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A METHOD FOR OBTAINING QUANTUM DOUBLES FROM THE YANG-BAXTER *R*-MATRICES

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Метод получения квантовых дублей из *R*-матриц Янга — Бакстера

В работах [1] и [2] был предложен метод, позволивший сопоставить каждому регулярному обратимому решению уравнений Янга — Бакстера без спектрального параметра некоторую квазитреугольную алгебру Хопфа. Мы развиваем этот подход и показываем, что эта алгебра Хопфа есть на самом деле квантовый дубль.

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A Method for Obtaining Quantum Doubles from the Yang-Baxter *R*-Matrices

We develop the approach of refs. [1] and [2] that enables one to associate a quasitriangular Hopf algebra to every regular invertible constant solution of the quantum Yang-Baxter equations. We show that such a Hopf algebra is actually a quantum double.

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1. It is well known [1] that any invertible constant matrix solution R of the quantum Yang-Baxter equation (QYBE)

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \tag{1}$$

naturally generates a bialgebra $A_R = \{1, t_{ij}\}$ defined by

$$R_{12}T_1T_2 = T_2T_1R_{12}, \quad \Delta(T_1) = T_1 \otimes T_1, \quad \varepsilon(T) = \mathbf{1},$$

(generators t_{ij} form a matrix T, Δ is a coproduct and ε a counit) and also another bialgebra $U_R = \{1, l^+_{ij}, l^-_{ij}\}$ with

$$R_{12}L^\pm_2 L^\pm_1 = L^\pm_1 L^\pm_2 R_{12}, \tag{2}$$

$$R_{12}L^+_2 L^-_1 = L^-_1 L^+_2 R_{12}, \tag{3}$$

$$\Delta(L^\pm_1) = L^\pm_1 \otimes L^\pm_1 \quad , \quad \varepsilon(L^\pm) = \mathbf{1}, \tag{4}$$

which is paired to A_R. This pairing [1, 2] is established by the relations

$$< T_1, L^+_2 >= R_{12}, \quad < T_1, L^-_2 >= R^{-1}_{21},$$

obeys the duality conditions

$$< \alpha\beta, a >=< \alpha \otimes \beta, \Delta(a) >, \quad < \Delta(\alpha), a \otimes b >=< \alpha, ab >,$$

and appears to be degenerate. With some additional effort (quotienting by appropriate null bi-ideals) these bialgebras can be made Hopf algebras $\check{A}_R$ and $\check{U}_R$, dual to each other. Their antipodes are defined by

$$< T_1, S(L^+_2) > = < S(T_1), L^+_2 >= R^{-1}_{12},$$

$$< T_1, S(L^-_2) > = < S(T_1), L^-_2 >= R_{21}.$$

With essential use of this duality Majid [2] showed that in fact, with a certain reservation, $\check{U}_R$ proves to be a quasitriangular Hopf algebra with the universal $\mathcal{R}$-matrix given by implicit formulas originated from $< T_1 \otimes T_2, \mathcal{R} >= R_{12}$. By the way, Majid claims [2] that $\check{U}_R$ is 'more or less' of the form of a quantum double. In the present note we argue that, modulo the same reservation, $\check{U}_R$ is actually a quantum double.

2. Recall t(i.e. the duality with opposite coproduct and inverse antipode) means $< e$

$^i, e_j >= \delta^i_j$ and

$$< \alpha\beta, a >=< \alpha \otimes \beta, \Delta(a) >, \quad < \Delta(\alpha), a \otimes b >=< \alpha, ba >, \tag{5}$$

$$\varepsilon(a) =< 1, a >, \quad \varepsilon(\alpha) =< \alpha, 1 >, \quad < S(\alpha), a >=< \alpha, S^{-1}(a) >,$$

where $a, b \in A$, $\alpha, \beta \in A^\circ$, and $\{e_j\}, \{e^i\}$ are the corresponding bases. To equip $A \otimes A^\circ$ with the Hopf algebra structure of the quantum double, one must define a very specific cross-multiplication recipe. If

$$e_i e_j = c_{ij}^k e_k, \quad \Delta(e_i) = f_i^{jk}(e_j \otimes e_k), \quad S(e^i) = \sigma_j^i e^j,$$

it reads

$$e^i e_j = \mathcal{O}_{jq}^{ip}\, e_p e^q, \qquad \text{where } \mathcal{O}_{jq}^{ip} = c_{nq}^t c_{ts}^i \sigma_r^s f_j^{rl} f_l^{pn}.$$

In invariant form this looks like

$$\alpha a = \sum \sum < S(\alpha_{(1)}), a_{(1)} >< \alpha_{(3)}, a_{(3)} > a_{(2)} \alpha_{(2)}, \tag{6}$$

where

$$\Delta^2(\alpha) = \sum \alpha_{(1)} \otimes \alpha_{(2)} \otimes \alpha_{(3)}, \quad \Delta^2(a) = \sum a_{(1)} \otimes a_{(2)} \otimes a_{(3)}.$$

Here the usual notation for coproducts (cf. $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$) is used. The resulting Hopf algebra proves to be quasitriangular with the universal $\mathcal{R}$-matrix

$$\mathcal{R} = \sum_i (e_i \otimes 1) \otimes (1 \otimes e^i).$$

One easily finds that $gl_q(2)$ and other simple examples of quantum universal enveloping algebras are both the $\breve{U}_R$-type algebras and quantum doubles. Can it happen that $\breve{U}_R$ would be a quantum double for any R? Majid's approach based on the $\breve{U}_R \leftrightarrow \breve{A}_R$ duality does not readily answer this question. That is why we choose another way: not to use A_R at all. The key observation is that there exists an inherent antiduality between U_R^+ and U_R^- which is precisely of the form required for the quantum-double construction.

3. Let us define bialgebras $U_R^+ = \{1, l_{ij}^+\}$ and $U_R^- = \{1, l_{ij}^-\}$ by eqs. (2),(4). Note that the cross-multiplication relation (3) is not yet imposed, so U_R^+ and U_R^- are considered to be independent so far. However, the very natural pairing between them can be introduced. It is generated by

$$< L_1^-, L_2^+ >= R_{12}^{-1}, \quad < L^-, 1 >=< 1, L^+ >=< 1, 1 >= 1 \tag{7}$$

and in the general case looks like

$$\langle L_{i_1}^- \dots L_{i_m}^-, L_{j_1}^+ \dots L_{j_n}^+ \rangle = R_{i_1 j_n}^{-1} \dots R_{i_q j_p}^{-1} \dots R_{i_m j_1}^{-1}\,, \tag{8}$$

where the r.h.s. is a product of mn R^{-1}-matrices corresponding to all pairs of indices $i_q j_p$ with j-indices ordered from right to left. The consistency of (8) and (5) with (4) is evident, while the proof of the consistency with (2) reduces to manipulations like

$$\begin{aligned} < L_0^-, R_{12} L_2^+ L_1^+ - L_1^+ L_2^+ R_{12} > &= < L_0^- \otimes L_0^-, R_{12}(L_1^+ \otimes L_2^+) - (L_2^+ \otimes L_1^+) R_{12} > \\ &= R_{12} R_{01}^{-1} R_{02}^{-1} - R_{02}^{-1} R_{01}^{-1} R_{12} = 0 \end{aligned}$$

and repeated use of QYBE (1).

For general R, this pairing is degenerate. To remove the degeneracy, i.e. to transform pairing into antiduality, one should factor out appropriate bi-ideals [2]. In simple cases this procedure is explicitly carried out and works well. For general R it is of course not under our control. The situation is quite similar to [2]: we are to rely on that the factorization procedure is "soft" in a sense that it does not destroy the whole construction.

4. Keeping this in mind, we observe that, being antidual, $U_R^\pm$ admit the Hopf algebra structure. Let us introduce an antipode S in U_R^- and an inverse antipode S^{-1} in U_R^+ by the relations

$$< S(L^-), 1 >=< 1, S^{-1}(L^+) >= 1, \; < S(L_1^-), L_2^+ >=< L_1^-, S^{-1}(L_2^+) >= R_{12}, \tag{9}$$

extending them on the whole of U_R^+ (or U_R^-) as antihomomorphisms of algebras and coalgebras. The definition is correct due to

$$\begin{aligned} < m \circ (S \otimes id) \circ \Delta(L_1^-), L_2^+ > &= < m \circ (S(L_1^-) \otimes L_1^-), L_2^+ > \\ =< S(L_1^-) \otimes L_1^-, L_2^+ \otimes L_2^+ > &= R_{12} R_{12}^{-1} = 1 =< \varepsilon(L_1^-), L_2^+ >, \end{aligned}$$

$$< S(L_0^-), R_{12} L_2^+ L_1^+ - L_1^+ L_2^+ R_{12} >=< \Delta \circ S(L_0^-), R_{12}(L_1^+ \otimes L_2^+) - (L_2^+ \otimes L_1^+) R_{12} >$$
$$=< S(L_0^-) \otimes S(L_0^-), R_{12}(L_2^+ \otimes L_1^+) - (L_1^+ \otimes L_2^+) R_{12} >= R_{12} R_{02} R_{01} - R_{01} R_{02} R_{12} = 0,$$
$$< S(R_{12} L_2^- L_1^- - L_1^- L_2^- R_{12}), L_0^+ >=< R_{12} S(L_1^-) S(L_2^-) - S(L_2^-) S(L_1^-) R_{12}, L_0^+ >$$
$$=< R_{12}(S(L_1^-) \otimes S(L_2^-)) - (S(L_2^-) \otimes S(L_1^-)) R_{12}, L_0^+ \otimes L_0^+ >= R_{12} R_{10} R_{20} - R_{20} R_{10} R_{12} = 0,$$

but there is no such a formula for $S(L^+)$ or $S^{-1}(L^-)$. Once again the factorization is hoped to be soft enough to allow the antipodes to be invertible.

If so, our bialgebras $U_R^\pm$ become the mutually antidual Hopf algebras $\breve{U}_R^\pm$, and it is possible to define the multiplicative structure of the quantum double upon $\breve{U}_R^+ \otimes \breve{U}_R^-$. The cross-multiplication rule is deduced from (6):

$$\begin{aligned} L_1^- L_2^+ R_{12} &= < S(L_1^-), L_2^+ > L_2^+ L_1^- < L_1^-, L_2^+ > R_{12} \\ &= R_{12} L_2^+ L_1^- R_{12}^{-1} R_{12} = R_{12} L_2^+ L_1^-. \end{aligned} \tag{10}$$

Thus we regain eq.(3) as the quantum-double cross-multiplication condition!

Our conclusion is that R-matrices obeying QYBE generate the algebraic structures of quantum double in quite a natural way.

5. To illustrate the proposed scheme, consider the $sl_q(2)$ R-matrix

$$R_q = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad R_q^{-1} = R_{q^{-1}}.$$

Here the bialgebras $U_{R_q}^\pm$ have 8 generators $l_{ij}^\pm$. The bi-ideals to be factored out are generated by the relations

$$l_{21}^- = 0, \quad l_{12}^+ = 0, \quad l_{11}^- l_{22}^- = l_{22}^- l_{11}^- = 1, \quad l_{11}^+ l_{22}^+ = l_{22}^+ l_{11}^+ = 1.$$

After factorization the number of independent generators is reduced to 4. We denote them $X^{\pm}, H, H'$ (note that $H' \neq H$ so far):

$$L^+ = \begin{pmatrix} q^{H/2} & 0 \\ (q^{1/2} - q^{-3/2})X^+ & q^{-H/2} \end{pmatrix}, \quad L^- = \begin{pmatrix} q^{-H'/2} & (q^{-1/2} - q^{3/2})X^- \\ 0 & q^{H'/2} \end{pmatrix}.$$

The multiplication rules (inside each algebra), coproducts and antipodes are:

$$\begin{aligned} [H, X^+] = 2X^+ \quad &, \quad [H', X^-] = -2X^-, \\ \Delta(H) = H \otimes 1 + 1 \otimes H \quad &, \quad \Delta(H') = H' \otimes 1 + 1 \otimes H', \\ \Delta(X^+) = X^+ \otimes q^{H/2} + q^{-H/2} \otimes X^+ \quad &, \quad \Delta(X^-) = X^- \otimes q^{H'/2} + q^{-H'/2} \otimes X^-, \\ S(X^{\pm}) = -q^{\pm 1} X^{\pm} \quad &, \quad S(H) = -H, \; S(H') = -H'. \end{aligned}$$

The quantum-double cross-multiplication rules (6) take the form

$$\begin{aligned} [H', X^+] = 2X^+ \quad &, \quad [H, X^-] = -2X^-, \; [H, H'] = 0, \\ [X^+, X^-] \quad &= \quad \left(q^{(H+H')/2} - q^{-(H+H')/2}\right)/(q - q^{-1}). \end{aligned}$$

The identification $H' \equiv H$ leads to the ordinary $sl_q(2)$.

6. To give one more illustration, let us consider a bialgebra introduced in [5]. In a slightly simplified form it has generators $\{1, t^i_j, u^i_j, E_j, F^i\}$ which obey the following relations (here we prefer to display all the indices):

$$\begin{aligned} R^{ij}_{mn}\, t^m_p\, t^n_q = R^{mn}_{pq}\, t^j_n\, t^i_m \quad &, \quad E_p\, t^j_q = R^{mn}_{pq}\, t^j_n\, E_m, && (11) \\ \Delta(t^i_j) = t^i_k \otimes t^k_j, \;\; \varepsilon(t^i_j) = \delta^i_j \quad &, \quad \Delta(E_j) = E_i \otimes t^i_j + 1 \otimes E_j, \;\; \varepsilon(E_j) = 0, && (12) \\ R^{ij}_{mn}\, u^m_p\, u^n_q = R^{mn}_{pq}\, u^j_n\, u^i_m \quad &, \quad F^i\, u^j_p = R^{ji}_{mn}\, u^m_p\, F^n, && (13) \\ \Delta(u^i_j) = u^i_k \otimes u^k_j, \;\; \varepsilon(u^i_j) = \delta^i_j \quad &, \quad \Delta(F^i) = F^i \otimes 1 + u^i_j \otimes F^j, \;\; \varepsilon(F^i) = 0, && (14) \\ R^{ij}_{mn}\, u^m_p\, t^n_q = R^{mn}_{pq}\, t^j_n\, u^i_m \quad &, \quad E_j F^i - F^i E_j = t^i_j - u^i_j, && (15) \\ u^i_p\, E_q = R^{mn}_{pq}\, E_n\, u^i_m \quad &, \quad t^i_p\, F^j = R^{ji}_{mn}\, F^m\, t^n_p, && (16) \end{aligned}$$

with R obeying QYBE (1). This is not a bialgebra of the form (2)-(4). Rather it is of the 'inhomogeneous quantum group' type [6]. Let us make sure that it is a quantum double as well.

Consider T, E-bialgebra (11),(12) and U, F-bialgebra (13),(14) firstly as being independent and fix nonzero pairings on the generators by

$$< u^i_j, t^p_q > = R^{ip}_{jq}, \quad < u^i_j, 1 > = < 1, t^i_j > = < F^i, E_j > = \delta^i_j, \tag{17}$$

extending them to the whole bialgebras with the help of (5). The definition is correct due to

$$\begin{aligned} < F^i, E_p\, t^j_q - R^{mn}_{pq}\, t^j_n\, E_m > \quad &= \quad < F^i \otimes 1 + u^i_k \otimes F^k \;,\; t^j_q \otimes E_p - R^{mn}_{pq}(E_m \otimes t^j_n) > \\ &= \quad R^{ij}_{kq}\, \delta^k_p - \delta^i_m R^{mn}_{pq}\, \delta^j_n = 0, \\ < F^i u^j_p - R^{ji}_{mn}\, u^m_p\, F^n, E_q > \quad &= \quad < F^i \otimes u^j_p - R^{ji}_{mn}(u^m_p \otimes F^n) \;,\; E_k \otimes t^k_q + 1 \otimes E_q > \\ &= \quad \delta^i_k R^{jk}_{pq} - R^{ji}_{mn}\, \delta^m_p\, \delta^n_q = 0. \end{aligned}$$

After factoring out the corresponding null bi-ideals, we may define antipodes on the generators as follows:

$$< S(u^i_j), t^p_q > = < u^i_j, S^{-1}(t^p_q) > = (R^{-1})^{ip}_{jq},$$

$$< S(u^i_j), 1 > = < 1, S^{-1}(t^i_j) > = \delta^i_j, \quad S(F^i) = -S(u^i_j) F^j, \quad S(E_j) = -E_i S(t^i_j).$$

The proof of correctness is in complete analogy with the $\breve{U}_R$-case.

Now a direct application of the recipe (6) exactly reproduces the cross-multiplication relations (15),(16). For example,

$$F^i E_j = < S(F^i), E_m >< 1, t^n_j > t^m_n + < S(u^i_n), 1 >< 1, t^m_j > E_m F^n$$

$$+ < S(u^i_n), 1 >< F^m, E_j > u^n_m = -t^i_j + E_j F^i + u^i_j,$$

because of

$$< S(F^i), E_m > = - < S(u^i_k) \otimes F^k \;,\; E_n \otimes t^n_m + 1 \otimes E_m > = -\delta^i_m.$$

Therefore, bialgebras of the type (11)-(14) are also transformed into the quantum double using our method.

7. Consider at last a bialgebra [7] that is known to be related [8] to bicovariant differential calculus on quantum groups. Its coalgebra structure is given by (12), whereas the multiplication relations (11) are to be supplemented by

$$\begin{aligned} t^i_p\, E_q + f^i_{nm}\, t^n_p\, t^m_q \quad &= \quad R^{nm}_{pq}\, E_m\, t^i_n + f^n_{pq}\, t^i_n, && (18) \\ E_i E_j - R^{mn}_{ij}\, E_n E_m \quad &= \quad f^m_{ij}\, E_m, && (19) \end{aligned}$$

f^i_{jk} being new structure constants. This bialgebra, unlike its ancestor (11),(12), exhibits the R-matrix-type representation

$$\mathbf{R}_{12}\mathbf{T}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{T}_1\mathbf{R}_{12}, \quad \Delta(\mathbf{T}) = \mathbf{T} \otimes \mathbf{T}, \tag{20}$$

where, in terms of multi-indices like $I = \{0, i\}$,

$$\mathbf{T}^I_J = \begin{pmatrix} 1 & E_j \\ 0 & t^i_j \end{pmatrix}, \quad \mathbf{R}^{IJ}_{MN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta^j_n & 0 & 0 \\ 0 & 0 & \delta^i_m & f^i_{mn} \\ 0 & 0 & 0 & R^{ij}_{mn} \end{pmatrix}.$$

Of course, $\mathbf{R}$ must satisfy the QYBE (1) which now involves the structure constants f^i_{mn} as well as R^{ij}_{mn}. Note that, due to (18),(19), the bialgebra (11),(12) is not restored from (20) by mere setting $f^i_{mn} \equiv 0$.

Now let us try to develop a quantum double from the bialgebra (20). However, it seems to be quite uneasy task. A natural Ansatz for the candidate antidual bialgebra is

$$\mathbf{U}^I_J = \begin{pmatrix} 1 & 0 \\ F^i & u^i_j \end{pmatrix},$$

which causes the corresponding R-matrix to be

$$\overline{\mathbf{R}}^{IJ}_{MN} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \delta^j_n & 0 & 0 \\ 0 & 0 & \delta^i_m & 0 \\ 0 & \bar{f}^{ij}_n & 0 & R^{ij}_{mn} \end{pmatrix}$$

with different structure constants $\bar{f}$ and another QYBE system involving R and $\bar{f}$. Now, attempting to fix a pairing in the form

$$< \mathbf{U}_1, \mathbf{T}_2 >= \mathbf{Q}_{12} \qquad (21)$$

with a certain numerical matrix $\mathbf{Q}$, we immediately arrive at the following general statement:

Let $\mathbf{R}$ and $\overline{\mathbf{R}}$ be invertible solutions of QYBE. If there exists an invertible solution $\mathbf{Q}$ of the equations

$$\begin{aligned} \mathbf{Q}_{12}\mathbf{Q}_{13}\mathbf{R}_{23} &= \mathbf{R}_{23}\mathbf{Q}_{13}\mathbf{Q}_{12}, \\ \overline{\mathbf{R}}_{12}\mathbf{Q}_{13}\mathbf{Q}_{23} &= \mathbf{Q}_{23}\mathbf{Q}_{13}\overline{\mathbf{R}}_{12}, \end{aligned}$$

then (21) is a correct pairing between the $\mathbf{T}$- and $\mathbf{U}$-bialgebras generated by $\mathbf{R}$ and $\overline{\mathbf{R}}$, respectively, and, assuming a proper quotienting procedure to be performed, the antipodes can be defined by the relations

$$< S(\mathbf{U}_1), \mathbf{T}_2 >=< \mathbf{U}_1, S^{-1}(\mathbf{T}_2) >= \mathbf{Q}^{-1}_{12}$$

and the quantum-double structure can be established on the tensor product of these bialgebras by the cross-multiplication formula

$$\mathbf{Q}_{12}\mathbf{U}_1\mathbf{T}_2 = \mathbf{T}_2\mathbf{U}_1\mathbf{Q}_{12}.$$

Whether such a program can really be carried through in interesting cases (e.g. for $\mathbf{R}$ and $\overline{\mathbf{R}}$ given above) is the subject of further investigation.

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