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ON THE SPHERICAL-SYMMETRIC METRIC ON THE BACKGROUND OF THE LOBACHEVSKY SPACE

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О сферически-симметричной метрике на фоне пространства Лобачевского

Исследуется сферически-симметричное решение уравнений гравитационного поля в пространстве Лобачевского. Показано, что в этом случае аналога теоремы Биркгофа не существует, но особенность типа Шварцшильда может присутствовать.

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Tentyukov M.N. E2-92-515

On the Spherical-Symmetric Metric on the Background of the Lobachevsky Space

Spherically symmetric solution of the gravitational equations on the background of the Lobachevsky space is found. It is shown that in this case there is no an analog of the Birkhoff theorem, but the Schwarzschild-like peculiarity can be present.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

Contemporary theory of gravitation is the Einstein General Relativity (GR). This theory describes the gravitation as a space-time metric g_{mn} which satisfies the Einstein equations

$$G_{mn} = \kappa T_{mn},$$

where $G_{mn} = R_{mn} - \frac{1}{2} R\, g_{mn}$ is the Einstein tensor.

These equations are very difficult and nonlinear. For investigation of these equations it would be useful to find the action functional.

Usually, the vacuum Einstein equations

$$G_{mn} = 0 \tag{1}$$

are derived from the Hilbert action

$$S_H = \int \sqrt{-g}\, R\, d^4x. \tag{2}$$

These equations are of the second order, and the Hilbert Lagrangian

$$L_H = \sqrt{-g}\, R \tag{3}$$

contains the second - order derivatives too. This leads to the known difficulties [1]. To avoid them, Gibbons and Hawking have suggested the surface term [1], but due to this term local gravitational invariants such as energy-momentum density became *quasi*local [2].

Another way consist in finding a suitable Lagrangian which will be local and contains only first-order derivatives. For a Lagrangian like that to exist, it is necessary to introduce the background object in the theory [3-5]. It should be mentioned that the well-known Einstein Lagrangian $L_E = \sqrt{-g}\, g^{mn}\left(\Gamma^a_{mb}\, \Gamma^b_{an} - \Gamma^a_{ba}\, \Gamma^b_{mn}\right)$ contains the background affine connection [6-8] whose coefficients $\check{\Gamma}^a_{mn}$ are zero in a chosen coordinate map (see sect.2).

Introduction of the background connection permits us to expand the GR by admitting a more general (nonflat) background connection. By comparing such an expanded theory with GR some interesting specific features of the Einstein equations can be found [7]. It would be interesting to find spherical-symmetric solutions and compare them with the Schwarzschild metric.

In the present paper, we shall investigate this solution of the gravitational equations on the background of the Riemannian space whose spatial submanifold is the three-dimensional Lobachevsky space. We shall see that in this case there is no an analog of the Birkhoff theorem but the Schwarzschild-like peculiarity can be present.

2. THE GRAVITATIONAL ACTION FUNCTIONAL AND EQUATIONS OF MOTION

Usually, equations (1) are derived from the Hilbert action (2) with the Hilbert Lagrangian (3). As it has been remarked above, this Lagrangian leads to the known difficulties.

Instead of (3) the noncovariant Einstein Lagrangian is often used

$$L_E = \sqrt{-g}\; g^{mn}(\Gamma^a_{mb}\Gamma^b_{an} - \Gamma^a_{sa}\Gamma^s_{mn}),$$

which differs from L_H by the divergence term

$$L_H - L_E = \partial_l \omega^l \tag{4}$$

where

$$\omega^l = \sqrt{-g}\;(g^{ln}\Gamma^m_{mn} - g^{mn}\Gamma^l_{mn}).$$

Now let us prove that noncovariance of L_E in fact means that the background object is present in the theory [6-8]. It is the affine connection without torsion. We shall denote the background connection coefficients by $\breve{\Gamma}^k_{mn}$.

The difference between the connection coefficients

$$P^k_{mn} = \breve{\Gamma}^k_{mn} - \Gamma^k_{mn}$$

is a tensor. It is named the affine - deformation tensor.

Let us consider the Lagrangian

$$\tilde{L} = \sqrt{-g}\; g^{mn}(P^a_{mb}P^b_{an} - P^a_{sa}P^s_{mn}).$$

For the action functional

$$\tilde{S} = \int \tilde{L}\, d^4x$$

the variational derivative

$$\tilde{\Psi}^{mn} = 2\,\frac{\delta\tilde{S}}{\delta g_{mn}}$$

has been calculated in [5]

$$\tilde{\Psi}^{mn} = \sqrt{-g}\; g^{ma}g^{nb}(\breve{R}_{ab} + \breve{R}_{ba} - \breve{R}_{ij}g^{ij}g_{ab} - 2G_{ab}), \tag{5}$$

where $\breve{R}_{ik} = \breve{R}^p_{pik}$ is the Ricci tensor; $\breve{R}^p_{lik} = \partial_i\breve{\Gamma}^p_{lk} - \partial_l\breve{\Gamma}^p_{ik} + \breve{\Gamma}^p_{ls}\breve{\Gamma}^s_{ik} - \breve{\Gamma}^p_{is}\breve{\Gamma}^s_{lk}$ is the Riemann tensor for the background connection. If $\breve{R}_{(ik)}=0$, then the equations

$$\tilde{\Psi}^{mn} = 0 \tag{6}$$

coincide with the Einstein equations (1), and

$$L_H - \tilde{L} = \breve{\nabla}_l F^l, \tag{7}$$

where $\breve{\nabla}_l$ is a covariant derivative with respect to the background connection and

$$F^l = \sqrt{-g}\;(g^{mn}P^l_{mn} - g^{ln}P^m_{mn})$$

is the vector density of weight one.

If $\breve{R}^l_{klm} = 0$, one can choose the coordinate map in which all $\breve{\Gamma}^l_{km} = 0$. Then, P^l_{km} turns into $-\Gamma^l_{km}$, $\tilde{L}$ turns into L_E, F^l into ω^l and (7) is transformed into (4).

Since L_E is noncovariant, converting L_H into L_E can be possible only after fixation of the coordinate map. Converting L_H into L_E by formula (4) is in fact converting L_H into $\tilde{L}$ with the fixation of the background connection whose coefficients in this map are assumed to be zero. Hence, it follows that in this theory it is necessary to use the Lagrangian $\tilde{L}$.

3. DEGENERATION OF EQUATIONS AND THE HARMONICITY CONDITIONS.

It is well-known that the Einstein equations (1) are degenerated in the sense that for ten unknown components of the metric tensor there are only six independent equations due to the Bianchi identities. This results from the invariance of (1) under the group of diffeomorphisms *Diff(M)* of the four-dimensional manifold *M*.

Degeneration of the equations leads to the existence of the functional arbitrariness in their solutions. To avoid this arbitrariness the suitable noncovariant conditions are used. Very popular are the "harmonicity conditions" $\partial_i(\sqrt{-g}\, g^{ij}) = 0$. Since we have separated the generally coordinate transformations and *Diff(M)*, they can be written in the covariant form

$$\check{\nabla}_i \mathfrak{g}^{ij} = 0. \tag{8}$$

Here $\mathfrak{g}^{ij} = \sqrt{-g}\, g^{ij}$ is the contravariant metric density. It is (8) that we will call the harmonicity conditions.

As it is clear from (5), the conditions $\check{R}^i_{klm} = 0$ are stronger than the necessary conditions for deriving the Einstein equations. If we put

$$\check{R}_{(ij)} = 0 \tag{9}$$

it would be enough. Although we cannot choose a coordinate map with nonzero $\check{\Gamma}^i_{km}$, the latter is not contained in the gravitational equations (6) that coincide with the Einstein ones.

If $\check{R}_{(ik)} \neq 0$, then the gravitational equations differ from the Einstein ones and in the general case degeneration is absent. But for some special background connections, degeneration can be present. It can be shown [7] that for equations to have these properties it is necessary that $\check{R}_{(mn)}$ should have such a null vector ξ

$$\xi^i \check{R}_{(ij)} = 0$$

that the Lie derivative of $\check{R}_{(ij)}$ should vanish

$$\underset{\xi}{\pounds}\check{R}_{(ij)} = 0.$$

If the background connection satisfies (9), then equations (6) coincide with the Einstein one. The harmonicity conditions are not connected with the equations and must be postulated as

external conditions. But if the background space is curved, then in the general case the gravitational equations differ from the Einstein one. If these equations are not degenerated, then (8) must be either consequences of (6) or inconsistent with them.

It can be shown [7] that the harmonicity conditions (8) follow from equations (6) only if the background space is the Einstein space. More exactly, the following statements are true:

1. For (8) to be consequences of (6),

a) it is necessary that the symmetric part of the Ricci tensor should satisfy the condition $\check{\nabla}_j \check{R}_{(ik)} = 0$;

b) it is necessary and sufficient that $\check{\nabla}_j \check{R}_{(ik)} = 0$ and $\det(\check{R}_{(ik)}) \neq 0$. In this case, (8) are equivalent to the conditions

$$\nabla_m \tilde{\Psi}^{mn} = 0. \tag{10}$$

2. If $\check{\nabla}_j \check{R}_{(ik)} = 0$ and $\det(\check{R}_{(ik)}) = 0$, then (8) is compatible with (6), but (6) without (8) remains degenerated. It means that

- the functional arbitrariness remains in the solutions;
- (8) can be postulated as external, but they are not consequences of (6).

The contents of 1 b) is equivalent to the statement that the background connection can be compatible with a certain metric of the Einstein space. In other words, $\check{R}_{(ij)} = \lambda\, \check{g}_{ij}$ where $\lambda \neq 0$ and $\check{g}_{ij}$ is a metric tensor with an *arbitrary* signature.

In [9] it has been shown that if the background space has a constant curvature, then (10) coincides with (8). In [10] it has been found that the coincidence of (10) with (8) takes place if $\check{\nabla}_j \check{R}_{(ik)} = 0$ and $\det(\check{R}_{(ik)}) \neq 0$. The general case has been investigated in [7].

4. THE EXACT SPHERICAL-SYMMETRIC SOLUTION

Let us consider the background connection which is the Christoffel connection derived from the background metric

$$d\breve{s}^2 = \breve{g}_{ij}dx^i dx^j = c^2dt^2 - h_{\alpha\beta}dx^\alpha dx^\beta.$$

Latin indices run from 0 to 3, Greek indices denote spatial components 1, 2, 3. Let $h_{\alpha\beta}$ be static:

$$\frac{\partial h_{\alpha\beta}}{\partial t} \equiv 0. \tag{11}$$

Tanks to (11) there is a Killing vector

$$\xi = \frac{\partial}{\partial t}$$

and, consequently, ξ forms the Ricci collineation:

$$\underset{\xi}{\pounds}\, \breve{R}_{ij} = 0.$$

In the component form

$$\xi^i = \{1, 0, 0, 0\}.$$

The background Ricci tensor is diagonal and

$$\breve{R}_{00} = 0.$$

Hence, $\xi^i \breve{R}_{ij} = 0$ and according to sect.3 equations (6) are degenerated. The solutions of equations (6) must be invariant under the transformations

$$g_{mn}(x) \longrightarrow \bar{g}_{mn}(x) \tag{12}$$

generated by the diffeomorphism

$$x^j = x^j(\bar{x}) \tag{13}$$

where

$$x^\alpha = \bar{x}^\alpha; \quad x^0 = f(\bar{x}^i).$$

Here $f(\bar{x})$ is an arbitrary function of the coordinates $\bar{x}^i$ which satisfy $f \equiv \bar{x}^0$ outside of the compact area.

Let us find these transformations. From the general tensor transformation law we have

$$\bar{g}_{mn}(\bar{x}) = \frac{\partial x^a}{\partial \bar{x}^m}\frac{\partial x^b}{\partial \bar{x}^n}\, g_{ab}\big(x(\bar{x})\big) \tag{14}$$

where derivatives $\dfrac{\partial x^a}{\partial \bar{x}^m}$ are determined from (13). To obtain (12)

we must replace $\bar{x}$ by x in the left-hand side of (14).

Now let us find $h_{\alpha\beta}$ such that never new degenerations would appear.

It is clear that $\det(\breve{R}_{\alpha\beta})$ must be nonzero. Consequently, a suitable choice is the Einstein space. Since $h_{\alpha\beta}$ is a three-dimensional metric, this space is a space of constant curvature. A very simple space like that is the Lobachevsky space. That is why we consider the background metric

$$d\breve{s}^2 = c^2dt^2 - dr^2 - k^2\sinh^2\frac{r}{k}\,(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{15}$$

where k is a constant, and will search for spherically symmetric static metric

$$ds^2 = V^2dt^2 - F^2dr^2 - H^2(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{16}$$

satisfying equations (6) which now can be written in the form

$$R_{ij} = \breve{R}_{ij}. \tag{17}$$

In (16) V, F, and H are the functions of r. We assume

$$x^0 = t,\ x^1 = r,\ x^2 = \theta,\ x^3 = \varphi.$$

The metric (15) has been considered in [11].

In our case, the alternative 2) from sect.3 takes place. But when we claim the metric be (16), the single "free" harmonicity condition $\breve{\nabla}_i g^{i0} = 0$ is satisfied automatically and the functional arbitrariness is absent. Indeed, from (14) we have the diagonal components

$$\bar{g}_{pp} = \frac{\partial x^a}{\partial \bar{x}^p}\frac{\partial x^b}{\partial \bar{x}^p}\, g_{ab}$$

For these components be independent of t it is necessary that

$$x^0 = A\bar{x}^0 + f(\bar{r}) \tag{18}$$

where A is a constant. But the nondiagonal term appears

$$\bar{g}_{01} = \bar{g}_{10} = \frac{\partial x^0}{\partial \bar{x}^1}\frac{\partial x^0}{\partial \bar{x}^0}\, g_{00}.$$

This term contradicts (16). For it to vanish, we should have $\dfrac{\partial x^0}{\partial \bar{x}^1} = 0$ and (18) should take the form

$$x^0 = A\bar{x}^0 + B \tag{19}$$

where B is a constant. Thus, the functional arbitrariness disappears and only a trivial linear (19) remains.

All nondiagonal components of both R_{ij} and $\breve{R}_{ij}$ are zero, $R_{33} = R_{22}\sin^2\theta$ and $\breve{R}_{33} = \breve{R}_{22}\sin^2\theta$, and we are only left with three equations for three unknown functions V, F and H:

$$R_{00} = \frac{VV'}{F^2}\left(\frac{F'}{F} + \frac{2H'}{H} - \frac{V'}{V}\right) + \frac{d}{dr}\left(\frac{VV'}{F^2}\right) = 0, \tag{20}$$

$$R_{11} = -\frac{V''}{V} - \frac{2H''}{H} + \frac{F'}{F}\left(\frac{V'}{V} + \frac{2H'}{H}\right) = -\frac{2}{k^2}, \tag{21}$$

$$R_{22} = 1 - \frac{1}{VF}\frac{d}{dr}\left(V\frac{HH'}{F}\right) = -2\sinh^2(r/k). \tag{22}$$

Here $'$ means $\frac{d}{dr}$.

Now the harmonicity condition $\breve{\nabla}_i g^{ii} = 0$

$$\frac{d}{dr}\left(\frac{VH^2}{F}\right) = VF\,k\sinh(2r/k) \tag{23}$$

is a consequence of all equations (20)-(22). We use (23) and the combination

$$\frac{1}{2}H\left(R_{11} + \frac{F^2}{V^2}R_{00}\right) = H'\frac{(VF)'}{VF} - H''.$$

Instead of (20)-(22) we obtain

$$H'' - H/k^2 = H'\frac{(VF)'}{VF}, \tag{24}$$

$$\frac{d}{dr}\left(\frac{VH^2}{F}\right) = VF\,k\sinh(2r/k), \tag{25}$$

$$\frac{d}{dr}\left(V\frac{HH'}{F}\right) = VF\cosh(2r/k). \tag{26}$$

Let us denote

$$\frac{HH'}{F^2} = \alpha, \quad \frac{H^2}{F^2} = \beta, \quad \frac{H'}{H} = \gamma, \quad \frac{(VF)'}{VF} = w. \tag{27}$$

Using (27) we write down (24)-(26) in the form

$$\gamma' + \gamma^2 - \gamma w = 1/k^2, \tag{28}$$

$$\beta' + \beta w = k\sinh(2r/k), \tag{29}$$

$$\alpha' + \alpha w = \cosh(2r/k), \tag{30}$$

$$\alpha = \beta\gamma. \tag{31}$$

Let w be considered as a parameter. Equations (29) and (30) are the linear equations, and therefore, can be easily solved. But equation (28) is the Riccati equation. We can't find its general solution. If we substitute α and β into (31) and then substitute γ in terms of w into (28), we obtain a nonlinear integro-differential equation. This equation cannot be solved too.

But we are able to find a particular solution of our system which correspond to $w = 0$. Indeed, if $w = 0$, then (28) becomes the equation with separated variables. From (29), (30) we have

$$\alpha = (k/2)\left(\sinh(2r/k) + a/2\right),$$

$$\beta = (k^2/2)\left(\cosh(2r/k) + b/2\right),$$

where a and b are the constants of integration. By separating variables in (28) we get

$$\frac{d\gamma}{1/k^2 - \gamma^2} = dr. \tag{32}$$

By integrating (32) we obtain

$$(k/2)\ln\left|\frac{1+\gamma k}{1-\gamma k}\right| = r + r_0 \tag{33}$$

where r_0 is the constant of integration.

Let us denote

$$D = \exp(r_0/k).$$

From (33) we have two branches. The first is

$$\frac{\gamma k + 1}{\gamma k - 1} = D^2\exp(2r/k); \tag{34}$$

$$\gamma < -(1/k);\ \gamma > (1/k) \tag{35}$$

and the second is

$$\frac{1+\gamma k}{1-\gamma k} = D^2\exp(2r/k); \tag{36}$$

$$-(1/k) < \gamma < 1/k. \tag{37}$$

In terms of hyperbolic functions we have from (34)

$$\gamma_1 = (1/k)\tanh^{-1}\left((r + r_0)/k\right)$$

and from (36)

$$\gamma_2 = (1/k)\ \tanh\big((r + r_0)/k\big). \tag{38}$$

As we can see, (35) and (37) are satisfied. But we must satisfy (31). Substituting γ, α and β into (31) we obtain

$$\alpha_1 = (k/2)\big(\sinh(2r/k) - \sinh(2r_0/k)\big),$$

$$\beta_1 = (k^2/2)\big(\cosh(2r/k) - \cosh(2r_0/k)\big),$$

$$\gamma_1 = (1/k)\ \tanh^{-1}\big((r + r_0)/k\big)$$

for the first branch, and

$$\alpha_2 = (k/2)\big(\sinh(2r/k) + \sinh(2r_0/k)\big),$$

$$\beta_2 = (k^2/2)\big(\cosh(2r/k) + \cosh(2r_0/k)\big),$$

$$\gamma_2 = (1/k)\ \tanh\big((r + r_0)/k\big)$$

for the second. Using definitions (27) we can obtain

$$H_1 = P\sinh\frac{r + r_0}{k};\quad F_1^2 = \frac{P^2}{k^2}\,\frac{\sinh\frac{r + r_0}{k}}{\sinh\frac{r - r_0}{k}};\quad V_1^2 = \frac{Q^2}{F_1^2},$$

$$H_2 = P\cosh\frac{r + r_0}{k};\quad F_2^2 = \frac{P^2}{k^2}\,\frac{\cosh\frac{r + r_0}{k}}{\cosh\frac{r - r_0}{k}};\quad V_2^2 = \frac{Q^2}{F_2^2}$$

where P and Q are the constants of integration. Therefore, the two branches of the solution are

$$ds_1^2 = \frac{k^2\sinh\frac{r - r_0}{k}}{P^2\sinh\frac{r + r_0}{k}}Q^2dt^2 - \frac{P^2\sinh\frac{r + r_0}{k}}{k^2\sinh\frac{r - r_0}{k}}dr^2 - P^2k^2\sinh^2\frac{r + r_0}{k}\,d\Omega^2,$$

$$ds_2^2 = \frac{k^2\cosh\frac{r - r_0}{k}}{P^2\cosh\frac{r + r_0}{k}}Q^2dt^2 - \frac{P^2\cosh\frac{r + r_0}{k}}{k^2\cosh\frac{r - r_0}{k}}dr^2 - P^2k^2\cosh^2\frac{r + r_0}{k}\,d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$. By demanding the metric g_{mn} to have the same asymptotics as $\breve{g}_{mn}$, we get $P^2 = k^2\exp(-2r_0/k)$, $Q^2 = c^2$. If we denote

$$\Lambda_1 = \exp(2r_0/k)\,\frac{\sinh\frac{r - r_0}{k}}{\sinh\frac{r + r_0}{k}},$$

$$\Lambda_2 = \exp(2r_0/k)\,\frac{\cosh\frac{r - r_0}{k}}{\cosh\frac{r + r_0}{k}},$$

two solutions of (24)-(26) can be written

$$ds_1^2 = \Lambda_1 c^2dt^2 - \Lambda_1^{-1}dr^2 - \exp(-2r_0/k)\ k^2\sinh^2\frac{r + r_0}{k}\,d\Omega^2, \tag{39}$$

$$ds_2^2 = \Lambda_2 c^2dt^2 - \Lambda_2^{-1}dr^2 - \exp(-2r_0/k)\ k^2\cosh^2\frac{r + r_0}{k}\,d\Omega^2. \tag{40}$$

The metric (39) has at first been found in [11].

For determining r_0 we postulate that the asymptotic behavior of g_{mn} must lead to the Newton gravitational law in the Lobachevsky space [12]. It leads to

$$\sinh\frac{2r_0}{k} = \frac{2\gamma M}{kc^2}.$$

Here γ is the Newton constant, M is the mass of the central source. If $r_0 << k$, then

$$r_0 \approx \frac{\gamma M}{c^2}$$

is the ordinary Schwarzschild radius.

The metric (39) is similar to the Fock metric

$$ds^2 = \frac{r - r_0}{r + r_0}\,c^2dt^2 - \frac{r + r_0}{r - r_0}\,dr^2 - (r + r_0)^2d\Omega^2 \tag{41}$$

and turns into (41) when $k \longrightarrow \infty$. As (41), (39) is singular when $r = r_0$.

But the second solution (40) doesn't have the Einstein limit if $k \longrightarrow \infty$. The solution (38) corresponding to (40) has the limit

$\lim_{k \to \infty} \gamma_2 = 0$ which is inconsistent with (24). On the other hand, it is (40) that violates the analogy of the Birkhoff theorem.

Notice that if $r_o \longrightarrow 0$, then (39) is turns into (15), but (40) remains

$$ds^2 = c^2dt^2 - dr^2 - k^2\cosh^2(r/k)\ d\Omega^2 \qquad (42)$$

as a solution of (17) in an absolutely empty space.

Radial movement of a photon may be determined from (16) by $dS = d\Omega = 0$. For the radial photon velocity v we obtain

$$v = \frac{dr}{dt} = \pm\frac{V}{F}. \qquad (43)$$

Then, we can get the law $r = r(t)$ by integrating (43). The time of radial motion τ from the initial radius R_{init} to $r < R_{init}$ is determined by the expression

$$\int_{\tau_{init}}^{\tau} t\ dt = -\int_{R_{init}}^{r} \frac{F}{V}\ dr$$

where τ_{init} is the time corresponding the photon position with $r = R_{init}$.

According to these formulas we have for (39):

The radial photon velocity

$$v_1 = dr/dt = \pm c\ \exp(2r_o/k)\ \frac{\sinh\frac{r - r_o}{k}}{\sinh\frac{r + r_o}{k}};$$

the time of radial motion from R_{init} to $r < R_{init}$

$$\tau_1 = A\ \ln\frac{\exp(2R_{init}/k) - \exp(2r_o/k)}{\exp(2r\ /k) - \exp(2r_o/k)} + B\ \frac{R_{init} - r}{c} + \tau_{init}$$

where $B = \exp(-\ 4r_o/k)$, $A = k(1 - B)/(2c)$. For (40) the same values are:

The radial photon velocity

$$v_2 = dr/dt = \pm c\ \exp(2r_o/k)\ \frac{\cosh\frac{r - r_o}{k}}{\cosh\frac{r + r_o}{k}};$$

the time of radial motion from R_{init} to $r < R_{init}$

$$\tau_2 = A\ \ln\frac{\exp(2R_{init}/k) + \exp(2r_o/k)}{\exp(2r\ /k) + \exp(2r_o/k)} + B\ \frac{R_{init} - r}{c} + \tau_{init}.$$

If $R_{init} \sim r >> r_o$, both τ_1 and τ_2 give the ordinary expression

$$\tau - \tau_{init} = (1 - B)\frac{R_{init} - r}{c} + B\ \frac{R_{init} - r}{c} = \frac{R_{init} - r}{c}.$$

As we can see, to arrive at $r = r_o$ the photon in (39) needs infinite time, but in (40) this time is finite. If $r \longrightarrow r_o$ then $v_1 \longrightarrow 0$, but from (40) we have monotonous increasing from $\lim_{r \to \infty} v_2 = c$ to $v_2 = c\ \exp(2r_o/k)$ if $r = 0$.

The physical time t_r which is at the point with the radial coordinate r is determined by the relations

$$dt_r = \frac{\sqrt{g_{oo}}}{c}\ dt = \begin{cases} \exp(-r_o/k)\left(\dfrac{\sinh\frac{r - r_o}{k}}{\sinh\frac{r + r_o}{k}}\right)^{1/2} dt & \text{for (39)},\\[2ex] \exp(-r_o/k)\left(\dfrac{\cosh\frac{r - r_o}{k}}{\cosh\frac{r + r_o}{k}}\right)^{1/2} dt & \text{for (40)}. \end{cases}$$

If $r \longrightarrow \infty$ we have

$$dt_r = dt$$

for both the metrics, but if $r \longrightarrow r_o$ then the physical time in (39) stops.

The physical speed of light is

$$\frac{dx_r}{dt_r} = \pm\frac{\sqrt{-g_{11}}\ dr}{\left(\sqrt{-g_{oo}}\ /\ c\right)\ dt} = \pm c$$

for both the metrics.

On the contrary, we can assume that the gravitation is an optical medium with the refraction coefficient different from 1 and the physical time interval dp_r is

$$dp_r = \frac{\sqrt{\breve{g}_{00}}}{c}\, dt = dt$$

in the spirit of the bimetric theories.

Notice that such an interpretation is possible only if we consider the background metric. But as has been shown, both equations and the Lagrangian contain only the background connection. That is why the last interpretation may contain certain arbitrariness. Just in our case the constant c is not contained in $\breve{\Gamma}^{l}_{jk}$, and for different c we get the same $\breve{\Gamma}^{l}_{jk}$. The result Q = c was obtained only due to demands of the asymptotic behavior of g_{mn}. The information about the background metric, contained in the background connection, is not complete and, hence, the background metric in our case is not observable.

5. Concluding Remarks

Although there was a number of investigations of the non - Einstein equations (6), their spherical-symmetric solutions as a rule were explored only as approximate ones. Apparently, integrability of the Einstein equations is closely connected with their degenerating. It seems to be probable that we have succeeded in finding the solution only due to degeneration of the equations.

In [13] N.Rosen has considered the case when the background space has a constant curvature. This theory was developed to remove the singularities existing in GR. Rosen succeeded in showing that in his case the gravitational field differs from the Schwarzschild field only very close to, and inside, the Schwarzschild sphere. The interior of this sphere is unphysical and impenetrable. It was suggested that the elementary particles may have the similar structure.

We have considered the solution which also may be interesting from the point of view of the elementary particle theory. In spite of the presence of the "classical" Schwarzschild peculiarity, the existence of two branches of the solution may be useful for constructing the models explaining some properties of the behavior of elementary particles.

There is the very strange solution (42) whose physical properties should be unusual. Since the asymptotics of (42) is the same as of (15), for a remote observer the metric (42) would seem to be the same as the metric (15), but close to r = 0 the metric (42) essentially differs from (15).

References

[1] C.W Gibbons, S.W.Hawking, *Phys. Rew.* **D15**, 2752 (1977).

[2] R.F.Polishcuk, *Positivity of Total Energy and Initial Value Problem in Gravitation* in: *On Relativity Theory*, World Scientific, Singapore 1985, v.2, pp. 153-160.

[3] N.A.Chernikov, Preprint JINR E2-92-394, Dubna 1992.

[4] N.A.Chernikov, Preprint JINR P2-88-27, Dubna 1988, in Russian.

[5] N.A.Chernikov, Preprint JINR P2-88-778, Dubna 1988, in Russian.

[6] M.N.Tentyukov, *Acta Phys. Pol.* **B20**, 911 (1989).

[7] M.N.Tentyukov, Preprint JINR E2-92-439, Dubna 1992

[8] M.N.Tentyukov, Preprint JINR E2-92-491, Dubna 1992

[9] N.Rosen, *Found. Phys.* **15**, 997 (1985).

[10] N.A.Chernikov, Communication JINR P2-90-399, Dubna 1990, in Russian.

[11] N.A.Chernikov, Comm. JINR P2-92-108, Dubna 1992, in Russian.

[12] N.A.Chernikov, *Acta Phys. Pol.* **B23**, 115 (1992).

[13] N.Rosen, *Found. Phys.* **10**, 673 (1980).

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