensor analysis (Schouten,1954) and the identity of Bianchi

$$D_iR_{jk} + D_jR_{ki} + D_kR_{ij} = 0$$

which can be easily obtained with the help of the relation $[D_i, D_j]B = [R_{ij}, B]$. According to (20) the metric tensor of the energy-momentum has the gauge type (0,0). From (22) and the gauge invariance of the metric tensor of energy-momentum it follows that the complete system of equations derived from the Lagrangian $L = L_g + L_B + L_\Gamma$, where L_g is the Einstein-Hilbert Lagrangian will be consistent and invariant both under gauge transformations and those of the group of diffeomorphisms. This invariance has the profound meaning that the group of diffeomorphisms as is mentioned above is the group of external automorphisms of the gauge group and thus the procedure of the derivation of the metric tensor of energy-momentum is gauge-invariant.

3.6 The Non-Linear Equations of The Affinor Field

The derived equations of the affinor field itself are linear. Let us show that it is possible to construct also a completely non-linear gauge-invariant system

of equations describing the interactions of the basis fields. As is known, the determinator $|g_{ij}| \neq 0$ that, actually, allows us to obtain for the tensor field g the equations invariant under the transformations of diffeomorphism group. By analogy let us consider the case when the determinator $|B^i{}_j| \neq 0$. Under this condition the affinor field B has the inverse one for which the non-linear gauge-invariant equations can be suggested. The simplest gauge-invariant Lagrangian has the form

$$L = -\frac{1}{2}Tr(D_iBD^iB^{-1}) + \lambda TrB - \frac{1}{4}Tr(R_{ij}R^{ij}), \tag{23}$$

where λ is a constant. Taking into account that $\delta B = -B(\delta B^{-1})B$ by varying with respect to B and Γ we obtain from (23) the following equations

$$D_i(\sqrt{|g|}R^{ij}) = \sqrt{|g|}J^j,$$

$$D_i(\sqrt{|g|}B^{-1}D^iB) = \lambda\sqrt{|g|}B,$$

$$J^i = [B^{-1}, D^iB].$$

It is easy to make certain that the tensor current $J = [B^{-1}, DB]$ satisfies the equation (19) and thus the written system of the equations is consistent. Varying the Lagrangian (23) with respect to g we obtain the gauge-invariant metric tensor of energy-momentum

$$T_{ij} = Tr(D_iBD_jB^{-1}) + Tr(R_{ik}R_j^k) + g_{ij}L$$

which satisfies the equation (22).

3.7 The Invariants of The Gauge Group

Some very interesting gauge-invariant quantities can be constructed from the fields B and R_{ij} . We dwell here on some of them. The invariants in particular are

$$\varphi = TrB,\ \Delta = |B^i_j|,\ \omega_{ij} = TrR_{ij}.$$

If B and R_{ij} satisfy the equations (15) and (17) then taking the trace we obtain that a scalar φ satisfies the Klein-Gordon equation and a bivector ω_{ij} satisfies the Maxwell equations without sources

$$\partial_i(\sqrt{|g|}\omega^{ij}) = 0,$$

since from (18) it follows that $TrJ = 0$.

Let $Q_i = Tr\Gamma_i = \Gamma^k_{ik}$. According to (5) and the differentiation rule for determinants, the transformation law for Q under gauge transformations has the form

$$\bar{Q}_i = Q_i - \partial_i ln|D|,$$

where $D = det(S^i_j)$. From (6) it follows that $\omega_{ij} = \partial_i Q_j - \partial_j Q_i$. Thus, the object Q represents in some sense a vector potential. It can be verified that as a topological object, the differential form $\omega = \frac{1}{2}\omega_{ij}dx^i \wedge dx^j$ is the so-called characteristic class $\omega = c_1$. If B obeys non-linear equation , then taking the trace of both sides of this equation we obtain that the invariants φ and $\triangle$ satisfy the equation

$$\partial_i(\sqrt{|g|}g^{ij}\partial_j ln|\triangle|) = \lambda\sqrt{|g|}\varphi.$$

From gauge invariance it follows that if there exists a state (B, R_{ij}), there should also exist states obtained from this state by applying to it all possible transformations of gauge symmetry. A state is said to be singlet if it is invariant under all the symmetry transformations. In our case a singlet state is given by the equations $B = SBS^{-1}$, $R_{ij} = SR_{ij}S^{-1}$ to be satisfied at any S . The first equation has the solution $B = \alpha E$, where α is a scalar field. If a gauge field obeys the equation, $R_{ij} = SR_{ij}S^{-1}$, it also obeys the equation $R_{ij} = \frac{1}{n}Tr(R_{ij})E$. Let $\hat{\Gamma}$ be components of the connection the Riemann tensor of which equals zero, $\hat{R}_{ij} = 0$. Then the connection components $\Gamma_i = \hat{\Gamma}_i + \omega_i E$, where ω_i are components of a covector field will satisfy the equation $R_{ij} = \frac{1}{n}Tr(R_{ij})E$. To complete the description of the singlet state note that in non-linear eq. for B it is convenient to set $B = e^{\alpha}E$, because in this case $B^{-1} = e^{-\alpha}E$. As in this case $\varphi = ne^{\alpha}, \triangle = e^{n\alpha}$,then from the equation for invariants φ and $\triangle$ we obtain that α will obey the Liouville equation

$$\partial_i(\sqrt{|g|}g^{ij}\partial_j\alpha) = \lambda\sqrt{|g|}e^{\alpha}$$

which establishes the link with the Liouville field theory.

3.8 On First-Order Equations

It is noteworthy that nontrivial gauge-invariant equations of the first order cannot be constructed for affinor fields of the boson type. Indeed, the most general tensor linear differential equation of the first order that can be satisfied by an affinor field is of the form

$$C^i\nabla_i B = MB,$$

where C^i and M are components of given tensor fields of the types (2,1) and (1,1) respectively. From (13) it follows that the field $\bar{B} = SBS^{-1}$ will obey the same equation when $C^i = SC^iS^{-1}$, $M = SMS^{-1}$ for all transformations of the gauge group. Hence it follows that $C^i = V^iE$, where V^i are components of the vector field X, and $M = mE$, where m is a scalar. As a result, for B we obtain a very special equation $\nabla_X B = mB$, that is not interest.

3.9 General Physical Interpretation

The Einstein equations of gravity reads

$$G_{ij} = \kappa T_{ij},$$

where

$$T_{ij} = (\epsilon + p)u_i u_j - pg_{ij}$$

is a hydrodynamical tensor of the energy-momentum (Landay and Lifshitz, 1971). This tensor energy- momentum is similar to the 4-vector of current in the Maxwell macroscopic theory of an electromagnetic field . In both cases the right hand side of the equations is not related with the symmetry principles. It is well known , that in the framework of quantum theory the equations of electrodynamics are completely defined by the laws of symmetry in full correspondence with the principle initiated by Einstein according to which symmetry dictates interactions (Yang, 1980). With this and on the basis of the results obtained above we can interpret equations (15), (17) together with the equations

$$G_{ij} = l^2 T_{ij}, \tag{24}$$

$$T_{ij} = Tr(D_iBD_jB) + Tr(R_{ik}R^k_j) + g_{ij}L, \tag{25}$$

where l is the Plank length, as the equations of Einstein gravity on a quantum level; we use the natural system of units ($G = \hbar = c = 1$). The equations in question are uniquely determined by the symmetry principles. So an affinor field of the boson type has the meaning of the "wave function "for gravitating particles. These particles, as it is shown earlier, are a single source of the gauge field Γ that is called the affine connection in terms of geometry. Now a central problem is to establish the role and status of the gauge forces in the Einstein gravitationl theory and to detect them at the laboratory. In this connection it must be noted that the equations of motion for the test particle in external gravitational and gauge fields have the form

$$\frac{d^2x^i}{ds^2} + \{^i_{jk}\}\frac{dx^j}{ds}\frac{dx^k}{ds} + \frac{f}{m}\omega^i_k\frac{dx^k}{ds} = 0, \tag{26}$$

where $\omega_{ij} = TrR_{ij}$, and f is the constant of interaction with the gauge field. It should be noted that the given physical interpretation is not complete without consideration affinor fields of the fermionic type.

4 Affinor fields of the Fermionic Type

4.1 Definition of the Gauge Derivative

In this section, we consider problems of deriving and analysing the second-order gauge-invariant equations for affinor fields of the fermionic type.

From (5),(8) and (9) it follows that it is necessary to define the gauge derivative of the affinor fields of the fermionic type for constructing their gauge-invariant theory, which is realized as follows. Let $\Lambda_j = (\Lambda^i_{jk})$ be components of a given affine connection Λ. Then components of any other connection Γ can be represented in the form $\Gamma^i_{jk} = \Lambda^i_{jk} + H^i_{jk}$, where $H_j = (H^i_{jk})$ are components of a tensor field of the type (1,2). We assume that the connection Λ is not changed under gauge transformations and accordingly we make the substitutions

$$\Gamma^i_{jk} = \Lambda^i_{jk} + H^i_{jk}\,,\ \bar{\Gamma}^i_{jk} = \Lambda^i_{jk} + \bar{H}^i_{jk}$$

in eq. (5). As a result, we find that under gauge transformations the tensor field H is transformed by the following law

$$\bar{H}_i = SH_iS^{-1} + S\nabla_iS^{-1}, \tag{27}$$

where ∇_i is a covariant derivative with respect to the connection Λ.

For affinor fields of the gauge type (1, 0) we assume

$$P_iF = \nabla_iF + H_iF. \tag{28}$$

Let $\bar{P}_iF = \nabla_iF + \bar{H}_iF$, then from (27) and (28) it follows that

$$\bar{P}_i\bar{F} = S(P_iF), \tag{29}$$

where $\bar{F} = SF$. Thus, F and P_iF are transformed by the same law. If an affinor field belongs to the gauge type (0, 1), we put

$$P_iF = \nabla_iF - FH_i. \tag{30}$$

From (27) and (30) we get $\bar{P}_i\bar{F} = (P_iF)S^{-1}$, where $\bar{F} = FS^{-1}$.

Therefore, for affinor fields of the fermionic type the gauge derivative P_i is determined by a tensor field of the type (1, 2) and a given affine connection Λ. This connection is not changed under gauge transformations and it will be called the background connection.

4.2 Reduction of the Gauge Group

The scalar $Tr(FF)$ is not invariant under the gauge transformations $\bar{F} = SF$ and $\bar{F} = FS^{-1}$. Therefore we define $\tilde{F} = (\tilde{F}^i_j) = (g^{ik}F^l_k g_{lj})$ and will call the affinor field $\tilde{F}$ conjugate to F through g. It can easily be verified that these fields obey the following relations

$$\tilde{\tilde{F}} = F\,,\ \widetilde{F_1F_2} = \tilde{F}_2\tilde{F}_1,,\ Tr(\tilde{F}) = TrF. \tag{31}$$

The scalar $Tr(F\tilde{F})$ is invariant under the gauge transformations satisfying the equation

$$S\tilde{S} = E. \tag{32}$$

From (31) it follows that the elements of the gauge group that obey equation (32) compose its subgroup. Rewriting (32) in the form $g_{ij}S^i_kS^j_l = g_{kl}$, we conclude that the structure of that subgroup is determined by the quadratic form $g(X,X) = g_{ij}V^iV^j$. Then we denote the gauge group we are interested in by $O_g(p,q)$ where p and q are positive and negative indices of the inertia of the quadratic form $g(X,X), p+q = n = dimM$.

It can be verified that in the case under investigation we cannot avoid the reduction of the gauge group, and we should fit the definitions of the gauge derivative (28) and (30) to this fact.

If F is of the gauge type (1,0), then from (31) and (32) it follows that $\tilde{F}$ is of the type (0,1). According to (30), it must be $P_i\tilde{F} = \nabla_i\tilde{F} - \tilde{F}H_i$. However, from (28) we obtain $\widetilde{P_iF} = \widetilde{\nabla_iF} + \tilde{F}\tilde{H}_i$. Therefore, the gauge field H_i and covariant derivative ∇_i should obey the conditions

$$H_i + \tilde{H}_i = 0 \tag{33}$$

and

$$\widetilde{\nabla_iF} = \nabla_i\tilde{F}. \tag{34}$$

As a result, for the gauge derivative we have $\widetilde{P_iF} = P_i\tilde{F}$. From (27) and (32) it follows that the relations (33) are gauge invariant. The condition (34) is fulfilled provided that $\nabla_ig_{jk} = 2\omega_ig_{jk}$, where ω_i are components of an arbitrary covector field. It is of interest to study the case when the background connection is the Weyl connection, however here we restrict our consideration to metric connections, the Levi-Civita connection and integrable connection;. the Riemann tensor of the latter is zero. The tensor gauge field H it is possible to consider as dynamical component of the affine connection.

4.3 Equations of Interacting Fields

The simplest gauge-invariant Lagrangian is of the form

$$L = -\frac{1}{2}Tr(P_i F P^i \tilde{F} - m^2 F\tilde{F}) - \frac{1}{4}Tr(R_{ij}R^{ij}), \tag{35}$$

where R_{ij} are components of the Riemann tensor of the connection $\Gamma = \Lambda + H$. According to (5),

$$R_{ij} = R_{ij}(\Lambda) + \nabla_i H_j - \nabla_j H_i + [H_i, H_j] + T^k_{ij} H_k, \tag{36}$$

where $R_{ij}(\Lambda)$ are components of the Riemann tensor of the background connection and $T^k_{ij} = \Lambda^k_{ij} - \Lambda^k_{ji}$ are components of the torsion tensor of this connection. For the integrable connection $R_{ij}(\Lambda) = 0$, and for the Levi-Civita connection $T^k_{ij} = 0$. The Lagrangian (35) is invariant under the gauge transformations $\bar{F} = SF, \bar{H} = SH\tilde{S} + S\nabla\tilde{S}$, $S\tilde{S} = E$.

Applying the variational principle to the Lagrangian (35) we arrive at the equations

$$(P_i - T_i)P^i F + m^2 F = 0, \tag{37}$$

$$D_i(\sqrt{|g|}R^{ij}) = \sqrt{|g|}J^j, \tag{38}$$

$$J^i = \frac{1}{2}\tilde{F}(P^i F) - \frac{1}{2}(P^i \tilde{F})F, \tag{39}$$

where $T_i = T^k_{ki}$ are components of the torsion covector of the background connection; for the Levi-Civita connection $T_i = 0$. The gauge derivative D_i in eq. (38) is taken with respect to the connection $\Gamma = \Lambda + H$. From (39) we obtain that $J_i + \tilde{J}_i = 0$, which is in agreement with the relation $R_{ij} + \tilde{R}_{ij} = 0$ following from (33) and (36). The tensor current (39) is of the gauge type (1,1) and obeys the equation $D_i(\sqrt{|g|}J^i) = 0$ if F is a solution of eq.(37). Thus, the system of tensor equations (37)-(39) is consistent.

Let H_{ijk} be components of a tensor field skew-symmetric in two indices, $H_{ijk} + H_{ikj} = 0$. Setting $H^i_{jk} = g^{il}H_{jlk}$, we may verify that this substitution identically satisfy equations of constraints (33). From (29) it follows that under gauge transformations the components H_{ijk} change by the law

$$\bar{H}_{ijk} = H_{ilm}S^l_j S^m_k + g_{lm}S^l_k \nabla_i S^m_j.$$

This completely determines the essence of equation (38) as an equation of the tensor field H_{ijk}. It is interesting to note that the connection Γ with the Christoffel symbols $\Gamma^i_{jk} = \{^{\,i}_{jk}\} + g^{il}H_{jlk}$ has a fine geometrical interpretation given by Weyl in 1921 (Schouten, 1954).

4.4 Reduction of the symmetry group of gravitational interactions

The theory formulated in subsections (4.1)-(4.3) is gauge invariant. Following the analysis made in subsect. (3.2) we will deal with the problem of the full symmetry group of this theory. First, we show that the gauge group $O_g(p,q)$ is not invariant under transformations of the diffeomorphism group. Let an affinor field S be an element of the group $O_g(p,q)$ and φ be a transformation in M. Setting as usual $\tilde{S} = \tilde{\varphi}S, \tilde{g} = \tilde{\varphi}\, g$, we obtain that $\tilde{g}_{ij}\tilde{S}^i_k\tilde{S}^j_l = \tilde{g}_{kl}$ if $g_{ij}S^i_k S^j_l = g_{kl}$. Hence it follows that the affinor field $\tilde{S}$ is an element of the gauge group $O_{\tilde{g}}(p,q)$. When $\tilde{g} = \lambda g$, where λ is a positive function on M, then the gauge groups $O_g(p,q)$ and $O_{\tilde{g}}(p,q)$ coincide. The inverse statement is also valid. Thus, only those transformations of the diffeomorphism group are external automorphisms of the gauge group $O_g(p,q)$, that obey the condition $\tilde{g} = \lambda g$. This condition in the local coordinate system, according to (2), is written as follows

$$g_{kl}(f(x))f^k_i(x)f^l_j(x) = \lambda(x)g_{ij}(x). \tag{40}$$

The geometrical interpretation of equations (40) is that they define the group of conformal transformations of the Riemann space V_n with the metric $ds^2 = g_{ij}dx^i dx^j$. The group of conformal transformations V_n is a finite continuous Lie group of dimension r not larger than $\frac{(n+1)(n+2)}{2}$ (Eisenhart, 1949). Consequently, the gauge group $O_g(p,q)$ is invariant under transformations of the conformal group. It is just these transformations that do not break the equivalence relationship established by the gauge group. In this sense the reduction of the gauge group $GL(n,R)$ automatically entails the reduction of the symmetry group of gravitational interactions.

4.5 Gauge Principle and Gravitational Interactions

Two consequences follow from the performed analysis. The first implies that the total symmetry group of the theory under consideration may contain also conformal transformations in addition to the gauge transformations. We do not examine it here as more important is the second consequence. Since variations of the field g do not obey the condition $\delta g = \lambda g$, the general method of construction of the metric tensor of energy-momentum is not invariant under gauge transformatiions (recall, this method results from the variational principle). The metric tensor of energy-momentum of the Lagrangian (35) being not invariant under transformations of the gauge group $O_g(p,q)$ can be verified in the following way. The invariant $Tr(F\tilde{F})$ characterizing the structure of the Lagrangian (35) and its symmetry properties varies when g varies. The relation $A = \tilde{F}F - F\tilde{F}$, where $A = (A^i_j)$, and

$$A^i_j = g^{ik}\frac{\partial Tr(F\tilde{F})}{\partial g^{jk}}$$

shows that the tensor field A is not invariant under transformations of the gauge group $O_g(p,q)$. Consequently, also the metric tensor of energy-momentum is not invariant under these transformations. Thus, if we add the Einstein-Hilbert Lagrangian to the Lagrangian (35), we arrive at the theory with one of its equations being not gauge invariant. In this case the gauge principle loses its meaning because the fields F and H can be made to obey equations not invariant under gauge transformations. So, in the theory of affinor fields of the fermionic type the diffeomorphism group cannot be considered as a symmetry group. In this connection we note the following . The basic idea of the Einstein theory of gravity consists in that the gravitational field is associated with a tensor field of second rank. If this correspondence is adequate to the nature of gravity, then the question is to be set concerning a special role of the gravitational interaction. The principle of universality of gravitational interactions requires experimental verification as it is not consistent with the gauge principle in the case of affinor fields of the fermionic type. Further discussion of gravitational interactions and experimental facts will be continued upon derivation of the Dirac equation. Now we consider one general problem of fundamental importance.

4.6 Energy-Momentum Canonical Tensor

Because of the contradiction of the principle of universality of gravitational interactions with the gauge invariance it is necessary to solve the problem concerning the energy conservation in the theory of affinor fields of the fermionic type. To this end we apply to the method of derivation of the energy conservation law in classical mechanics (Landau and Lifshitz, 1960). First consider the case of integrable background connection. The Lagrangian (35) does not depend on the coordinates $x^0, \cdots, x^{n-1}$ explicitly. Therefore using the relation $\partial_i TrF = Tr(D_i F)$ and the condition (33) we obtain

$$\partial_i L = -Tr(P^i\tilde{F}P_jP_iF) + m^2Tr(FP_j\tilde{F}) - \frac{1}{2}Tr(R^{ik}D_jR_{ik}) - \frac{1}{2}(\partial_j g^{ik})Tr(R_{il}R^l_k) \tag{41}$$

Now we employ the commutation relations $[P_i, P_j]F = -FR_{ij} - T^k_{ij}P_kF$, the Bianchi identity $D_iR_{jk} + D_jR_{ki} + D_kR_{ij} = 0$, and the metricity condition for

the background connection

$$\partial_j g^{ik} + \Lambda^i_{jl}g^{lk} + \Lambda^k_{jl}g^{il} = 0.$$

Then, upon some computations we may write equation (41) in the form

$$\nabla_i W^i_j - T_i W^i_j + T^k_{ij}W^i_k = Tr(P_jF\frac{\delta L}{\delta F}) + Tr(R_{jk}\frac{\delta L}{\delta H_k}),$$

where

$$W^i_j = Tr(P^i\tilde{F}P_jF) + Tr(R^{ik}R_{jk}) + \delta^i_j L$$

is the canonical gauge-invariant tensor of the energy-momentum of our system, from which and from (37), (38) we obtain the local energy conservation law

$$\nabla_i W^i_j - T_i W^i_j + T^k_{ij}W^i_k = 0. \tag{42}$$

If the background connection is the Levi-Civita connection, we can construct a symmetric gauge-invariant tensor of the energy-momentum in a similar way. Here we will present the final result

$$\Theta^{ij} = Tr(P^i\tilde{F}P^jF) + Tr(R^{ik}R^j_k) + g^{ij}L + S^{ikj}{}_{;k} + S^{jki}{}_{;k},$$

where

$$S^{ijk} = g^{il}S^j_{lm}g^{mk}, \ (S^j_{lm}) = S_l = \frac{1}{2}F(P_l\tilde{F}) - \frac{1}{2}(P_lF)\tilde{F}.$$

The energy-momentum tensor Θ^{ij} obeys the equation $\Theta^{ij}{}_{;i} = 0$. However, we cannot replace the metric energy-momentum tensor in the Einstein equation by the energy- momentum tensor Θ^{ij} as this contradicts the variational principle.

So, it is shown that there exists the local gauge-invariant energy conservation law, and an interesting connection is found between the torsion tensor and energy-momentum canonical tensor. We will study this connection in more details.

4.7 On the Physical Meaning of the Torsion Tensor

As to the torsion tensor, we should first of all notice the following. If Γ^i_{jk} are components of an arbitrary affine connection Γ, then $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$ are components of a skew-symmetric tensor field of the type (1,2) called the torsion tensor of affine connection Γ. The torsion tensor was introduced by Cartan in 1922 and since then has been extensively studied by many scientists (Hehl et al., 1976).

From definition it follows that the torsion tensor is a geometrical object with respect to the diffeomorphism group since according to (11) and definition

$$\tilde{T}^i_{jk}(x) = T^l_{mn}(f(x))\varphi^i_l(f(x))f^m_j(x)f^n_k(x).$$

On the other hand, from formula (5) it follows that the torsion tensor is not a geometrical object with respect to the gauge group. The tensor T^i_{jk} defines no representation of the gauge group. It may be said that the concept of the torsion tensor is not gauge-covariant. This somewhat unexpected fact is of basic importance as it is a reason unknown till now for which all attempts to assign a physical meaning to the torsion tensor have not led to any heuristic result. Therefore, the solution to this problem is to be looked for along another direction to be shown below.

Consider the integral laws of energy conservation following from (42). Components of a vector field X obey the equality

$$\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}V^i) = (\nabla_i - T_i)V^i,$$

that can easily be derived from the metricity condition of the integrable connection. This equality results in the relationship

$$\frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}V^jW^i_j) = (\nabla_i W^i_j - T_i W^i_j + T^k_{ij}W^i_k)V^j + (\nabla_i V^j - T^j_{ik}V^k)W^i_j.$$

from which and from (42) it follows that the integral

$$I = \int (V^j W^i_j)dS_i$$

does not depend on the hypersurface chosen provided the components of the vector field X obey the equation

$$\nabla_i V^j - T^j_{ik}V^k = 0. \tag{43}$$

Here we have take into account that the canonical tensor of energy-momentum is generally not symmetric. For equations (43) being integrable we first derive the condition $V^l\nabla_l T^i_{jk} = 0$. Consequently, equations (43) are completely integrable provided $\nabla_l T^i_{jk} = 0$. As it is known (Eisenhart, 1933) the affine connection defines a simply transitive group of transformations if its Riemann tensor equals zero and the torsion tensor obeys the equation $\nabla_l T^i_{jk} = 0$. If the manifold M admits the simply transitive group, it is parallelizable and hence is a spinor manifold (Geroch, 1968). As a result, we arrive at the following sequence: the torsion tensor, the canonical energy-momentum tensor,

integral conservation laws, simply transitive group of transformations, spinor structure. What is then the physical meaning of the torsion tensor?

Let not all structure constants of a simply transitive group be zero. Then the torsion tensor defined by this group is also nonzero. Minkowski space-time admits the simply transitive group with all the structure constants being zero; this is the group of translations. So, to establish the physical meaning of the torsion tensor implies to solve the problem of existence of physical space-time admitting the non-Abelian simply transitive group of transformations. The answer will be positive.

4.8 Energy Vector

Untill now we were based on the general assumption that space-time has structure of a differentiable manifold. Now we have some indications at our disposal that allow us to specify this general structure. In what follows, space-time will be taken to be a differentiable manifold M admitting a simply transitive group of transformations.

Let

$$E_a = e^i_a\frac{\partial}{\partial x^i}\,,\ a = 0, 1, \cdots, n-1 \tag{44}$$

be vector fields on the manifold being generators of a simply transitive group. Then

$$[E_a, E_b] = f^c_{ab}E_c, \tag{45}$$

where f^c_{ab} are structure constants of the group. The vector fields (44) uniquely define the system of covector fields $\omega^a = e^a_i dx^i$ such that

$$e^i_a e^a_j = \delta^i_j\,,\ e^a_i e^i_b = \delta^a_b.$$

The simply transitive group induces a natural integrable connection Λ on M whose Christoffel symbols are

$$\Lambda^i_{jk} = e^i_a\partial_j e^a_k. \tag{46}$$

From the metricity condition of the connection (46) we have the natural metric on M

$$g_{ij} = \eta_{ab}e^a_i e^b_j\,,\ g^{ij} = \eta^{ab}e^i_a e^j_b, \tag{47}$$

where η_{ab} is a matrix whose elements are numbers. From (45) and (46) we obtain for the torsion tensor

$$T^i_{jk} = -f^a_{bc}e^i_a e^b_j e^c_k\,,\ T_i = T^k_{ki} = -f^a_{ab}e^b_j. \tag{48}$$

By virtue of (45) and (47) we have

$$e_a^i e_{b;i}^j = (f_{ab}^c - \eta_{ad} f_{be}^d \eta^{ce} - \eta_{bd} f_{ae}^d \eta^{ce}) e_c^j, \tag{49}$$

where the semicolon means the covariant derivative with respect to the Levi-Civita connection of the metric (47). Equation (43) allows determination of generators H_a of a mutual group.

In the Minkowski space-time $E_0 = \frac{\partial}{\partial x^0}$, and hence, the energy operator $H = -i\hbar E_0$. For this reason, the vector field $E_0 = e_0^i \partial_i$ defined by (44) will be called the energy vector. In this way, we also suggest that the physical energy is, generally speaking, described not only by the well-known operator $H = -i\hbar \frac{\partial}{\partial x^0}$. It can be understood that the problem of existence of a new energy operator is inseparable from the physical meaning of the torsion.

5 First-Order Equations

5.1 Gauge Invariance

Earlier it was shown that for affinor fields of the bosonic type, no nontrivial gauge-invariant equations of the first order exist. Let us now consider the problem of existence of such equations for real affinor fields of the fermionic type.

Let $C_i = (C_{ik}^j)$ be components of a tensor field of the type (1,2). Consider the equation

$$C^i P_i F = mF, \tag{50}$$

where F is an affinor field of the gauge type (0,1),m is a scalar, P_i is a gauge derivative defined in subsect. (4.1), $C^i = g^{ij} C_j$. According to (30) and (50),

$$C^i \bar{P}_i \bar{F} = C^i (P_i F) S^{-1} = m\bar{F}.$$

Hence equation(50) is gauge invariant. If the field F is of the gauge type (1,0), then the equation

$$(P_i F) C^i = mF,$$

is also gauge invariant, which can easily be verified with the use of (29). Thus, it is shown that there exist nontrivial gauge-invariant tensor equations of the first order for affinor fields of the fermionic type.

Unlike affinor fields of the bosonic type, the requirement of gauge invariance does not here impose any constraints on the tensor field C, which allows us to specialize it through its association with the problem of finding real tensor representations of Clifford algebras. The so-called spinor representations of Clifford algebras are well known. The Dirac wave function is the carrier space of a complex, faithful and irreducible representation of the Clifford algebra (Cartan, 1966).

5.2 Clifford Tensor

The real tensor field C of the type (1,2) will be called the Clifford tensor if its components $C_i = (C_{ik}^j)$ satisfy the equation

$$C_{im}^k C_{jl}^m + C_{jm}^k C_{il}^m = 2g_{ij}\delta_l^k,$$

or in the matrix notation, $C_i C_j + C_j C_i = 2g_{ij}E$. We will demonstrate that the class of Clifford tensors is not empty. At every point p of space-time the Clifford tensor should define a real irreducible representation of a Clifford algebra $Cl(g)$. Then, based on the theory of representations of Clifford algebras (Chevalley, 1954) we conclude that the Clifford tensor can exist only on a manifold M of even dimension, $dimM = 2m$, as all tensor indices run over n values. Next we find that m should obey the equation $2m = 2^m$ that has two solutions, $m = 1, m = 2$. Consequently, the Clifford tensor can exist only on manifolds of dimension $n = 2$ or $n = 4$. The case $n = 2$ will be considered separately. Therefore, in what follows $dimM = 4$ throughout. To satisfy the reality condition, it is necessary to require that the quadratic form $g(X,X) = g_{ij}V^iV^j$ be of the signature $(-,+,+,+)$ or $(-,-,+,+)$. We will dwell upon the first of them for the reason to be explained in the next subsection.

The literature on physics deals mainly with the signature $(+,-,-,-)$, therefore, we will write the relation defining the Clifford tensor in the form

$$C_i C_j + C_j C_i = -2g_{ij}E. \tag{51}$$

By definition, space-time admits a simply transitive group of transformations. Therefore, we set $C_{ik}^j = C_{ad}^b e_k^d e_b^j e_i^a$ and substitute it into equation (51). From (44), (47) and (51) it follows that components of the Clifford tensor in the basis E_a should obey the equation $\{C_a, C_b\} = -2\eta_{ab}E$, where $\eta_{ab} = (+1,-1,-1,-1)$, $C_a = (C_{ad}^b)$. We will solve this equation defining the components of the Clifford tensor in the basis E_a explicitly:

$$C_0 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$C_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence, the Clifford tensors are proved to exist on a 4-manifold admitting a simply transitive group of transformations. The study of a general case of the 4-manifold with the metric of signature $(+,-,-,-)$ is beyond the scope of the present paper. According to (46) the constructed Clifford tensor obeys the equation

$$\nabla_i C^k_{jl} = 0 \tag{52}$$

that will be useful in what follows.

5.3 Almost Complex Structure

Now let us explain the choice of the metric signature. Tensor fields of the type $(1, p+1), p = 0,1,2,3,4$,

$$(C_{i_1 \cdots i_p}{}^k{}_l) = C_{i_1 \cdots i_p} = \frac{1}{p!}\delta^{j_1 \cdots j_p}_{i_1 \cdots i_p} C_{j_1} \cdots C_{j_p},$$

where $\delta^{j_1 \cdots j_p}_{i_1 \cdots i_p}$ is the generalized Kronecker delta, will be called the Clifford basis that contains a unit affinor field $E = (\delta^k_i), p = 0$. For $p = 2$ we have $C_{i_1 i_2}{}^k{}_l = \frac{1}{2}(C^k_{i_1 m} C^m_{i_2 l} - C^k_{i_2 m} C^m_{i_1 l})$ and so on. Let e_{ijkl} be components of a completely antisymmetric Levi-Civita tensor field with $e_{0123} = \varepsilon\sqrt{|g|}$, where ε is a chosen orientation of M, i.e. an even scalar is such that $\varepsilon^2 = 1$ (de Rham, 1955). We introduce the affinor field J with the equation

$$J = \frac{1}{4!} e^{ijkl} C_{ijkl}, \tag{53}$$

where $e^{ijkl} = g^{im} g^{jn} g^{kp} g^{lq} e_{mnpq}$. Then $J^2 = -E$ only when the metric has the signature $(+,-,-,-)$. As it is known, the almost complex structure on a real differentiable manifold M is an affinor field J such that $J^2 = -E$ (Flaherty, 1976). Consequently, the affinor field we have introduced (53) is the almost

complex structure in space-time. Thus, the choice of correct signature of the metric is connected with a definite property of space-time. In what follows it will be shown what are the laws predetermined by this property of space-time. Here we will only consider the general characteristics of the almost complex structure.

Let space-time be a real differentiable manifold M of dimension $n = 4$. The system of complex local coordinates on M will be defined as a topological mapping of an open region V of the manifold M onto an open region of the number complex space C^2. This mapping makes every point p from V correspond to two complex numbers $z^k = x^k + iy^k, k = 1,2$. The manifold is said to admit the complex analytic structure provided there exists a set of systems of complex local coordinates V_a satisfying the following conditions: The union of all V_a coincides with $M, M = \cup V_a$; in the intersection of two domains $V_a \cap V_b$, the complex local coordinates z^k_a and z^k_b, are holomorphic functions of each other, $z^k_a = z^k_a(z^k_b)$ and $z^k_b = z^k_b(z^k_a)$ (i.e. z^k_a are functions of z^k_b and not of their conjugates $\bar{z}^k_b$).

There is an important correspondence between the almost complex structure J on M and complex analytic structure (Flaherty, 1976). Let

$$N^i_{jk} = J^l_j \partial_l J^i_k - J^l_k \partial_l J^i_j - J^i_l \partial_j J^l_k + J^i_l \partial_k J^l_j$$

be components of the Nijenhuis tensor of the almost complex structure J on M. If $N^i_{jk} = 0$, then M admits the complex analytic structure, and vice versa, every complex analytic structure on M induces the almost complex structure J whose torsion tensor equals zero. So, intuitively it is clear that the almost complex structure should have a direct relationship to the complex affinor fields and electromagnetic interactions.

5.4 Variational Principle

Here we will study a Dirac-like equation

$$C^i P_i F = mF, \tag{54}$$

where C is the Clifford tensor (51) and F is a real affinor field of the gauge type (0,1). As equation (54) is gauge-invariant, it remains to be seen how this equation can be derived from the variational principle. To this end, we first consider some properties of the Clifford basis following from its definition and equations (51). We have the relations

$$C_j C_{i_1 \cdots i_p} = C_{j i_1 \cdots i_p} + \frac{1}{p!}(p g_{jk_1} C_{k_2 \cdots k_p} \delta^{k_1 \cdots k_p}_{i_1 \cdots i_p}), \tag{55}$$

$p = 0,1,2,3,4$. From' (51),(55) it followes that a affinor field may be represented as an expantion on Clifford basis

$$F = \sum_{p=1}^{4} \frac{1}{p!} f_{i_1 \cdots i_p} C^{i_1 \cdots i_p},$$

where

$$f_{i_1 \cdots i_p} = \frac{1}{4}(-1)^{\frac{p(p+1)}{2}} Tr(C_{i_1 \cdots i_p} F).$$

A completely skew-symmetric covariant tensor field of the type $(0,p)$ is called the p -vector, whereas the 0 -vector is simply a scalar field; and the 1 -vector, a covector field. So, it may be said that components of p -vectors are coordinates of the affinor field in the Clifford basis. Then it follows that in the given case the affinor field can be made to correspond to an inhomogeneous differential form ω. The formal expression $\omega_p = \frac{1}{p!} f_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ is called the differential form of the degree p, then $\omega = \sum_{p=0}^{4} \omega_p$ (Eguchi, Gilkey and Hanson, 1980). Since in the general case ω has 2^n components and F has n^2 components, we find that under the condition $n^2 = 2^n$ we get again $n = 2$ and $n = 4$. We define the operation of conjugation so that the conjugate affinor field $\hat{F}$ obey the equation

$$Tr(F\hat{F}) = \sum_{p=0}^{4} \frac{1}{p!} f_{i_1 \cdots i_p} f^{i_1 \cdots i_p}.$$

Applying the operation of conjugation by means of g to the relation (51) we find that $\tilde{C}$ will be a Clifford tensor, if C is a Clifford tensor. Consequently, there exists an affinor field T defined up to a scalar uniquely such that $\tilde{C}_i = T^{-1} C_i T$. Further, because $JC_i + C_i J = 0$ then $J^{-1} C_i J = -C_i$ The operation of conjugation we need is given by the equation $\hat{F} = JTFT^{-1}J^{-1}$. By definition, $\hat{C}_i = -C_i$, and hence, $\hat{F} = \sum_{p=0}^{4} \frac{1}{p!}(-1)^{\frac{p(p+1)}{2}} f_{i_1 \cdots i_p} C^{i_1 \cdots i_p}$ We also note the relations

$$\widehat{F_1 F_2} = \widehat{F_2}\widehat{F_1}\,, \; TrF = Tr(\widehat{F})\,, \; J = \widehat{J}\,, \; \widehat{\widehat{F}} = F.$$

The scalar product $Tr(F\widehat{F})$ is invariant under the gauge transformations obeying the equation

$$S\widehat{S} = E. \tag{56}$$

From the (56) it follows that the gauge field H_i must satisfy the equation

$$H_i + \widehat{H_i} = 0. \tag{57}$$

From the (54),(57) it followes that $\widehat{F}$ obey the equation

$$(\nabla_i \widehat{F}) C^i + H_i \widehat{F} C^i = -m\widehat{F}.$$

Equation (54) follows from the variational principle with the Lagrangian

$$L_F = \frac{1}{2} Tr(\widehat{F} C^i P_i F) - \frac{m}{2} Tr(\widehat{F} F)$$

under the condition that the torsion covector of the ground connection is zero. In the general case the Euler-Lagrange equation for L_F is as follows

$$C^i (P_i - \frac{1}{2} T_i) F = mF, \tag{58}$$

where $T_i = T^k_{ki}$. The Lagrangian of interacting fields

$$L = \frac{1}{2} Tr(\widehat{F} C^i P_i F) - \frac{m}{2} Tr(\widehat{F} F) + \frac{1}{4} Tr(R_{ij} R^{ij}) \tag{59}$$

is invariant under the gauge transformations of type (56). It is interesting that the energy-momentum canonical tensor of the Lagrangian (59) is not symmetric because

$$W^i_j = \frac{1}{2} Tr(\widehat{F} C^i P_j F) + Tr(R^{il} R_{jl}) - L\delta^i_j$$

but it is gauge invariant.

5.5 Structure of the Gauge group

Equation (56) defines a subgroup G_c of a principal gauge group. We shall now describe its structure and then vary the Lagrangian (59). Let an infinitesimal transformation of the group G_c be of the form $S = E + A$; then from (56) we get $A + \widehat{A} = 0$. Because of the equality $C_i + \widehat{C}_i = 0$ we have $A = \frac{1}{2} e_i C^i + \frac{1}{4} e_{ij} C^{ij}$, where e_i are components of a covector field, and e_{ij} are components of a 2 -vector. If $B = \frac{1}{2} f_i C^i + \frac{1}{4} f_{ij} C^{ij}$, then $D = [A, B]$ will also obey the equation $D + \widehat{D} = 0$. Let $D = \frac{1}{2} h_i C^i + \frac{1}{4} h_{ij} C^{ij}$, then $h_i = e_j f^j_i - f_j e^j_i$, $h_{ij} = e_i f_j - e_j f_i + e_{ik} f^k_j - e_{jk} f^k_i$. Thus, we can say that the gauge group G_c has the structure of conformal group.To canonical parameters of that group there correspond p- vectors, $p = 1, 2$, on the manifold M.

Considering transformations of the gauge group G_c in the Clifford basis we may deduce that external automorphisms of the group G_c are only those transformations of the diffeomorphism group for which $\lambda(x) = 1$ in eq. (40). Hence it follows that the gauge group G_c is invariant under isometric transformations of the metric $ds^2 = g_{ij} dx^i dx^j$. Therefore, the group of isometric transformations may generally be a symmetry group of the theory along with the group G_c. It is interesting to note that the isometry group of a given metric is a finite continuous Lie group of dimension r not larger than $\frac{n(n+1)}{2}$ .If $r = \frac{n(n+1)}{2}$, then g is a metrical tensor of the space of constant curvature (Eisenhart, 1933). So we here meet with almost the same situation as in the previous section when the second-order equation of fermionic type was considered.

5.6 Equations of a Gauge Field

To derive equations for a gauge field from (59), we solve the equation of constraints (57). If we set

$$(H_{ij}^k) = H_i = \frac{1}{2}X_{im}C^m + \frac{1}{4}Y_{iml}C^{ml}, \tag{60}$$

where X_{im} are components of a covariant tensor field of the second rank and Y_{iml} are components of a covariant tensor field of the third rank that is skew-symmetric in the last two indices, $Y_{iml}+Y_{ilm}=0$, we obtain a general solution to equation (57). According to (52) and (60) for the Riemann tensor of the connection $\Gamma_i = \Lambda_i + H_i$ we have

$$R_{ij} = \frac{1}{2}F_{ijk}C^k + \frac{1}{4}H_{ijkl}C^{kl}, \tag{61}$$

where

$$F_{ijk} = \nabla_i X_{jk} + X_{il}Y_{jk}^l - (i \leftrightarrow j) + T_{ij}^l X_{lk}, \tag{62}$$

$$H_{ijkl} = \nabla_i Y_{jkl} + Y_{ikm}Y_{jl}^m + X_{ik}X_{jl} - (i \leftrightarrow j) + T_{ij}^m Y_{mkl}. \tag{63}$$

So, F_{ijk} and H_{ijkl} are coordinates of the Riemann tensor of the connection $\Gamma = \Lambda + H$ in the Clifford basis. From (60) and (61) it follows that the Lagrangian (59) can be represented by a sum of three Lagrangians

$$L_F = \frac{1}{2}Tr(\widehat{F}C^i\nabla_i F) - \frac{m}{2}Tr(\widehat{F}F)$$

$$L_H = -\frac{1}{4}F_{ijk}F^{ijk} - \frac{1}{8}H_{ijkl}H^{ijkl}$$

$$L_I = -X_{ij}J^{ij} - \frac{1}{2}Y_{ijk}J^{ijk},$$

where

$$J^{ij} = \frac{1}{4}Tr(C^iFC^j\widehat{F}), J^{ijk} = \frac{1}{4}Tr(C^iFC^{jk}\hat{F}) \tag{64}$$

are tensor fields of currents.

By a variational procedure, from the Lagrangian $L_H + L_I$ we obtain equations of the gauge fields X_{ij} and Y_{ijk} in the form

$$(\nabla_i - T_i)F^{ijk} + Y_{il}^k F^{ijl} + X_{il}H^{ijkl} - \frac{1}{2}T_{il}^j F^{ikl} = J^{jk}, \tag{65}$$

$$(\nabla_i - T_i)H^{ijkl} + Y_{im}^k H^{ijml} + X_i^k F^{ijl} - (k \leftrightarrow l) - \frac{1}{2}T_{im}^j H^{imkl} = J^{jkl}. \tag{66}$$

If F satisfies equation (58), the tensor currents J^{ij} and J^{ijk} obey the equations

$$\nabla_i J^{ij} + X_{ik}J^{ijk} + Y_{ik}^j J^{ik} = 0,$$

$$\nabla_i J^{ijk} + Y_{il}^j J^{ilk} + X_i^j J^{ik} - (j \leftrightarrow k) = 0$$

which may be verified with the use of (58) and (64). The same equations follow from (65), (66) and the identities

$$\nabla_i\nabla_j F^{ijk} = -\frac{1}{2}T_{il}^j\nabla_l F^{ijk}, \ \nabla_i\nabla_j H^{ijkl} = -\frac{1}{2}T_{ij}^l\nabla_l H^{ijkl}.$$

Hence, the system of equations (58),(64),(65),(66) is consistent.

Thus, the simplest laws describing the dynamics of a real affinor field within the framework of gauge principle have been established and quite definite extra indications of the structure of space-time have been found. Now we are ready to determine electromagnetic interactions and to derive the Dirac equation.

6 Electromagnetic field and Almost Complex Structure

6.1 Interactions of electromagnetic type

Almost complex structure being an essential property of a real space-time manifold defines interactions of electromagnetic type. This notion is specified by the following natural conditions:

i. The almost complex structure J is an element of the gauge group and generates its two subgroups $G(J)$ and $U(1)$. The subgroup $G(J)$ is composed of the elements of the gauge group obeying the equation

$$SJS^{-1} = J,$$

whereas the second subgroup $U(1)$ consists of elements of the form $Q = \exp(\frac{1}{2}\alpha J) = \cos\frac{\alpha}{2} + J\sin\frac{\alpha}{2}$, where α is a scalar. The reasons for half angles will be clear from further consideration.

ii.The Lagrangian describing interactions of electromagnetic type should be invariant under transformations of the gauge groups $G(J)$ and $U(1)$.

iii. The Riemann tensor should also be invariant under transformations of the gauge group $U(1)$.

Consider interactions of the electromagnetic type in framework of the gauge groups $O_g(p,q)$ and G_c. Let J be an element of the first gauge group , then it has to satisfy the equations

$$J\tilde{J} = E\,,\ J^2 = -E.$$

When $p = 1$, $q = 3$, i.e. for the gauge group $O_g(1,3)$ these equations have no solution. A solution exists for $p = 2m$ and $q = 0$. Thus, second-order equations of the fermionic type are the entity that does not admit interactions of the electromagnetic type.

The almost complex structure for the group G_c will be denoted by J_0. From equations $J_0\hat{J}_0 = E$ and $J_0^2 = -E$ it follows that $J_0 = V^iC_i$, where V^i are components of the vector field X satisfying the equation $g(X,X) = 1$. It is interesting that such a nonzero continuous vector field exists owing to the normal hyperbolic type of metric $ds^2 = g_{ij}dx^idx^j$ (Lichnerowicz, 1955). According to the results of subsect. (5.6), this vector field is to be identified with the energy vector $E_0 = e_0^i\partial_i$, so that $V^i = e_0^i$. It is not difficult to construct infinitesimal transformations of the gauge group $G(J)$ induced in G_c by the almost complex structure J_0. Let $S = E + A$ be an infinitesimal transformation of the group G_c, then it can be shown that $[J_0, A - J_0AJ_0] = 0$. As a result, with the requirement of gauge invariance in mind, we obtain that the affine connection associated with the almost complex structure J_0 can be written in the form $\Gamma_i = \Lambda_i + H_i$, where

$$H_i = \frac{1}{2}\omega_i J_0 + \frac{1}{4}(Y_{ijkl}V^l)C^{jk}. \qquad (67)$$

Here ω_i are components of a covector field, and Y_{ijkl} are components of a covariant tensor field of the type (0,4) skew-symmetric in the last three indices. As $\nabla J_0 = 0$,then under the gauge transformations $S = \exp(\frac{1}{2}\alpha J_0)$ we have $\bar{H}_i = \frac{1}{2}(\omega_i + \partial_i\alpha)J_0 + \frac{1}{4}(Y_{ijkl}V^l)C^{jk}$. Computing the Riemann tensor components we get

$$R_{ij} = \frac{1}{2}\omega_{ij}J_0 + \frac{1}{4}H_{ijkl}C^{kl}, \qquad (68)$$

where $\omega_{ij} = \partial_i\omega_j - \partial_j\omega_i$,

$$H_{ijkl} = \nabla_i Z_{jkl} + Z_{ikm}Z^m_{jl} - (i \leftrightarrow j) + T^m_{ij}Z_{mkl} \qquad (69)$$

$$Z_{ijk} = Y_{ijkl}V^l. \qquad (70)$$

From (68) and (69) it follows that the Lagrangian (59) can be written in the form

$$L = \frac{1}{2}Tr(\widehat{F}C^i\nabla_i F) - \frac{m}{2}Tr(\widehat{F}F) - \frac{1}{4}\omega_{ij}\omega^{ij} - \frac{1}{8}H_{ijkl}H^{ijkl} - \omega_i J^i - \frac{1}{2}Z_{ijk}Z^{ijk}, \qquad (71)$$

where

$$J^i = \frac{1}{4}Tr(\widehat{F}C^iFJ_0),\ J^{ijk} = \frac{1}{4}(\widehat{F}C^iFC^{jk}).$$

The Lagrangian (71) describes interactions of the electromagnetic type in framework of the gauge group G_c.

Consider the vector current $J^i = \frac{1}{4}(\widehat{F}C^iFJ_0)$. As $J_0 = V_iC^i$, then $J^i = V_kJ^{ik}$, where $J^{ij} = \frac{1}{4}Tr(\widehat{F}C^iFC^j)$. An interesting result holds true giving evidence for the theory being self-consistent: if the metric $ds^2 = g_{ij}dx^idx^j$ has signature $(+,-,-,-)$, then the component 00 of tensor fei ld J^{ij} is nonnegative, $J^{00} \geq 0$, and $J^{00} = 0$ if $F = 0$. This result can be proved by expressing the components J^{ij} through the coordinates of the affinor field F in the Clifford basis. The time component of the current J^i may be considered to be the probability density.

Now let us ascertain the physical meaning of complex affinor fields of the fermionic type, $\Psi = \Psi_1 + i\Psi_2$. Introducing the projection operators $P_1 = \frac{1}{2}(E - iJ_0)$, $P_2 = \frac{1}{2}(E + iJ_0)$, $P_1 + P_2 = E, P_1^2 = P_1, P_2^2 = P_2$, we can represent Ψ as a superposition of two pure states, $\Psi = \Psi_+ + \Psi_-$, where $\Psi_+ = \Psi P_1$ and $\Psi_- = \Psi P_2$. It is not difficult to write the gauge invariant Lagrangian for Ψ_+, by setting

$$\overset{*}{\Psi}_+ = \hat{\bar{\Psi}}_+.$$

where the bar means complex conjugation, and taking scalar product in the form $Tr(\overset{*}{\Psi}_+\ \Psi_+)$. The equation for Ψ_+ will contain the vector potential ω_j in the form $i\omega_j$. The equation for Ψ_- will include the vector potential ω_j in the form $-i\omega_j$. Thus, it is clearly seen haw the almost complex structure is connected with the notion of a particle-antiparticle, $\Psi_+ \leftrightarrow \Psi_-$. Doubling of the number of degrees of freedom of the field acquires a clear physical meaning: If we restrict ourselves to the real affinor fields, the states of particle-antiparticle cannot be introduced because $J_0^2 = -E$ though interactions of the electromagnetic type do exist.

6.2 Connection with Dirac Theory

Recall that the correct signature of the metric was found on the basis of the almost complex structure (44). Therefore, it seems of interest to find the corresponding interactions of the electromagnetic type. From (44) it follows that J is not an element of the gauge group G_c. As $JJ_0 + J_0J = 0, \widehat{J} = J$, then $\widehat{J}J_0J = J_0$. Hence, we conclude that J is an element of the gauge group given by the equation

$$\widehat{S}J_0S = J_0. \tag{72}$$

From equation (72) and $[J, S] = 0$ it follows that the gauge field H_i should be subjected to the constraints $H_iJ_0 + J_0H_i = 0$ and $H_iJ - JH_i = 0$. We will write the general solution of these equations as follows

$$H_i = \frac{1}{2}A_iJ + \frac{1}{4}(Y_{ijkl}V^l)C^{jk}, \tag{73}$$

where A_i is the 4-vector potential and Y_{ijkl} are components of the tensor field of the same type as in equation (67). Now we shall construct a gauge invariant Lagrangian of interacting fields. The scalar $Tr(FJ_0\widehat{H})$ is invariant under gauge transformations satisfying equation (72). As $Tr(\widehat{F}) = Tr(F)$ and $\widehat{J_0} = -J_0$, then $Tr(FJ_0\widehat{H}) = -Tr(HJ_0\widehat{F})$. From this it followes that $Tr(FJ_0\widehat{F}) = 0$. So, in the case of real affinor fields the Lagrangian will be massless. Owing to this interesting property we should immediately go over to complex affinor fields $\Psi = \Psi_1 + i\Psi_2$. Setting, as above, $\Psi_+ = \frac{1}{2}\Psi(E - iJ)$ and $\Psi_- = \frac{1}{2}\Psi(E + iJ)$, let us now consider an expression $\varphi = Tr(\Psi_+J_0\overset{*}{\Psi}_+)$, where $\bar{\Psi} = \Psi_1 - i\Psi_2$ and $\overset{*}{\Psi}_+ = \widehat{\bar{\Psi}}_+$. Performing complex conjugation we obtain $\bar{\varphi} = -\varphi$. Consequently, the invariant $iTr(\Psi_+J_0\overset{*}{\Psi}_+)$ is a real scalar.

As a result, we arrive at a gauge invariant Lagrangian of the form

$$L = \frac{i}{2}Tr(C^iP_i\Psi_+J_0\overset{*}{\Psi}_+) - \frac{im}{2}Tr(\Psi_+J_0\overset{*}{\Psi}_+) + c.c. + \frac{1}{4}Tr(R_{ij}R^{ij}), \tag{74}$$

where R_{ij} are components of the Riemann tensor of the connection $\Gamma_i = \Lambda_i + H_i$ and Λ is the integrable connection. From (73) it follows that the Lagrangian (74) can be rewritten as

$$L = L_\Psi - A_iJ^i - \frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}Z_{ijk}J^{ijk} - \frac{1}{8}H_{ijkl}H^{ijkl}, \tag{75}$$

where Z_{ijk} obeys equation (70); and H_{ijkl} equation (69), and

$$F_{ij} = \partial_iA_j - \partial_jA_i,$$

$$L_\Psi = \frac{i}{2}Tr(C^i\nabla_i\Psi_+J_0\overset{*}{\Psi}_+) - \frac{i}{2}Tr(\Psi_+J_0\nabla_i\overset{*}{\Psi}_+C^i) - imTr(\Psi_+J_0\overset{*}{\Psi}_+),$$

$$J^i = \frac{1}{2}Tr(C^i\Psi_+J_0\overset{*}{\Psi}_+), J^{ikl} = \frac{1}{2}Tr(C^i\Psi_+C^{kl}J_0\overset{*}{\Psi}_+).$$

The proof that the 00-component of the field $J^{ij} = \frac{1}{2}Tr(C^i\Psi_+C^j\overset{*}{\Psi}_+)$ being positive definite proceeds in the same way as for $J^{00} = \frac{1}{4}Tr(C^0FC^0\widehat{F})$. The variational principle produces for Ψ_+ the following equation

$$C^i(\nabla_i - \frac{1}{2}T_i - iA_i)\Psi_+ - C^i\Psi_+H_i = m\Psi_+, \tag{76}$$

where T_i is the torsion covector of the background connection Λ,and

$$H_i = \frac{1}{4}(Y_{ijkl}V^l)C^{jk}.$$

For reasons to be expounded in detail in the next section, we set Y_{ijkl} in (76) to be zero and study the equation

$$C^i(\nabla_i - \frac{1}{2}T_i - iA_i)\Psi_+ = m\Psi_+ \tag{77}$$

Consider the affinor fields $Q_a = e^i_aC_i, a = 0, 1, 2, 3$. From (47) and (51) we get $Q_aQ_b + Q_bQ_a = -2\eta_{ab}E$. Since $JQ_a + Q_aJ = 0$ and, according to (46), (52), $\nabla_iQ_a = 0$, the operator $\Sigma = iQ_1Q_2$ commutes with J and acts in the space of solutions to equation (77). Let us now solve the equation $\Psi_+\Sigma = \Psi_+$. The affinor field Ψ_+ in the Clifford basis can be written in the form

$$\Psi_+ = \phi E + \phi_iC^i + \frac{1}{2}\phi_{ij}C^{ij} + \frac{i}{3!}(e_{ijkl}\phi^i)C^{jkl} - i\phi J \tag{78}$$

provided that the bivector components satisfy the equation $\phi_{ij} = \frac{i}{2}e_{ijkl}\phi^{kl}$. Setting $\phi_i = z_ae^a_i$,and $\phi_{ij} = z_{ab}e^a_ie^b_j$, we find that the equation $\Psi_+\Sigma = \Psi_+$ is satisfied if $z_0 = z_3, z_1 = -iz_2$ and

$$z_{03} = \phi, z_{01} = -iz_{02}, z_{12} = -iz_{03}, z_{23} = -iz_{01}, z_{31} = -iz_{02}.$$

Therefore, putting

$$\phi_i = \psi_1l_i + \psi_2m_i, \phi = \psi_4, \phi_{ij} = \psi_3(l_in_j - l_jn_i - m_i\bar{m}_j + m_j\bar{m}_i) + \psi_4(l_im_j - l_jm_i), \tag{79}$$

where

$$l_i = e^0_i + e^3_i, n_i = e^0_i - e^3_i, m_i = e^1_i + ie^2_i, \bar{m}_i = e^1_i - ie^2_i$$

and $\psi_1, \psi_2, \psi_3, \psi_4$ are arbitrary complex functions, we derive a general solution of the equation $\Psi_+\Sigma = \Psi_+$. It is worth-while to notice that the known basis $l_i, n_i, m_i, \bar{m}_i$ (Newman and Penrose, 1965) arises here in algebraic context naturally. Inserting (79) into (78) and (78) into (77) we derive for $\psi_1, \psi_2, \psi_3, \psi_4$ the following system of equations

$$(-D_0 - D_3)\psi_3 + (D_1 + iD_2)\psi_4 = m\psi_1,$$
$$(-D_1 + iD_2)\psi_3 + (D_0 - D_3)\psi_4 = m\psi_2,$$
$$(D_0 - D_3)\psi_1 + (-D_1 - iD_2)\psi_2 = m\psi_3,$$
$$(D_1 - iD_2)\psi_1 + (-D_0 - D_3)\psi_2 = m\psi_4,$$

where
$$D_a = e^i_a(\partial_i - \frac{1}{2}T_i - iA_i) = E_a - \frac{1}{2}T_a - iA_a, a = 0,1,2,3.$$

If we introduce the matrices

$$\gamma^0 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

then the system of equations for $\psi_1, \psi_2, \psi_3, \psi_4$ can be rewritten in a symmetric matrix form

$$\gamma^a D_a \psi = m\psi, \qquad (80)$$

where ψ is a matrix column. The matrices γ^a obey the standard relations

$$\gamma^a\gamma^b + \gamma^b\gamma^a = -2\eta^{ab}.$$

Thus, equation (80) is nothing else than the Dirac equation. The Dirac Lagrangian follows from the Lagrangian (74) with the use of the above described substitutions. Introducing the projection operator $P = \frac{1}{4}(E - iJ)(E + iQ_1Q_2)$ we can write the affinor field equivalent to the Dirac wave function ψ in the form

$$\Psi_D = \Psi P = \frac{1}{4}\Psi(E - iJ)(E + iQ_1Q_2). \qquad (81)$$

So, on a formal level, the connection with the Dirac theory may be considered to be established. However, it must sheds new light upon the Dirac theory. Consequently, it is necessary to analyze this problem more thoroughly.

6.3 The Yang-Mills Field

First we will demonstrate that the tensor gauge field $Y_{ijkl}V^l$ we have neglected in deriving the Dirac equation is the Yang-Mills field. The reasons are as follows: The group of gauge symmetry of the Lagrangian (71) is defined by the equations $S\hat{S} = E, [J_0, S] = 0$ from which it follows that this group has the structure of the group $SU(2) \times U(1)$. In similar fashion, from the equations $\hat{S}J_0S = J_0, [J, S] = 0$ we find that the group of gauge symmetry of the Lagrangian (74) is the group $SU(2) \times U(1)$. Because of this fact we should look for the correspondence between the tensor gauge field $Y_{ijkl}V^l$ and vector gauge Yang-Mills field or, which is the same, between the Lagrangian $L_\Gamma = \frac{1}{4}Tr(R_{ij}R^{ij})$ given by the Riemann tensor of affine connection $\Gamma_i = \Lambda_i + \frac{1}{4}(Y_{ijkl}V^l)C^{jk}$ and the original Yang- Mills Lagrangian (1954).

In the Minkowski space-time the torsion tensor is zero. In the Cartesian coordinates $\Lambda = 0$ and components of the vector field $E_0 = e^i_0\partial_i = V^i\partial_i$ can be taken in the form $V^i = \delta^i_0$. Thus, $Y_{ijkl}V^l = Y_{ijk0}$. Since $Y_{ijkl} = -Y_{ikjl} = -Y_{ijlk}$, then nonzero are only the components $Y_{i120}, Y_{i230}, Y_{i310}$. From this and (69), (70) it follows that the strength tensor has nonzero components $H_{ij12}, H_{ij23}, H_{ij31}$. If we set $B^1_i = -Y_{i230}, B^2_i = -Y_{i310}, B^3_i = -Y_{i120}$, and respectively, $F^1_{ij} = H_{ij23}, F^2_{ij} = H_{ij31}, F^3_{ij} = H_{ij12}$, then from (69) we obtain $F^1_{ij} = \partial_i B^1_j - \partial_j B^1_i + B^2_i B^3_j - B^2_j B^3_i$ and a similar equations for F^2_{ij}, F^3_{ij}. With this identification the Lagrangian $L_\Gamma = \frac{1}{4}Tr(R_{ij}R^{ij}) = -\frac{1}{8}H_{ijkl}H^{ijkl}$ coincides with the original Yang-Mills Lagrangian.

So we conclude that geometrically the Yang-Mills field is the Yang-Mills affine connection. In principle it is also important to emphasize that the so-called isotopic index has masked not only the tensor nature of the gauge Yang-Mills field but also the energy vector $E_0 = e^i_0\partial_i = V^i\partial_i$. Let us show that this fact along with the requirement of relativistic invariance allow us to understand why the Yang-Mills field is not so usual as the electromagnetic field.

6.4 Relativistic Invariance

Up till now we have mainly considered the gauge symmetry. Now let us examine the Lagrangian in the light of space-time symmetry. As is seen from (73) and (74), the Lagrangian (74), besides the quantities describing the state of the system, contains the Minkowski metric and energy vector $E_0 = V^i\partial_i = \frac{\partial}{\partial x^0}$. Therefore, this Lagrangian will be invariant under those space-time transformations which leave unchanged not only the metric tensor but also the vector field E_0. This group of space-time transformations will not be the Poincare group because it will not include boosts (pure Lorentz transformations). Thus, the requirement of relativistic invariance does not hold valid. Here we have only the relativistic covariance. Consider what happens if we switch off interactions with the Yang-Mills field. As in this case $J_0 = V^i C_i$, the Lagrangian will nevertheless contain the vector field E_0, but equation (77) will be relativistic-invariant. And which is more, it will also be invariant under gauge transformations of the type $\Psi_+ \Rightarrow \Psi_+ S$ under the condition that $[S, J] = 0, \nabla_i S = 0$. Just the latter was used in deriving the Dirac equation. Let us now show how we can achieve the relativistic invariance of the Lagrangian (74) by combining space-time and gauge transformations. Let $P_a = \delta_a^i \partial_i$ and $M_{ab} = (x_a \delta_b^i - x_b \delta_a^i)\partial_i$, where $x_a = \eta_{ab} x^b$, are vector fields that are generators of the Poincare group and $Q_a = \delta_a^i C_i$ are affinor fields utilized in deriving the Dirac equation. We have the following commutation relations between P_a, M_{ab} and Q_a

$$[P_a, Q_b] = 0, [Q_a, M_{bc}] = \eta_{ab} Q_c - \eta_{ac} Q_b, \tag{82}$$

that can be most easily computed in the Clifford basis. If we set $Q_{ab} = \frac{1}{4}(Q_a Q_b - Q_b Q_a)$, it is not difficult to verify that $[Q_a, Q_{bc}] = -\eta_{ab} Q_c + \eta_{ac} Q_b$. From this relation and (82) it follows that the operators $L_{ab} = M_{ab} + Q_{ab}$ commute with $Q_a, [Q_a, L_{bc}] = 0$. Since $J_0 = Q_0$, the Lagrangian (74) will be invariant under transformations induced by the generators P_a, L_{bc} which, like P_a, M_{bc}, satisfy the structure relations of the Lie algebra of the Poincare group. If we consider the action of operators L_{ab} in space $\psi_1, \psi_2, \psi_3, \psi_4$, we obtain standard generators of the Lorentz group in the Dirac theory. So, the relativistic invariance of the Lagrangian (74) and the Dirac theory is achieved at the expense of gauge transformations. It is, however, still unclear why the Yang-Mills field is sacrificed to this, so to say, hybrid relativistic invariance. In this direction, certain studies are to be made because it is interesting to deal with unusual properties of seemingly well-known objects. Here we note the following. As it may be deduced from the shape of the electron Dirac wave function defined by equation (81), electrons are splitted into two classes differing in the quantum number with values ± 1. We denote the states with quantum numbers $+1$ and -1 by $|++>$ and $|+->$ respectively. Superposition of these states $|+\varphi> = \cos\varphi|++> + \sin\varphi|+->$ should possess interesting properties dependent on the angle φ. If this new state is realizable as an unstable particle, then interaction of this object (may be μ or τ) with the Yang -Mills field is possible. Of special interest is that the isotropy of space-time should break in this case and this violation will obviously be maximal at $\varphi = \frac{\pi}{4}$.

Let us imagine the space-time in which the group of space-time transformations preserving the metric (47) coincides with the group of space-time transformations conserving the energy vector E_0. In this space-time there is no necessity to remove the Yang-Mills field and it should "work" to the full strength. In what follows we will show that space-time of that kind does exist. It is quite possible that it is of fundamental importance not less than the Minkowski space-time.

From (81) and (82) it may be concluded that in the space of Dirac wave functions not only the hybrid operators L_{ab} act, but also an operator of the proper Lorentz group, M_{12}, that defines rotations about the z axis. In this way, the Dirac wave function retains one "genuine" rotational degree of freedom. It can freely rotate about the z axis. Such is quantum-mechanical picture of a "rotating electron"; it shows that along with phenomena of interference and diffraction, the proper magnetic moment of the electron demonstrates its wave properties.

And finally, note that if we go over from the Lagrangian (74) to the Dirac Lagrangian, the almost complex structure J_0 transforms into the matrix γ^0 that camouflages the energy vector E_0.

6.5 On Gravitational Interactions of Electrons

Once the connection with the Dirac theory is established, we turn again to the question about the universal character of the gravitational interactions in direct relation to concrete stable physical objects, electrons. In Wigner's opinion (1964), it is conceivable that the special role of the gravitational interaction may dissolve in higher harmony. Now there is no question that higher harmony predicted by Wigner is the gauge symmetry. As has been shown in subsect. (4.4), reduction of the principal gauge group leads to reduction of the diffeomorphism group, the group of symmetry of gravitational interactions. We see that without reduction of the gauge group it is impossible to derive the Dirac equation. Therefore, the problem of gravitational interactions of electrons assumes a fundamental importance. In this connection let us discuss the measurements of gravitational acceleration of electrons (Wittenborn and Fairbank, 1967) and similar planned experiments with positrons (Fairbank and Wittenborn, 1988).

The experiment has shown that even if electrons suffer from the gravity force, this is not observed experimentally. This result has been interpreted in context of the work by Schiff and Barnhill (1966), from which it follows that positrons in similar experiments should fall with the acceleration $a = 2g$, where g is the gravity acceleration. Another explanation we propose here consists in that electrons and positrons do not interact with the gravity field at all and thus the same results of experiments should also follow for positrons, i.e. positrons like electrons should fall with a zero acceleration in the gravitational field of the earth. We can say that electrons and positrons represent such a form of energy which does not gravitate. In any case, the experiment with positrons will be of great importance for understanding the nature of gravity forces. Unfortunately, the measurement of gravitational acceleration of positrons in the group of Fairbank is open to question (Nieto and Goldman, 1991), therefore it remains to believe that such experiments will be undertaken in other groups of experimentalists. Results of these measurements may have the same effect as the measurement of the proton magnetic moment.

7 Hyperbolic Space-Time

7.1 Definition And Some Properties

As mentioned above, the Minkowski space-time permits the Abelian simply transitive group of transformations. Here the definition will be given for the hyperbolic space-time which permits the non-Abelian simply transitive group of transformations and differs from the Minkowski space-time both in geometric and in topological properties. In the next subsections there will be brought the arguments indicating that the hyperbolic space-time together with the Minkowski space-time is interesting from the physical point of view.

Let the indices marked by capital letters run over five labels $0, 1, 2, 3, 4$. In the five-dimentional Minkowski space-time $M^5_{1,4}$ with Cartesian coordinates x^A and metric

$$ds^2 = \eta_{AB}dx^A dx^B = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 - (dx^4)^2 \qquad (83)$$

let us consider the one sheet hyperboloid H^4

$$\eta_{AB}x^A x^B = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 = -a^2, \qquad (84)$$

where a is a positive constant,radius of H^4.

If we equip H^4 with the metric, induced by the quadratic differential form (83), then H^4 will transform into the Riemann manifold called the de Sitter space-time in physical literature . For escaping the confusion let us emphasize that here we won't use the de Sitter metric as the one-sheet hyperboloid (84) permits the simply transitive group of transformations and thus it is the parallelized manifold. Let us prove this.

We will use the scalar product $(X, Y) = \eta_{AB}V^A U^B$ for any vector fields $X = V^A\partial_A$, $Y = U^A\partial_A$ on $M^5_{1,4}$. The vector fields

$$P_A = \delta^C_B\partial_C, M_{AB} = (x_A\delta^C_B - x_B\delta^C_A)\partial_C,$$

where $x_A = \eta_{AB}x^B$, are the generators of the Poincare group of the five-dimensional Minkowski space-time. All the vector fields M_{AB} are orthogonal to the radius vector $R = x^C\partial_C$ at each point $M^5_{1,4}$, $(R, M_{AB}) = 0$, whereas it is not so for the vector fields P_A.

The expansion $P_A = P_A + \frac{1}{a^2}(R, P_A)R - \frac{1}{a^2}(R, P_A)R$ forms the vector fields

$$M_A = aP_A + \frac{1}{a}(R, P_A)R = (a\delta^C_A + \frac{1}{a}x_A x^C)\partial_C$$

which are tangent to H^4 because from (85) it follows that $(R, M_A) = 0$ at each point of H^4. The vector fields M_A, M_{AB} form the Lie algebra of the group of the conformal transformations H^4 because

$$[M_A, M_B] = -M_{AB}, [M_A, M_{BC}] = \eta_{AB}M_C - \eta_{AC}M_B. \qquad (85)$$

Let us introduce the vector fields $E_0 = M_0$, $E_1 = M_{14} + M_{23}$,$E_2 = M_{24} + M_{31}$, $E_3 = M_{34} + M_{12}$ and write out their components

$$E_0 = (a + \frac{1}{a}x_0^2, \frac{1}{a}x_0x^1, \frac{1}{a}x_0x^2, \frac{1}{a}x_0x^3, \frac{1}{a}x_0x^4)$$
$$E_1 = (0, -x_4, -x_3, x_2, x_1)$$
$$E_2 = (0, x_3, -x_4, -x_1, x_2)$$
$$E_3 = (0, -x_2, x_1, -x_4, x_3)$$

It is easy to see that the vector fields E_0, E_1, E_2, E_3 don't become zero at any point of H^4 and they are obviously continuous. Because of $(E_a, E_b) = 0$ for $a \neq b$, $a, b = 0, 1, 2, 3$ and

$$(E_0, E_0) = -(E_1, E_1) = -(E_2, E_2) = -(E_3, E_3) = a^2 + x_0^2$$

the vector fields $E_a, a = 0, 1, 2, 3$ are linearly independent at each point H^4, which was to be proved.

Thus it is shown that hyperquadric (84) is the parallelized manifold. Owing to this we equip H^4 with the metric such as (47) which differs from the de Sitter metric and thus we transform H^4 into the hyperbolic space-time $H^4_{1,3}$. From (85) it follows that

$$[E_0, E_i] = 0, [E_i, E_j] = 2e_{ijk}E_k\ , i, j, k = 1, 2, 3,$$

where e_{ijk} is the Levi-Civita completely antisymmetric symbol with $e_{123} = 1$. Thus among the structure constants of the found simply transitive group of transformations of the space-time $H^4_{1,3}$ only

$$f^1_{23} = f^2_{31} = f^3_{12} = 2 \tag{86}$$

are not equal zero. From (49), (86) it follows that the vector field E_0 is absolutely parallel relatively to the Levi-Civita connection on $H^4_{1,3}$ which is formed by the vector fields $E_a, a = 0,1,2,3$. For comparison let us notice that the de Sitter metric does not permit obviously the absolutely parallel vector fields.

The vector fields $F_0 = M_0, F_1 = M_{14} - M_{23}, F_2 = M_{24} - M_{31}, F_3 = M_{34} - M_{12}$ also form the global basis on $H^4_{1,3}$, mutual to the basis of E_a. The matrix of the transition $T = (E_i, F_j)$ as easy to check coincides with the matrix of rotations in three dimensional vector space. From (85) it follows that the vector fields F_a satisfy the following structure relations

$$[F_0, F_i] = 0,\ [F_i, F_j] = -2e_{ijk}F_k,\ i,j,k = 1,2,3;\ [E_a, F_b] = 0.$$

Let us notice that on the two-sheet hyperboloid $\eta_{AB}x^Ax^B = a^2$ the vector fields , similar to E_a and F_a become zero at the points $x^0 = \pm a, x^1 = x^2 = x^3 = x^4 = 0$.

7.2 The pointlike particle and rigid body

Now let us directly prove that the hyperbolic space-time $H^4_{1,3}$, like the Minkowski spacetime, is a physical space-time. As the section of the hypersurface (84) by the hyperplane $x^0 = const$ is a three-dimensional sphere S^3 we will first of all discuss a movement of material point (particle) along a trajectory on the three dimensional sphere

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2 \tag{87}$$

in the four-dimensional Euclidean space R^4.

Let E^4 is a four-dimensional vector Euclidean space with the orthonormalized basis e_1, e_2, e_3, e_4, $(e_i, e_j) = \delta_{ij},\ i,j = 1,2,3,4$. If the vector $A \in E^4$, then $A = \sum_{i=1}^4 A^ie_i$, $(A,B) = \sum_{i=1}^4 A^iB^i$. Vectors from E^4 orthogonal to a certain vector A of unit length form a three-dimensional vector space, $E^3(A)$. The vector spaces $E^3(A)$ and $E^3(B)$ are isomorphic; isomorphism is given by the the mapping σ

$$Q = \sigma P = (B,P)A - (B,A)P + [B \times A \times P],$$

where $P \in E^3(A), Q \in E^3(B)$ and

$$[A \times B \times C]_i = e_{ijkl}A^jB^kC^l,$$

where e_{ijkl} is the Levi-Civita completely antysymmetric symbol with $e_{1234} = 1$. The mapping σ conserves the length $(P,P) = (Q,Q)$ and is obviously linear. The inverse mapping is given by

$$P = \sigma^{-1}Q = (A,Q)B - (A,B)Q + [A \times B \times Q].$$

If $R = \sum_{i=1}^4 x^ie_i$ is a radius-vector of points on the three-dimensional sphere(87), then $(R,R) = a^2$, and consequently, the velocity vector $V = \dot{R}$ belongs to $E^3(N)$,where $N = \frac{1}{a}R$. The Newton equation describing the dynamics of a particle in the space of a constant positive curvature (88) can be written in the four-dimensional form as follows

$$m(\ddot{R} + \frac{1}{a^2}V^2R) = F, \tag{88}$$

where $V^2 = (V,V) = (\dot{R},\dot{R}), (R,F) = 0$. The physical space can be considered to be a hyperplane $(A,R) = 0$ in R^4. Hence it follows that the Euler equation for the spherical top can be written in the four-dimensional form:

$$I\dot{\Omega} = M, \tag{89}$$

where M is the moment of external forces, Ω is the vector of angular velocity, I is the moment of inertia of the top. By condition, the vectors Ω and M are orthogonal to the constant normalized vector A giving R^3, $(\Omega, A) = (M,A) = 0$, $(A,A) = 1$. For simplicity, the vector A can be taken in the form $A = e_4$. As the vector Ω belongs to $E^3(A)$ and $V \in E^3(N)$, we can establish a one-to-one correspondence between Ω and $V = \dot{R}$ by using the above-described mapping σ

$$a^2\Omega = (A,\dot{R})R - (A,R)\dot{R} + [A \times R \times \dot{R}], \tag{90}$$

$$\dot{R} = (R,\Omega)A - (R,A)\Omega + [R \times A \times \Omega]. \tag{91}$$

From (90) and (91) it follows that when R obeys equation (88), then Ω will obey equation (89) and vice versa, where $aF = (N,M)A - (N,A)M + [N \times A \times M]$ and

$$I = ma^2. \tag{92}$$

Thus, we may say that an intuitive image of a material point of the space of constant positive curvature is a rigid body; and from the above consideration it follows that the concept of a rigid body can be reduced to the concept of a material point. In some sense the rotation is the movement in fourth dimention. Then, a four-dimensional observer will perceive that, we perceive as rotation of a spherical top, as motion of a material point in four-dimensional space with an imposed holonomic constraint $(R, R) = a^2$.

7.3 The Dirac and Maxwell Equations in the Hyperbolic Space-Time

The consideration in subsect (7.2) was made within the framework of classical mechanics. We will complement it with the quantum-mechanical characteristic of the hyperbolic space-time. To this end we will make a comparative analysis of the Minkowski space-time and hyperbolic space-time in terms of the Dirac and Maxwell equations. It is convenient to begin this analysis in a general form without specifying the structure constants f^a_{bc} of simply transitive groups of transformations $M^4_{1,3}$ and $H^4_{1,3}$. In accordance with (80) and (48) we will write the Dirac equation in the form

$$\gamma^a D_a \psi = \mu\psi, \tag{93}$$

where $D_a = E_a + \frac{iea}{\hbar c}A_a - f_a$, $f_a = f^e_{ea}$. Since $[E_a, E_b] = f^c_{ab}E_c$ then $[D_a, D_b] = f^c_{ab}D_c + \frac{iea}{\hbar c}F_{ab}$, where

$$F_{ab} = E_a A_b - E_b A_a - f^c_{ab}A_c \tag{94}$$

are components of the electromagnetic field tensor in the basis E_a. The Jacobi identity $[D_a, [D_b, D_c]] + [D_b, [D_c, D_a]] + [D_c, [D_a, D_b]] = 0$ leads to the first four Maxwell equations

$$E_a F_{bc} + f^d_{ab}F_{cd} + E_b F_{ca} + f^d_{bc}F_{ad} + E_c F_{ab} + f^d_{ca}F_{bd} = 0.$$

Setting $\tilde{F}^{ab} = \frac{1}{2}e^{abcd}F_{cd}$, where e^{abcd} are components of the completely skew-symmetric unit Levi-Civita tensor with $e^{0123} = 1$, it is not difficult to verify that these equations can be written as follows

$$E_a\tilde{F}^{ab} + f_a\tilde{F}^{ab} + \frac{1}{2}f^b_{ad}\tilde{F}^{ad} = 0. \tag{95}$$

Hence, other Maxwell equations are of the form

$$E_a F^{ab} + f_a F^{ab} + \frac{1}{2}f^b_{ad}F^{ad} = \frac{4\pi a}{c}j^b, \tag{96}$$

where j^a are components of the current vector in the basis E_a. In the Minkowski space-time all structure constants are zero and therefore the Maxwell equations assume the form

$$E_a\tilde{F}^{ab} = 0\,,\; E_a F^{ab} = \frac{4\pi}{c}j^b.$$

This form simplifies rather complicated constructions in passing to curvilinear coordinates.

Now, we will proceed to the hyperbolic space-time where not all the structure functions equal zero and therefore we come to three-dimensional vector notation.Putting

$$j^a = (c\rho, \vec{j})\,,\; A_a = (\varphi, -\vec{A})\,,\; F_{0i} = e_i\,,\; \frac{1}{2}e_{ijk}F_{jk} = h_i,$$

we obtain in vector notation,

$$\vec{E} = -\partial_0\vec{A} - \text{ grad }\varphi\,,\; \vec{H} = \text{ rot }\vec{A},$$

where $\partial_0 = E_0, \nabla_i = E_i$. Considering that div $\vec{A} = \sum_{i=1}^3 \nabla_i A_i$, the Maxwell equations (95) and (96) assume a familiar form,

$$-\partial_0\vec{H} = \text{ rot }\vec{E}\,,\; \text{div }\vec{H} = 0\,,\; \text{rot }\vec{H} = \partial_0\vec{E} + \frac{4\pi a}{c}\vec{j}\,,\; \text{div }\vec{E} = 4\pi a\rho,$$

where a is the radius $H^4_{1,3}$. Using the commutation relations

$$[\nabla_i, \nabla_j] = 2e_{ijk}\nabla_k\,,\; i, j, k = 1, 2, 3$$

we can prove the validity of the known identities

$$\text{div rot } = 0,\; \text{rot grad } = 0.$$

Besides, we have

$$\text{div grad } = \Delta,$$

where Δ is the Laplacian on a three-dimensional sphere. Noncommutativity of the basis vector fields E_a can be seen not only in the definition of the operator rot but also in the identity

$$\text{rot rot } = -\Delta + \text{ grad div } - 2\text{ rot }.$$

In this connection it is important to derive the spectrum of the operator rot . We introduce Hermitian operators $M_i = \frac{i}{2}E_i + S_i$ and $N_i = \frac{i}{2}F_i$ where, as usual, $(S_i\vec{A})_j = -ie_{ijk}A_k$. Also there hold valid the following relations $M^2 = -\frac{1}{4}\Delta -$ rot $, N^2 = -\frac{1}{4}\Delta$ and $[M_i, \text{rot}\ ] = [N_i, \text{rot}\ ] = 0$, $[M_i, M_j] = ie_{ijk}M_k$, $[N_i, N_j] = ie_{ijk}N_k$. Thus,

$$2(M^2 + N^2) = -\Delta - 2\ \text{rot}$$

and

$$M^2 - N^2 = -\ \text{rot}\ .$$

Hence it follows that the spectrum of the operator rot is discrete and runs over the values $p = \pm 2, \pm 3, \pm 4, \cdots$. The spectrum does not start from zero as from the equations div $\vec{A} = 0$ and rot $\vec{A} = 0$ it follows that $\vec{A} = 0$ because the Betti number $b_1(S^3) = 0$.

Now we will continue the analysis of the Dirac equation. For the Dirac equation in $M^4_{1,3}$ we only mention the known fact that the description of spinor functions of a point in terms of curvilinear coordinates does not require any extra conceptions if we take into account parallelizability of the Minkowski space-time. Therefore, we will shortly characterize the Dirac equation in $H^4_{1,3}$, where

$$\mu = \frac{mca}{\hbar}.$$

Squaring (93) with the use of (86) we obtain

$$-D_a D^a \psi + P\psi + \frac{iea}{\hbar c} F_{ab} S^{ab} \psi = \mu^2 \psi,$$

where

$$S^{ab} = \frac{1}{4}(\gamma^a \gamma^b - \gamma^b \gamma^a), P = \Sigma_1 D_1 + \Sigma_2 D_2 + \Sigma_3 D_3$$

and $\Sigma_i = \frac{1}{2} e_{ijk} \gamma^j \gamma^k$. The operator P has properties similar to those of rot . Indeed, because

$$\Sigma_i \Sigma_j = -\delta_{ij} - e_{ijk}\Sigma_k,$$

$P^2 = -\Delta - 2P$ at $A_a = 0$. Introducing Hermitian operators $M_i = \frac{i}{2}E_i - \frac{i}{2}\Sigma_i$ and $N_i = -\frac{i}{2}F_i$, we obtain

$$M^2 = -\frac{1}{4}\Delta + \frac{1}{2}(P + \frac{3}{2}),\ N^2 = -\frac{1}{4}\Delta.$$

As a result,

$$2(M^2 + N^2) = -\Delta + (P + \frac{3}{2}), 2(M^2 - N^2) = P + \frac{3}{2}.$$

The above relations allow us to derive the following formula for energy levels when there is no electromagnetic field:

$$E^2 = m^2 c^4 + n^2 \frac{c^2 \hbar^2}{a^2}$$

with integer n. If we rewrite this formula in the form

$$E^2 = m^2 c^4 (1 + n^2 \frac{\lambda^2}{a^2}),$$

where $\lambda = \hbar\ mc$ is the Compton wavelength, then it follows from it that in the limit of large a, when $a \gg \lambda$,

$$E = mc^2 + n^2 \frac{\hbar^2}{2ma^2}$$

or

$$E = mc^2 + \frac{L^2}{2I},$$

where $L = n\hbar$ is the angular momentum and $I = ma^2$ is the moment of inertia. The last relation is consistent with the classical formula

$$E = \frac{L^2}{2I}$$

for the energy of a top.

Therefore, there are definite quantum-mechanical arguments for the hyperbolic space-time being a physical space-time. In this connection, we stress once more that here we are not talking about the replacement of the Minkowski space-time by the hyperbolic space-time for description of the same physical reality. On the contrary, we conclude that the hyperbolic space-time determines the laws of behavior of a new physical entity, the rotating particle or a top. The rotating particles have additional property, the moment of inertia, I, which is connected with their mass by the relation $I = ma^2$, where a is an "absolute length", that is "the trace of the rigid body". The notion of a rotating particle characterized not only by the mass, spin and charge but also the moment of inertia is very important because from the physical point of view the notion of a rigid body is not less fundamental than the concept of a material point. Here we must remark that the idea of a rotating particle is not new; the complete classification of particles with rigid internal structure was given by Finkelstein (1955).

7.4 Yang-Mills field and hyperbolic space-time

The seven-parameter group of transformations G_7 with generators

$$E_0, E_1, E_2, E_3, F_1, F_2, F_3$$

defined above is the group of space-time symmetry of hyperbolic space-time. As the energy vector E_0 commutes with all generators of the group G_7, then it remains unchanged under all the transformations of that group. Hence, it follows that the Lagrangian (74) in $H^4_{1,3}$ will be invariant under the transformations of group G_7 even if we do not switch off the interaction with the Yang-Mills field. From this result and those derived in subsects. (6.2), (6.3) and (6.4) it follows that the Yang-Mills field in the hyperbolic space-time should be of no less importance than the electromagnetic field. It is just this circumstance that allows us to understand the nature of strong interactions between rotating particles. Thus, the construction of quantum field theory in the hyperbolic space-time on the basis of the Lagrangian (74) is of fundamental importance for understanding the physical nature of the Yang-Mills field and hadron structure.

Note also that owing to the existence of two energy operators

$$H = -i\hbar\frac{\partial}{\partial x^0} \text{ and } \bar{H} = -i\hbar(\frac{\partial}{\partial x^0} + \frac{x^0}{a^2}x^C\frac{\partial}{\partial x^C})$$

we can question about the physical meaning of time, i.e. how time is connected with the nature of physical phenomena. Having just one energy operator $H = -i\hbar\frac{\partial}{\partial x^0}$ at our disposal, we could use the time conjugated to it only for describing the evolution of material objects as this time could be compared to nothing.

Let the proton be a top. Then we have

$$\frac{\hbar}{2} = I\omega = m_p a^2\omega,$$

where ω is the angular velocity of the proton. Since $a\omega = c$, where c is the light velocity, then for the diameter of gyration $d = 2a$ we have

$$d = \frac{\hbar}{m_p c} = \lambda_p.$$

If $a \to \infty$,then $\omega \to 0$. So, we may to say that electron dosn't rotate. That is why there is no separation between the Minkowski space-time and such entity as electron.

7.5 Coulomb law in the hyperbolic space-time

Our goal here is to find the form of the Coulomb law in the $H^4_{1,3}$. For a constant electric field we have

$$\text{div } \vec{E} = 4\pi a\rho,\ \vec{E} = -\frac{1}{a} \text{ grad } \varphi$$

and consequently φ obeys the equation

$$\Delta\varphi = -4\pi a^2\rho.$$

An invariant of rotations on a three-dimensional sphere is either the arc length or angle between the radii

$$\cos\theta = \frac{1}{a^2}(x^1y^1 + x^2y^2 + x^3y^3 + x^4y^4).$$

Setting $\rho = 0, \varphi = \varphi(\theta)$ we get the equation

$$\frac{d^2\varphi}{d\theta^2} + 2\cot\theta\frac{d\varphi}{d\theta} = 0$$

for $\varphi(\theta)$ .The general solution of this equation has the form

$$\varphi(\theta) = c_1\cot\theta + c_2,$$

where c_1 and c_2 are constants.

Let us consider the stereographic projection of a three-dimensional sphere $(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = a^2$ onto the hyperplane $x^4 = 0$ from the point $(0, 0, 0, a)$. Let x, y, z are Cartesian coordinate of the space which we identify with the hyperplane $x^4 = 0$. In this case we have

$$x^1 = x\frac{2a^2}{r^2 + a^2},\ x^2 = y\frac{2a^2}{r^2 + a^2}$$

$$x^3 = z\frac{2a^2}{r^2 + a^2},\ x^4 = a\frac{r^2 - a^2}{r^2 + a^2},$$

where $r^2 = x^2 + y^2 + z^2$. If the charged rotating particle is at the point $(0, 0, 0, -a)$ then we have

$$\cot\theta = \frac{a}{2r} - \frac{r}{2a}.$$

When $r \geq a$ we should have the usual Coulomb potential

$$\varphi(r) = -\frac{\alpha}{r}.$$

In order to have φ continuous,it is then necessary to set $c_2 = -\frac{\alpha}{a}$. Hence, for $r < a$, φ has the form

$$\varphi(r) = (-\frac{\alpha}{r} + \frac{\alpha r}{a^2})\beta - \frac{\alpha}{a},$$

where β is a dimensionless constant. Thus, we have found the Coulomb potential for a charged rotating particle.

We see that the notions of hyperbolic space-time and a rotating particle leads to the change of the laws of electrodynamics at short distances and gives the exact form of these laws. In this connection it must be noted that the Cornel potential(Eichten et al., 1980) has a more profound meaning than usually suggested. It is quite possible that evidences for the existence of tops were already obtained in the Krisch experiment with polarized protons. So the continuation of this experiment is very important and has the principal meaning. This experiment must show whether the proton is a single rotating particle or it consists of rotating particles with the laws of interactions determined above. It is very interesting that the angular velocity of the proton may vary without changing the spin and this should have important consequences.

7.6 Concluding remarks

The theory of fundamental interactions presented here is based on the assumption of an intimate relation between the properties of space-time and phenomena in reality. The theory has obviously two aspects. One of them is mathematical and has an absolute character. The other aspect is the development of consequences from the mathematical formalism and their comparison with experiment. In view of the latter, it is of fundamental importance to establish the physical meaning both of the principal gauge field and of the Yang-Mills gauge field. The latter is localized in a region of rotation beyond which the strength of that field is zero. However, the connection (or a gauge potential) of the Yang-Mills field beyond that region should not be zero. So, we may assume that the physics of mesons (pions,rhos,omegas,etc.) is determined by the Yang-Mills gauge potential having zero strength.

It must be noted that the rotating particles do not need confinement because confinement simply means that it rotates. We would like to note also that the concept of the string has a nonstandard interpretation in the framework of the developed theory but this question cannot be treated in this paper.

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