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LOCAL SYMMETRIES IN SYSTEMS WITH CONSTRAINTS

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Локальные симметрии в системах со связями

Для произвольных систем со связями первого рода дан метод построения локальных калибровочных преобразований в соответствии с гипотезой Дирака как в фазовом, так и в конфигурационном пространстве.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Local Symmetries in Systems with Constraints

For arbitrary systems with first-class constraints a method is given for constructing local gauge transformations in accordance with the Dirac hypothesis in phase and configuration spaces.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1 Introduction

The approach Dirac proposed for describing systems with constraints [1] attracted much attention again in view of the fundamental role of gauge theories in elementary particle physics because these theories belong to the class of degenerate theories. The Dirac hypothesis is under discussion for a long time according to which all the first-class constraints are generators of gauge transformations. In the existing literature there are many divergent opinions, some of them [2 and 3] totally reject this hypothesis and some of them completely accept it [4–7], which signifies that in the general case no method does yet exist for finding gauge transformations in theories with constraints. Knowledge of the explicit form of gauge transformations is necessary in many cases, for instance, in BRST [8] and $Sp(2)$ [9] quantization, for deriving improper conservation laws and for studying the connection between various gauges.

Gauge transformations were constructed by two approaches (however, in the general case the problem was not solved). One of them [6] is based on a generalized Hamiltonian H_E that is a sum of the canonical Hamiltonian and all the first-class constraints with their Lagrange multipliers, there the phase space is formally extended by making the Lagrange multipliers to be extra coordinates. This extension is required for removing the terms proportional to the Lagrange multipliers from the action variation by ascribing the corresponding transformations to the multipliers. This approach differs from the Dirac approach and, moreover, the space thus extended has no symplectic structure of the phase space. The other approach [3, 10, 11] (without extending the phase space) also did not permit one to obtain gauge transformations in the general case. The reason is that the group structure of the generator of gauge transformations was given a priori: the number of arbitrary parameters was fixed beforehand (it was equal to the amount of primary first-class constraints), which did not follow from the Dirac hypothesis.

In our earlier papers [4 and 5], we have suggested a method of constructing infinitesimal gauge transformations on the basis of the variational principle for the action. We proceeded in accordance with the Dirac hypothesis and on the algebra of constraints we imposed the restriction consisting in that the Poisson brackets of primary constraints with all constraints are linear combinations of primary constraints. Then it is natural to ask to what extent this restriction reduces the class of theories for which gauge transformations can be constructed and what is the nature of degeneracy of Lagrangians because there are examples [11 and 12] when this restriction is broken up. Notes that the mentioned restriction on the constraints applies also to the aforecited articles. Moreover, these approaches do not embrace the cases when higher derivatives are present in the symmetry transformation law. The latter has also to do with the Lagrangian formalism [13].

In this paper, following the method applied in ref.[3] (i.e. requiring the transformed coordinates to be solutions of the Hamiltonian equations of motion) and the Dirac hypothesis, we derive infinitesimal gauge transformations in phase and configuration space for arbitrary degenerated Lagrangians. We also show that the difficulty due to restriction on the algebra of constraints can be removed by passing to an equivalent set of constraints and that the degeneracy of Lagrangians stems from their being gauge-invariant. (We will consider only the first-class constraints as only these constraints are responsible for gauge degrees of freedom [1].) The method will be applied to an example not solved yet [11].

2 Definitions, Derivation of Infinitesimal Gauge Transformations, Algebra of Constraints

Subsequent considerations may be extended to the field theory, but here we restrict for simplicity ourselves to a system with a finite number of the degrees of freedom described by a degenerate Lagrangian $L(q,\dot{q})$, where $q=(q_1,\cdots,q_N)$ and $\dot{q}=\frac{dq}{dt}=(\dot{q}_1,\cdots,\dot{q}_N)$ are generalized coordinates and velocities, respectively. Degeneracy of the Lagrangian implies that

$$\mathrm{rank}\left\|\frac{\partial^2 L}{\partial\dot{q}_i\partial\dot{q}_j}\right\| = R < N, \qquad i,j=1,\cdots,N. \tag{1}$$

To pass into the Hamiltonian formalism, we introduce the momentum

variables

$$p_i(q,\dot{q}) = \frac{\partial L}{\partial\dot{q}_i},$$

which are not all independent due to the condition (1). As a result, there appear $N-R$ relationships in the phase space,

$$\phi^1_\alpha(q,p)\approx 0, \qquad \alpha=1,\cdots,N-R. \tag{2}$$

By the Dirac terminology, ϕ^1_α in (2) are primary constraints and $\approx$ means weak equality.

The Hamiltonian equations of motion are written in terms of the standard Poisson brackets as follows [1]:

$$\dot{q}_i=\{q_i,H_T\}, \qquad p_i=\{p_i,H_T\}, \tag{3}$$
$$\phi^1_\alpha(q,p)\approx 0,$$

where the total Hamiltonian H_T is

$$H_T = H_c + u_\alpha\phi^1_\alpha. \tag{4}$$

In (4) H_c is the canonical Hamiltonian and u_α are the arbitrary functions of time.

For the system of equations (3) being self-consistent, the primary constraints should be conserved in time. As a result, there arise secondary constraints $\phi^2_\alpha(q,p)\approx 0$ that should also be conserved in time and lead to constraints of the next stage. This process is to be continued up to trivial fulfillment of the conditions of stationarity to be occurred at a certain stage M_α. Following Dirac, we denote the whole set of constraints, both primary and secondary if all stages, as follows

$$\phi^{m_\alpha}_\alpha, \qquad \alpha=1,\cdots,N-R, \qquad m_\alpha=1,2,\cdots,M_\alpha. \tag{5}$$

We assume that the system (5) is a complete set of independent functions[1].

In accordance with the Dirac hypothesis, we look for infinitesimal gauge transformations in the form

$$\begin{aligned} q'_i &= q_i+\delta q_i, \quad \delta q_i=\{q_i,G\}, \\ p'_i &= p_i+\delta p_i, \quad \delta p_i=\{p_i,G\}, \end{aligned} \tag{6}$$

where the generating function G is given by

$$G=\varepsilon^{m_\alpha}_\alpha\phi^{m_\alpha}_\alpha, \qquad \alpha=1,\cdots,N-R, \qquad m_\alpha=1,\cdots,M_\alpha \tag{7}$$

with $\varepsilon_\alpha^{m_\alpha}$ being arbitrary functions of time.

Like in ref.[3], we will require the transformed quantities q_i' and p_i' defined by (6) to be solutions of the Hamiltonian equations of motion,

$$q_i' = \left.\frac{\partial H_T}{\partial p_i}\right|_{q+\delta q, p+\delta p}, \qquad p_i' = -\left.\frac{\partial H_T}{\partial q_i}\right|_{q+\delta q, p+\delta p}, \tag{8}$$

$$\phi_\alpha^1(q+\delta q, p+\delta p) \approx 0.$$

Expanding the first of eqs. (8) in a series in δq_i and δp_i and taking account of (6) we obtain

$$\dot q_i + \frac{d}{dt}\{q_i, G\} = \frac{\partial H_T}{\partial p_i} + \{\{q_i, H_T\}, G\}. \tag{9}$$

Using the definition of the total derivative with respect to time and the Jacobi identity for q_i, G and H_T we find from (9)

$$\left\{q_i, \frac{\partial G}{\partial t} + \{G, H_T\}\right\} = 0 \tag{10}$$

and analogously for p_i

$$\left\{p_i, \frac{\partial G}{\partial t} + \{G, H_T\}\right\} = 0. \tag{11}$$

From equations (10) and (11) it follows that

$$\left[\frac{\partial G}{\partial t} + \{G, H_T\}\right]_{\phi_\alpha^1\approx 0} = 0. \tag{12}$$

We recall that we consider only the first-class constraints, which implies the following relations

$$\{\phi_\alpha^{m_\alpha}, \phi_\beta^{m_\beta}\} = f_\alpha^{m_\alpha}{}_\beta^{m_\beta}{}_\gamma^{m_\gamma}\phi_\gamma^{m_\gamma}, \tag{13}$$

$$\{\phi_\sigma^{m_\sigma}, H_c\} = g_\sigma^{m_\sigma}{}_\tau^{m_\tau}\phi_\tau^{m_\tau}, \tag{14}$$

$\alpha, \beta, \gamma, \sigma, \tau = 1, \cdots, N-R$; $m_{\alpha,\beta,\gamma,\sigma} = 1, \cdots, M_{\alpha,\beta,\gamma,\sigma}$; $m_\tau = 1, \cdots, m_\sigma+1$. (here and in what follows, summation runs over repeated upper and lower indices). Using these relations and the function G defined by (7) we rewrite equation (12) in the form

$$\Big[(\dot\varepsilon_\alpha^{m_\alpha} + g_\beta^{m_\beta}{}^{m_\alpha}_{\alpha}\varepsilon_\beta^{m_\beta})\phi_\alpha^{m_\alpha} + (\dot\varepsilon_\alpha^1 + g_\beta^{m_\beta}{}^1_\alpha\varepsilon_\beta^{m_\beta} - u_\gamma f_\gamma^1{}_\beta^{m_\beta}{}_\alpha^1\varepsilon_\beta^{m_\beta})\phi_\alpha^1$$
$$+ u_\gamma f_\beta^{m_\beta}{}_\gamma^1{}_\alpha^{m_\alpha}\varepsilon_\beta^{m_\beta}\phi_\alpha^{m_\alpha}\Big]_{\phi_\alpha^1\approx 0} = 0, \qquad m_\alpha \geq 2, \quad m_\beta \geq m_\alpha - 1. \tag{15}$$

Owing to the constraints being independent, the equality (15) can be satisfied if the coefficients of secondary constraints of all stages vanish, i.e.

$$(\dot\varepsilon_\alpha^{m_\alpha} + g_\beta^{m_\beta}{}^{m_\alpha}_{\alpha}\varepsilon_\beta^{m_\beta}) + u_\gamma f_\beta^{m_\beta}{}_\gamma^1{}_\alpha^{m_\alpha}\varepsilon_\beta^{m_\beta} = 0, \quad m_\alpha \geq 2. \tag{16}$$

This equality cannot be satisfied by any selection of functions $\varepsilon_\alpha^{m_\alpha}$ because the Lagrange multipliers $u_\gamma(t)$ are arbitrary. However, when

$$f_\beta^{m_\beta}{}_\gamma^1{}_\alpha^{m_\alpha} = 0 \qquad \text{for} \quad m_\alpha \geq 2, \tag{17}$$

we obtain [4]

$$\dot\varepsilon_\alpha^{m_\alpha} + g_\beta^{m_\beta}{}^{m_\alpha}_{\alpha}\varepsilon_\beta^{m_\beta} = 0, \qquad m_\beta \geq m_\alpha - 1. \tag{18}$$

Note that the condition (17) is equivalent to the relation

$$\{\phi_\beta^1, \phi_\gamma^{m_\gamma}\} = f_\beta^1{}_\gamma^{m_\gamma}{}_\alpha^1\phi_\alpha^1. \tag{19}$$

In our previous papers [4 and 5], on the basis of variational principle, we have derived the relation (18) between the parameters $\varepsilon_\alpha^{m_\alpha}$ for systems with constraints obeying the condition (19).

The system of equations (18) is not complete; the number of unknown functions exceeds the number of equations by the number of primary constraints. Therefore, introducing arbitrary functions ε_α in the amount equal to that of primary constraints and applying the iteration procedure to eqs.(18), we can express all $\varepsilon_\alpha^{m_\alpha}$ in terms of the introduced functions and $g_\beta^{m_\beta}{}^{m_\alpha}_\alpha$ and their derivatives [5]

$$\varepsilon_\alpha^{m_\alpha} = B_\alpha^{m_\alpha}{}_\beta^{m_\beta}\varepsilon_\beta^{(M_\alpha - m_\beta)}, \qquad m_\beta = m_\alpha, \cdots, M_\alpha \tag{20}$$

(here summation runs also over m_β), where

$$\varepsilon_\beta^{(M_\alpha - m_\beta)} \equiv \frac{d^{M_\alpha - m_\beta}}{dt^{M_\alpha - m_\beta}}\varepsilon_\beta(t), \qquad \varepsilon_\beta(t) \equiv \varepsilon_\beta^{M_\beta}$$

and $B_\alpha^{m_\alpha}{}_\beta^{m_\beta}$ are, generally speaking, functions of q and p and their derivatives up to the order $M_\alpha - m_\alpha - 1$. Then the generating function of gauge transformations G assume the form

$$G = B_\alpha^{m_\alpha}{}_\beta^{m_\beta}\phi_\alpha^{m_\alpha}\varepsilon_\beta^{(M_\alpha - m_\beta)}, \qquad m_\beta = m_\alpha, \cdots, M_\alpha. \tag{21}$$

Owing to the derivatives of q and p with respect to time being present in $B_\alpha^{m_\alpha}{}_\beta^{m_\beta}$, the Poisson brackets are not determined in the transformations

(6). However, this problem was solved in our previous paper [5] where it was shown that gauge transformations generated by the generating function G (21) are canonical in the extended phase space. The action is invariant under these transformations and the corresponding gauge transformations in the configuration space (see Appendix A). [2] The amount of arbitrary functions, the function G is dependent on, equals to the number of primary constraints. As can be seen from formula (21), the transformation law may include both arbitrary functions $\varepsilon_\alpha(t)$ and their derivatives up to and including the order $M_\alpha - 1$; the highest derivatives $\varepsilon_\alpha^{(M_\alpha-1)}$ should be always present.

Note that derivation of formula (21) presents no extra difficulties and does not require further assumptions, as compared with the Dirac approach.

Also we mention that since the first-class constraints compose a pseudoalgebra [14], the condition (19) means that primary constraints represent the ideal of that pseudoalgebra. Below, we will show that for arbitrary Lagrangians (even when the condition (19) does not hold) we can always to pass to an equivalent set of constraints for which the condition (19) will be valid.

3 Gauge Transformations for an Arbitrary Degenerate Lagrangian

To generalize the method of construction of gauge transformations to an arbitrary degenerate Lagrangian, it is necessary to analyze the situation when the condition (19) is not fulfilled. To this end, let us recall the inherent arbitrariness of the Dirac Hamiltonian formalism. When there is a complete set of constraints defined by the Dirac procedure ($\phi_\alpha^{m_\alpha}$, $\alpha = 1, \cdots, N-R$; $m_\alpha = 1, \cdots, M_\alpha$) and functionally independent, we can

[2]The corresponding gauge transformations in the configuration space are defined as follows:

$$\delta q_i(t) = \{q_i(t), G\}\Big|_{p=\frac{\partial L}{\partial \dot q}}, \qquad \delta \dot q(t) = \frac{d}{dt} q(t).$$

always pass to an equivalent set of constraints by the transformation

$$\bar\phi_\beta^{m_\beta} = C_{\beta}^{m_\beta}{}_{\alpha}^{m_\alpha} \phi_\alpha^{m_\alpha}, \tag{22}$$

where

$$\det \left\| C_{\beta}^{m_\beta}{}_{\alpha}^{m_\alpha} \right\|_\Sigma \neq 0, \tag{23}$$

i.e. this determinant is not zero on the surface Σ given by the complete set of constraints.

Now consider a particular case of the transformation (22) when primary constraints remain unchanged, i.e.

$$C_{\beta}^{1}{}_{\alpha}^{m_\alpha} = \delta_{\beta\alpha} \qquad \text{for any} \quad m_\alpha.$$

It is not difficult to see that taking account of (13) and (14) we obtain

$$\{\phi_\alpha^1, \bar\phi_\beta^{m_\beta}\} = \left[\{\phi_\alpha^1, C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}\} + f_{\alpha}^{1}{}_{\delta}^{m_\delta}{}_{\gamma}^{m_\gamma} C_{\beta}^{m_\beta}{}_{\delta}^{m_\delta}\right] \phi_\gamma^{m_\gamma} + f_{\alpha}^{1}{}_{\delta}^{m_\delta}{}_{\gamma}^{1} C_{\beta}^{m_\beta}{}_{\delta}^{m_\delta} \phi_\beta^1, \tag{24}$$

$$m_\beta, m_\delta, m_\gamma \geq 2.$$

From the expression (24) it is clear that if we could choose $C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}$ so that the coefficients of secondary constraints vanish

$$\{\phi_\alpha^1, C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}\} + f_{\alpha}^{1}{}_{\delta}^{m_\delta}{}_{\gamma}^{m_\gamma} C_{\beta}^{m_\beta}{}_{\delta}^{m_\delta} = 0, \tag{25}$$

the condition (19) will be valid for the new set of constraints $\bar\phi_\beta^{m_\beta}$. Thus, for $C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}$ we have derived the system of linear inhomogeneous equations in the first-order partial derivatives (25). This system can be shown to be fully integrable. The condition of integrability for systems of the type (25) looks as follows [15]

$$\{\phi_\sigma^1, \{\phi_\alpha^1, C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}\}\} - \{\phi_\alpha^1, \{\phi_\sigma^1, C_{\beta}^{m_\beta}{}_{\gamma}^{m_\gamma}\}\} = 0. \tag{26}$$

Using eq.(25), properties of the Poisson brackets and making some transformations we rewrite the relation (26) in the form

$$\Big[\{\phi_\alpha^1, f_{\sigma}^{1}{}_{\delta}^{m_\delta}{}_{\gamma}^{m_\gamma}\} - f_{\alpha}^{1}{}_{\delta}^{m_\delta}{}_{\tau}^{m_\tau} f_{\sigma}^{1}{}_{\tau}^{m_\tau}{}_{\gamma}^{m_\gamma} - \{\phi_\sigma^1, f_{\alpha}^{1}{}_{\delta}^{m_\delta}{}_{\gamma}^{m_\gamma}\}$$
$$+ f_{\sigma}^{1}{}_{\delta}^{m_\delta}{}_{\tau}^{m_\tau} f_{\alpha}^{1}{}_{\tau}^{m_\tau}{}_{\gamma}^{m_\gamma}\Big] C_{\beta}^{m_\beta}{}_{\delta}^{m_\delta} = 0, \tag{27}$$

$$m_\beta, m_\delta, m_\tau \geq 2.$$

Utilizing the Jacobi identity

$$\{\phi_\alpha^1, \{\phi_\sigma^1, \phi_\beta^{m_\beta}\}\} + \{\phi_\beta^{m_\beta}, \{\phi_\alpha^1, \phi_\sigma^1\}\} + \{\phi_\sigma^1, \{\phi_\beta^{m_\beta}, \phi_\alpha^1\}\} = 0, \qquad m_\beta \geq 2$$

and relation (13) we obtain

$$\Big[\{\phi^1_\alpha, f^{1\ m_\delta m_\gamma}_{\sigma\ \delta\ \ \gamma}\} - f^{1\ m_\delta m_\tau}_{\alpha\ \delta\ \ \tau} f^{1\ m_\tau m_\gamma}_{\sigma\ \tau\ \ \gamma} - \{\phi^1_\sigma, f^{1\ m_\delta m_\gamma}_{\alpha\ \delta\ \ \gamma}\}$$
$$+ f^{1\ m_\delta m_\tau}_{\sigma\ \delta\ \ \tau} f^{1\ m_\tau m_\gamma}_{\alpha\ \tau\ \ \gamma}\Big]\phi^{m_\gamma}_\gamma = \{\{\phi^1_\alpha, \phi^1_\sigma\}, \phi^{m_\delta}_\delta\}, \qquad (28)$$

$$m_\beta \geq 2, \quad m_\gamma, m_\delta, m_\tau \geq 1.$$

Note the Poisson brackets between primary constraints may, without loss of generality, be considered to be strictly zero in the whole phase space. As every primary constraint contains at least one momentum variable, there always exist canonical transformations transforming the primary constraints into new momentum variables (see below). Therefore, the expressions in square brackets in front of the constraints $\phi^{m_\gamma}_\gamma$ on the left-hand side of the identity (28) being coefficients of the functionally independent quantities disappear each separately. As the condition (27) contains the same coefficients of $C^{m_\beta m_\delta}_{\beta\ \ \delta}$, it is satisfied identically, which proves the system of equations (25) to be fully integrable. Therefore, there always exists a set of constraints equivalent to the initial set for which the condition (19) holds valid.

Now we shall describe the way of passing to, at least, one separated set of equivalent constraints $\bar\phi^{m_\alpha}_\alpha$ when all the primary constraints are momentum variables. This can be done by the iteration procedure provided that we take into account the property of primary constraints

$$\{\phi^1_\alpha, \phi^1_\beta\} = f^{1\ 1\ 1}_{\alpha\ \beta\ \gamma}\phi^1_\gamma$$

that follows from the stationarity condition for ϕ^1_α and from the fact that we are dealing only with the first-class constraints. There always exist canonical transformations of the form [16 and 17]

$$\bar P_1 = \phi^1_1(q,p), \qquad \{\bar Q_1, \bar P_1\} = 1, \qquad \{\bar Q_\sigma, \bar P_\tau\} = \delta_{\sigma\tau},$$
$$\{\bar P_1, \bar P_\tau\} = \{\bar Q_1, \bar P_\tau\} = \{\bar P_1, \bar Q_\tau\} = \{\bar Q_1, \bar Q_\tau\} = 0, \qquad (29)$$
$$\sigma, \tau = 2, \cdots, N.$$

(The bar over a letter means the first stage of the iteration procedure.) All the remaining primary constraints assume the form

$$\Phi^1_\alpha(\bar Q, \bar P) = \phi^1_\alpha\big(q(\bar Q, \bar P), p(\bar Q, \bar P)\big)\Big|_{\bar P_1 = 0}, \qquad \alpha = 2, \cdots, N - R.$$

In view of the transformation being canonical, we can write

$$\{\bar P_1, \Phi^1_\alpha\} = -\frac{\partial\Phi^1_\alpha}{\partial\bar Q_1} = \bar f^{1\ 1\ 1}_{1\ \alpha\ \gamma}\Phi^1_\gamma, \qquad \alpha, \gamma \geq 2,$$

with Φ^1_α having the structure [17]

$$\Phi^1_\alpha = \bar D^{1\ 1}_{\alpha\ \gamma}\bar\Phi^1_\gamma, \qquad \det\bar D\Big|_\Sigma \neq 0 \qquad (30)$$

and obeying the conditions

$$\frac{\partial\bar\Phi^1_\gamma}{\partial\bar Q_1} = \frac{\partial\bar\Phi^1_\gamma}{\partial\bar P_1} = 0, \qquad \gamma \geq 2.$$

As all the constraints $\bar\Phi^1_\gamma$ do not depend upon $\bar Q_1$ and $\bar P_1$, we perform an analogous procedure for the constraint $\bar\Phi^1_2$ in the $2N-2$-dimensional subspace $(\bar Q_\sigma, \bar P_\sigma)(\sigma = 2, \cdots, N)$, i.e. without affecting $\bar Q_1$ and $\bar P_1$. Then the constraints $\bar{\bar\Phi}{}^1_\alpha(\alpha = 3, \cdots, N - R)$ arising in a formula analogous to formula (30) are independent of $\bar Q_1, \bar P_1$ and $\bar{\bar Q}_2, \bar{\bar P}_2$. Next, making this procedure step by step $N - R - 2$ times we finally obtain the primary constraints to be momenta, and therefore they commute with each other (final momenta and coordinates will be denoted by Q_α and P_α, respectively, $\alpha = 1, \cdots, N - R$). All secondary constraints will then assume the form

$$\Phi^{m_\alpha}_\alpha(Q, P) = \phi^{m_\alpha}_\alpha\big(q(Q, P), p(Q, P)\big)\Big|_{P_\alpha = 0},$$
$$\alpha = 1, \cdots, N - R, \quad m_\alpha = 2, \cdots, M_\alpha.$$

As the transformations are canonical, we can write

$$\{P_\alpha, \Phi^{m_\beta}_\beta\} = -\frac{\partial\Phi^{m_\beta}_\beta}{\partial Q_\alpha} = f^{1\ m_\beta m_\gamma}_{\alpha\ \beta\ \ \gamma}\Phi^{m_\gamma}_\gamma$$

with $\Phi^{m_\alpha}_\alpha$ having the structure [17]

$$\Phi^{m_\alpha}_\alpha = A^{m_\alpha m_\beta}_{\alpha\ \ \beta}\widetilde\Phi^{m_\beta}_\beta, \qquad \det A\Big|_\Sigma \neq 0 \qquad (31)$$

and obeying the conditions

$$\frac{\partial\widetilde\Phi^{m_\alpha}_\alpha}{\partial Q_\beta} = \frac{\partial\widetilde\Phi^{m_\alpha}_\alpha}{\partial P_\beta} = 0, \qquad \alpha, \beta = 1, \cdots, N - R, \quad m_\alpha \geq 2.$$

The set of constraints thus constructed (primary constraints being momenta and secondary $\tilde{\Phi}^{m_\alpha}_\alpha$) satisfies the condition (19) with vanishing right-hand side, i.e. we have derived the searched set of constraints. Note that $(A^{-1})^{m_\alpha m_\beta}_{\alpha\ \ \beta}$ in (31) is a solution to the system of equations (25).

So, we may conclude that the difficulty associated with the condition (19) being not valid for a certain degenerate Lagrangian can be overcome by passing to an equivalent set of constraints. Therefore, the method we proposed earlier for constructing gauge transformations [4 and 5] is applicable in the general case.

4 Example

Consider the Lagrangian [11]

$$L = \frac{1}{\alpha}\dot{\mathbf{y}} \cdot (\dot{\mathbf{x}} + \beta \mathbf{y}), \tag{32}$$

where α and β are scalar coordinates; $\mathbf{x}$ and $\mathbf{y}$ are n- dimensional vectors.

In the Hamiltonian formalism there are the following primary constraints

$$\phi^1_1 = p_\alpha \approx 0, \qquad \phi^1_2 = p_\beta \approx 0$$

and the total Hamiltonian

$$H_T = \alpha \mathbf{p}_x \cdot \mathbf{p}_y - \beta \mathbf{y} \cdot \mathbf{p}_x + u_1 p_\alpha + u_2 p_\beta.$$

From the condition of self-consistency of the theory we obtain two secondary and one tertiary constraint:

$$\phi^2_1 = -\mathbf{p}_x \cdot \mathbf{p}_y \approx 0, \qquad \phi^2_2 = \mathbf{y} \cdot \mathbf{p}_x \approx 0, \qquad \phi^3_1 = -\beta \mathbf{p}_x^2 \approx 0.$$

It may be verified directly that all these constraints are of the first class and they do not obey the condition (19)

$$\{\phi^1_2, \phi^3_1\} = -\frac{1}{\beta}\phi^3_1.$$

As the primary constraints in the example are momentum variable, we can turn into an equivalent set of constraints by formula (31):

$$\bar{\phi}^1_1 = \phi^1_1 = p_\alpha, \qquad \bar{\phi}^1_2 = \phi^1_2 = p_\beta,$$

$$\bar{\phi}^2_1 = \phi^2_1 = -\mathbf{p}_x \cdot \mathbf{p}_y, \qquad \bar{\phi}^2_2 = \phi^2_2 = \mathbf{y} \cdot \mathbf{p}_x, \qquad \bar{\phi}^3_1 = -\frac{1}{\beta}\phi^3_1 = \mathbf{p}_x^2,$$

which do satisfy the condition (19)

$$\{\bar{\phi}^1_2, \bar{\phi}^3_1\} = 0.$$

All $\bar{g}^{m_\beta m_\alpha}_{\beta\ \ \alpha}$ in formula (14) for the Poisson brackets of these constraints with the canonical Hamiltonian vanish except for

$$\bar{g}^{1\ 2}_{1\ 1} = \bar{g}^{1\ 2}_{2\ 2} = 1, \quad \bar{g}^{2\ 3}_{1\ 1} = -\beta, \quad \bar{g}^{2\ 3}_{2\ 1} = \alpha.$$

Then the system of equations (18) takes the form

$$\begin{aligned} &\dot{\varepsilon}^3_1 - \beta\varepsilon^2_1 + \alpha\varepsilon^2_2 = 0, \\ &\dot{\varepsilon}^2_2 + \varepsilon^1_2 = 0, \\ &\dot{\varepsilon}^2_1 + \varepsilon^1_1 = 0. \end{aligned} \tag{33}$$

With the redifinition $\varepsilon^3_1 \equiv \varepsilon_1$ and $\dot{\varepsilon}^2_2 \equiv \varepsilon_2$ we obtain

$$\varepsilon^2_1 = \frac{1}{\beta}(\dot{\varepsilon}_1 + \alpha\varepsilon_2), \qquad \varepsilon^1_2 = -\dot{\varepsilon}_2, \qquad \varepsilon^1_1 = -\frac{d}{dt}\left(\frac{\dot{\varepsilon}_1 + \alpha\varepsilon_2}{\beta}\right).$$

As the parameters in the generating function depend on $\dot{\alpha}$ and $\dot{\beta}$, we can derive the "well-defined" Poisson brackets by applying the procedure of extending the phase space described in our previous paper [5]. It is sufficient to make the following extension of the phase space: Introduce the coordinates

$$q_{1\ i} = \begin{cases} \alpha, & i = 1, \\ \beta, & i = 2, \\ x_i, & i = 3, \cdots, n+2, \\ y_i, & i = n+3, \cdots, 2n+2, \end{cases}$$

$$q_{2\ i} \equiv \dot{q}_{1\ i}, \tag{34}$$

whereas momenta follow from the definition of momentum variables in the theories with higher derivatives [17–19](see formula (36) in Appendix A: in our case, $K = 2, \quad r = 1, 2, \quad i = 1, \cdots, 2n+2$). Note that by this definition, apart from the existing primary constraints $\phi^1_1 = p_{1\ 1} = p_\alpha$ and $\phi^1_2 = p_{1\ 2} = p_\beta$, there appear extra momenta $\phi^1_i = p_{2\ i} = 0$. Then, in the extended phase space we have

$$\bar{H}_c = H_c(q_{1\ i}, p_{1\ i}), \qquad \bar{H}_T = H_T + v_i\, p_{2\ i},$$

where v_i are the Lagrange multipliers and H_c does not contain $q_{2\ i}$ and $p_{2\ i}$. As the extra coordinates and momenta which extend the phase space enter into $\bar{H}_T$ as separate terms, the structure of algebra of constraints and the system of equations (33) are not changed. Due to the definition (34) we can rewrite the generating function G (7) in the form

$$G = \left[-\ddot{\varepsilon}_1 + \frac{q_{2\ 2}}{q_{1\ 2}}\dot{\varepsilon}_1 - q_{1\ 1}\dot{\varepsilon}_2 + \left(\frac{q_{2\ 2}}{q_{1\ 2}} - q_{2\ 1}\right)\varepsilon_2\right]\frac{p_{1\ 1}}{q_{1\ 2}} - \dot{\varepsilon}_2\, p_{1\ 2} \quad (35)$$
$$+ \sum_{m=3}^{n+2}\left[\frac{1}{q_{1\ 2}}(\dot{\varepsilon}_1 + q_{1\ 1}\,\varepsilon_2)p_{1\ m}\,p_{1\ m+n} - \varepsilon_1(p_{1\ m})^2 - \varepsilon_2\, q_{1\ m+n}\, p_{1\ m}\right],$$

from which it is clear that the corresponding transformations are canonical in the expanded phase space.

Using the explicit form of coordinate transformations in the configuration space (see footnote 2 on page 6), we can write

$$\delta\alpha = -\frac{d}{dt}\left(\frac{\dot{\varepsilon}_1 + \alpha\varepsilon_2}{\beta}\right), \qquad \delta\beta = -\dot{\varepsilon}_2,$$
$$\delta\mathbf{x} = -\frac{1}{\alpha\beta}(\dot{\mathbf{x}} + \beta\mathbf{y})(\dot{\varepsilon}_1 + \alpha\varepsilon_2) + 2\frac{\dot{\mathbf{y}}}{\alpha}\varepsilon_1 + \mathbf{y}\varepsilon_2, \quad \delta\mathbf{y} = -\frac{\dot{\mathbf{y}}}{\alpha\beta}(\dot{\varepsilon}_1 + \alpha\varepsilon_2),$$

and it is not difficult to obtain that

$$\delta L = \frac{d}{dt}\left[-\frac{\dot{\mathbf{y}}}{\alpha^2\beta}(\dot{\mathbf{x}} + \beta\mathbf{y})(\dot{\varepsilon}_1 + \alpha\varepsilon_2) + \left(\frac{\dot{\mathbf{y}}}{\alpha}\right)^2\varepsilon_1\right],$$

i.e. the action is invariant under the gauge transformations we have derived.

The transformation law (35) in this example is consistent with the requirements we have discussed above, i.e. it is a two- parameter transformation as there are only two primary constraints, and it contains $\ddot{\varepsilon}_1$ and $\dot{\varepsilon}_2$, i.e. $M_1 = 3$ and $M_2 = 2$. Note that the Noether transformations derived for the Lagrangian (32) in paper [11] are a particular case of our transformations (they are one- parameter transformations).

5 Conclusion

We have suggested the method of constructing gauge transformations for arbitrary degenerate Lagrangians (without restrictions on the algebra of constraints) in the generalized Hamiltonian formalism; and they can be obtained explicitly on the basis of a specific form of the Lagrangian. The generating function given by (21) is derived from the requirement that the transformed quantities q_i' and p_i' (6) be solutions of the same system of equations (3) as the initial quantities q_i and p_i do. As to deriving gauge transformations, this requirement is equivalent to the invariance of the action under these transformations.

In our previous papers [4 and 5] we have constructed gauge transformations for Lagrangians with the only restriction on the algebra of constraints (formula (19)) that is satisfied by a wide class of theories. In this paper we have proved that there always exist equivalent sets of constraints for which the condition (19) holds valid. We have shown the way of transition to one of these sets when all the primary constraints are momentum variables.

The generating function (21) in form corresponds to the Dirac hypothesis in the sense that all the first-class constraints generate gauge transformations. The amount of arbitrary functions (important parameters) which the function G depends on is equal to the number of primary constraints. Note an essential peculiarity: the transformation law contains essential parameters and their derivatives, but the leading derivative is always present and is of an order by one smaller than the number of stages in deriving secondary constraints by the Dirac procedure.

By formulae in the footnote 2 on page 6, we have obtained the Noether transformations (i.e. with respect to which the action is invariant) in the configuration space. The mechanism of appearance of higher-order derivatives with respect to coordinates established earlier [5] in the class of theories with restriction on the algebra of constraints is now applicable in the general case.

As it is known, gauge-invariant theories belong to the class of degenerate theories. In this paper we have shown that the degeneracy of theories with the first-class constraints is due to their invariance under gauge transformations we have here constructed.

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Appendix A

We will now show invariance of the action under the gauge transformations(6) and (21) in the phase space expanded by the method of ref.

[5]. The coordinates are defined as follows:

$$q_{1\ i}=q_i,\qquad q_{s\ i}=\frac{d^{s-1}}{dt^{s-1}}q_i,\quad s=2,\cdots,K,\quad i=1,\cdots,N$$

(K equals the highest order of derivatives of q and p) and the conjugate momenta defined by the formula [17 and 19]

$$p_{r\ i}=\sum_{l=r}^{K}(-1)^{l-r}\frac{d^{l-r}}{dt^{l-r}}\frac{\partial L}{\partial q_{r+1\ i}} \tag{36}$$

are

$$p_{1\ i}=p_i,\qquad p_{s\ i}=0\quad\text{for}\quad s=2,\cdots,K.$$

The generalized momenta for $s\geq 2$ are extra primary constraints.

The total Hamiltonian is of the form

$$\bar{H}_T=H_T(q_{1\ i},p_{1\ i})+\lambda_{s\ i}\,p_{s\ i},\quad s\geq 2, \tag{37}$$

where H_T is given by (4) and $\lambda_{s\ i}$ are arbitrary functions of time. From (37) we may conclude that there do not appear additional secondary constraints corresponding to $p_{s\ i}$ for $s\geq 2$.

The set of constraints (5) in the initial phase space remains the same in the extended phase space, obeys the same algebra (13), (14), and does not depend on the new coordinates and momenta. The action is of the form

$$S=\int_{t_1}^{t_2}dt\big[p_{r\ i}\,q_{r+1\ i}+p_{K\ i}\,\dot{q}_{K\ i}-\bar{H}_T\big],\qquad r=1,\cdots,K-1. \tag{38}$$

and the generating function in the extended phase space is [5]

$$G=B_{\alpha\ \beta}^{m_\alpha m_\beta}(q_{1\ i},q_{2\ i},\cdots,q_{K\ i};p_{1\ i})\phi_\alpha^{m_\alpha}(q_{1\ i},p_{1\ i})\varepsilon_\beta^{(M_\alpha-m_\beta)}(t).$$

The coordinates and momenta are then transformed in the following way

$$\delta q_{1\ i}=\frac{\partial\big(B_{\alpha\ \beta}^{m_\alpha m_\beta}\phi_\alpha^{m_\alpha}\big)}{\partial p_{1\ i}}\varepsilon_\beta^{(M_\alpha-m_\beta)},\qquad \delta q_{s\ i}=0,$$

$$\delta p_{1\ i}=-\frac{\partial\big(B_{\alpha\ \beta}^{m_\alpha m_\beta}\phi_\alpha^{m_\alpha}\big)}{\partial q_{1\ i}}\varepsilon_\beta^{(M_\alpha-m_\beta)},\qquad \delta p_{s\ i}=-\frac{\partial B_{\alpha\ \beta}^{m_\alpha m_\beta}}{\partial q_{s\ i}}\phi_\alpha^{m_\alpha}\varepsilon_\beta^{(M_\alpha-m_\beta)}.$$

Using this equation and the equality

$$\begin{aligned}\frac{dG}{dt}&=\frac{\partial G}{\partial t}+\Bigg[\frac{\partial\big(B_{\alpha\ \beta}^{m_\alpha m_\beta}\phi_\alpha^{m_\alpha}\big)}{\partial q_{1\ i}}q_{2\ i}+\frac{\partial\big(B_{\alpha\ \beta}^{m_\alpha m_\beta}\phi_\alpha^{m_\alpha}\big)}{\partial p_{1\ i}}\dot{p}_{1\ i}\\&\quad+\left(\frac{\partial B_{\alpha\ \beta}^{m_\alpha m_\beta}}{\partial q_{s\ i}}q_{s+1\ i}+\frac{\partial B_{\alpha\ \beta}^{m_\alpha m_\beta}}{\partial q_{K\ i}}\dot{q}_{K\ i}\right)\phi_\alpha^{m_\alpha}\Bigg]\varepsilon_\beta^{(M_\alpha-m_\beta)},\end{aligned}$$

we obtain

$$\delta S=\left[p_{1\ i}\frac{\partial G}{\partial p_{1\ i}}-G\right]_{t_1}^{t_2}+\int_{t_1}^{t_2}dt\left[\frac{\partial G}{\partial t}+\{G,\bar{H}_T\}\right]. \tag{39}$$

The first term in (39) vanishes due to the boundary conditions on ε_α and their derivatives. The second term of (39), in view of (12), can be written in the form

$$\int_{t_1}^{t_2}dt\ v_\alpha\phi_\alpha^1. \tag{40}$$

and, therefore,

$$\delta S\Big|_{\phi_\alpha^1\approx 0}=0.$$

As a result of (36), (40) and $\phi_\alpha^1(q,p(q,\dot{q}))\equiv 0$ we obtain from (39) $\delta S=0$ in the configuration space.

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