сообщения
объединенного
института
ядерных
исследований
дубна

E2-92-497

A.N.Vall*, V.M.Leviant*, A.V.Sinitskaya*

SOLUTION OF THE HEISENBERG EQUATION FOR THE FOUR-FERMION CONTACT INTERACTION BY THE METHOD OF DYNAMICAL MAPPINGS

*Department of Theoretical Physics, Irkutsk State University,
Irkutsk, Russia

1992

Introduction

The present paper is devoted to the solution of the Heisenberg equation for four-fermion contact interaction (non-relativistic analog of the Nambu-Jona-Lasinio (NJL) model without form factor[1, 2]) in representation of physical particles and their states. The idea of "physical" states was first introduced by Heisenberg[3]. However, the most rigorous and consistent realization of the physical states is given in the papers[4]. The Hamiltonian of the interacting particles system on these states by definition has the diagonal form:

$$\langle a \mid H \mid b \rangle = < a \mid \int d^3q A^+(\mathbf{q}) A(\mathbf{q}) E(\mathbf{q}) \mid b > + W_0 < a \mid b > . \qquad (1)$$

The important fact is that the states $A^+(\mathbf{k}) \mid 0\rangle$ are eigenstates of the Hamiltonian H, that is

$$\begin{aligned} \left[H, A^+(\mathbf{k})\right] \mid 0\rangle &= E(\mathbf{k}) A^+(\mathbf{k}) \mid 0\rangle \\ A(\mathbf{k}) \mid 0\rangle &= 0, \end{aligned} \qquad (2)$$

where $E(\mathbf{k})$ determines thus the spectrum of one particle excitations of "physical" vacuum, defined by the second line in (2).

Since all interaction characteristics are reduced to the calculating of matrix elements of Heisenberg field combinations over the "physical" vacuum, it is necessery to have a connection between the Heisenberg fields and the "physical" ones, whose excitations are described by the $A^+(\mathbf{k})$ operators. We will call such a connection, following the papers[4], the dynamical mapping of Heisenberg fields on "physical" fields. The regular procedure of seeking the dynamical mappings is based on the fact that they must be consistent both with the Heisenberg equation for interacting fields and with relation (2). The meaning of the latters, in fact, is led to the statement that the state $A^+(\mathbf{k}) \mid 0\rangle$ describes free quasiparticle with the momentum $\mathbf{k}$ and the energy $E(\mathbf{k})$. It is worth noting that the mehtod of the dynamical mappings is naturally adjusted to describing the phase transition phenomena and the processes with the spontaneous symmetry breaking. This fact is connected with the known case, when the dynamical mappings correspond to the unitary non-equivalent transformations of the Hilbert space, transfering it to the space ortogonal to the initial one, accompanied by the reconstruction of the Hamiltonian spectrum and, in particular, by changing the vacuum (ground state) energy. We show below that the vacuum energy density depends on the parameters contained in the dynamical mapping. By varying the values of these parameters we "enumerate" all possible mutually ortogonal Hilbert spaces. One of these spaces is the "physical" space, and the criterion of its choosing is the condition for the vacuum state energy to be minimal.

Further on we consider the bound state of two excitations (of two "clad" fermions). It is well known, that while investigating the bound states properties in quantum field theory the main quantity is the Bethe-Solpither (BS) amplitude[5], given by the following matrix element:

$$\Phi_{abc}(x, y; \mathbf{k}) = \langle 0 \mid T A_a(x) B_b(y) \mid A_c(\mathbf{k})\rangle , \qquad (3)$$

where $A_a(x)$ and $B_b(y)$ are field operators in the Heisenberg representation, and $| A_c(\mathbf{k})\rangle$ is the bound state vector, a, b, c - generalized indices (spin, flavour, colour, *ect*), depending on the choice of studied theory. The usual approach is to use the BS equation[5] ether on $\Phi_{abc}(x, y; \mathbf{k})$ or on vertex function:

$$\Gamma_{ab\ldots}(x_1, x_2, \ldots) = \langle 0 \mid TA_a(x_1)B_b(x_2)\ldots \mid 0\rangle . \tag{4}$$

The type of the vertex is fixed by the interaction. It is worth mentioning, that reactions of composite particles can be described by the effective Lagrangians (see, for example, reviews[6, 7]). Development of QCD initiated the interest to the models with four-fermion interaction[8](non-local prototypes of the NJL model) in which gluons are effectively accounted by introducing the non-local quark interaction kernel[8].

The equations on the function (3) or (4) are usually solved by using some perturbative theory (loop expansion, "ladder" approximation, *ect*). This leaves the set of crucial points out of consideration. Firstly, the matter of vacuum state choice. Secondly, connection of initial field operators in Heisenberg representation with operator $\hat{A}^{+}(\mathbf{k})$, forming the bound state. At last, for the solution of BS equation it is necessery to know total two-point Green functions, i.e. the solution of the Schwinger-Dyson (SD) equation. The method of dynamical mappings allows to overcome all these dificulties, because it gives the opportunity to do direct calculations of all interaction physical characteristics.

1. Dynamical mapping of the Heisenberg fields

The Hamiltonian of the four-fermion interaction can be written in the following form:

$$H = \int d^3x \left[\psi^{+}_{\alpha}(x)\epsilon(\nabla)\psi_{\alpha}(x) + \frac{\lambda}{4}\chi^{+}(x)\chi(x)\right], \tag{5}$$

where α is spin index running over $1, 2, \epsilon(\nabla)$ - energy spectrum of free fermions, defined by the condition

$$\epsilon(\nabla)e^{i\mathbf{kx}} = \epsilon(\mathbf{k})e^{i\mathbf{kx}} \tag{6}$$

and

$$\chi(x) = \epsilon_{\alpha\beta}\psi_{\alpha}(x)\psi_{\beta}(x), \quad \chi^{+}(x) = \epsilon_{\alpha\beta}\psi^{+}_{\beta}(x)\psi^{+}_{\alpha}(x).$$

We need to find the relation between the Heisenberg fields $\psi_{\alpha}(x)$ and fields $\phi_{\alpha}(x)$, whose excitations would be eigenstates of the Hamiltonian H. In other words, it is necessary to find out the dynamical mapping [4]. As a starting point we consider the following case. Let the fields $\psi_{\alpha}(x)$ satisfy the canonical equal-time anticommutation relations:

$$\{\psi^{+}_{\alpha}(x), \psi_{\beta}(y)\}_{t_x=t_y} = \delta_{\alpha\beta}\delta(x-y). \tag{7}$$

Defining generator W as:

$$W = \int d^3x \left(f(x)\chi(x) - \overline{f}(x)\chi^{+}(x)\right), \tag{8}$$

where $f(x)$ is arbitrary complex function, we can write down the transformation of the fields $\psi_{\alpha}(x) \to \phi_{\alpha}(x)$

$$\phi_{\alpha}(x) = e^{W}\psi_{\alpha}(x)e^{-W}. \tag{9}$$

In this equation all the fields are taken at equal times. Using equation (7), we obtain:

$$\begin{aligned}\phi_{\alpha}(x) &= u(x)\psi_{\alpha}(x) - v(x)\epsilon_{\alpha\beta}\psi^{+}_{\beta}(x)\\ \psi_{\alpha}(x) &= u(x)\phi_{\alpha}(x) + v(x)\epsilon_{\alpha\beta}\phi^{+}_{\beta}(x)\end{aligned} \tag{10}$$

where

$$u(x) = cos \mid 2f(x) \mid, \quad v(x) = \frac{\overline{f}(x)}{f(x)} sin \mid 2f(x) \mid .$$

From the explicit form of $u(x)$ and $v(x)$ there follows the normalization condition:

$$u^2(x) + \mid v^2(x) \mid = 1 . \tag{11}$$

Second equation in (10) gives rise to the dynamical mapping of the Heisenberg fields $\psi_{\alpha}(x)$ on the physical fields $\phi_{\alpha}(x)$ if, certainly, it is consistent with the Hamiltonian (5). By the next step we postulate the set of properties for the field $\phi_{\alpha}(x)$. That is, let

$$\phi_{\alpha}(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k g(\mathbf{k})e^{i\mathbf{kx}-iE(\mathbf{k})t}A_{\alpha}(\mathbf{k}). \tag{12}$$

be a representation for $\phi_{\alpha}(x)$, where $A_{\alpha}(\mathbf{k})$and $A^{+}_{\alpha}(\mathbf{k})$are annihilation and creation operators for the particles with momentum $\mathbf{k}$, satisfying the following canonical relations:

$$\{A_{\alpha}(\mathbf{k}), A^{+}_{\beta}(\mathbf{q})\} = \delta_{\alpha\beta}\delta(\mathbf{k}-\mathbf{q}). \tag{13}$$

$E(\mathbf{k})$ is unknown yet excitation spectrum of $A^{+}_{\alpha}(\mathbf{k})$; $g(\mathbf{k})$ is a distribution function over the momenta inside the packet, and $g(-\mathbf{k}) = \overline{g}(\mathbf{k})$. Here we should note the following important circumstance. Representation (12) completely breaks the initial properties of transformations (9) and (10) - they cease to be canonical, i.e. they do not preserve the anticommutation relations. It is the presence of $g(\mathbf{k})$ function that causes the field anticommutation relations of $\phi_{\alpha}(x)$not to be local. Indeed, from equation (12) with account of relation (13) we have:

$$\begin{aligned}\{\phi_{\alpha}(x), \phi^{+}_{\beta}(y)\} &= \frac{\delta_{\alpha\beta}}{(2\pi)^3}\int d^3k \mid g(\mathbf{k}) \mid^2 e^{[i\mathbf{k}(\mathbf{x}-\mathbf{y})-iE(\mathbf{k})(t_x-t_y)]}\\ \{\phi_{\alpha}(x), \phi^{+}_{\beta}(y)\}_{t_x=t_y} &= \frac{\delta_{\alpha\beta}}{(2\pi)^3}\int d^3k \mid g(\mathbf{k}) \mid^2 e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\\ \{\phi_{\alpha}(x), \phi^{+}_{\beta}(y)\}_{x=y} &= \frac{\delta_{\alpha\beta}}{V^{*}}\\ \{\phi_{\alpha}(x), A^{+}_{\beta}(\mathbf{k})\} &= \frac{\delta_{\alpha\beta}}{(2\pi)^{\frac{3}{2}}}g(\mathbf{k})e^{i\mathbf{kx}}.\end{aligned} \tag{14}$$

where

$$\frac{1}{V^{*}} = \frac{1}{(2\pi)^3}\int d^3k \mid g(\mathbf{k}) \mid^2 . \tag{15}$$

So, the input of $g(\mathbf{k})$ allows to modify the product of fields to be defined at one point, thus breaking the locality. Transformation (9) plays auxiliary role for obtaining (10) leading to the fact that relations (12) and (13) have no sense any longer.

To find the meaning of normalization (15) we can use the dimensions of the fields (as it follows from (13)) $[A^2_\alpha(\mathrm{k})] = L^3, [\psi^2_\alpha(x)] = L^{-3}$, therefore, $g(\mathrm{k})$ is dimensionless quantity, and integral in the l.h.s of (15) has the dimension L^{-3}. Further on, since $| g(\mathrm{k}) |^2$ characterizes the momentum fluctuations inside the excitation, volume V^* has to be connected with a characteristic spatial size of the excitation. If one accepts that in $(\Delta k)^3$ region, where $| g(\mathrm{k}) |^2$ essentially differs from zero, it is equal to 1 ($| g(\mathrm{k}) |^2 \approx 1$), then

$$\frac{1}{(2\pi)^3}\int d^3k \mid g(\mathrm{k}) \mid^2 = \frac{1}{(2\pi)^3}(\Delta k)^3 = \frac{1}{L^{*3}} \tag{16}$$

where $L^* = \frac{2\pi}{\Delta k}$ is linear size of the packet. Accounting this fact we take for $| g(\mathrm{k}) |^2$ the normalization (15) with the additional condition:

$$<\mid g(\mathrm{k}) \mid^2>_g = \frac{\int d^3k \mid g(\mathrm{k}) \mid^4}{\int d^3k \mid g(\mathrm{k}) \mid^2} = 1. \tag{17}$$

Thus, in momentum space typical for the wave packet $\phi^+_\alpha(x)$, $| g(\mathrm{k}) |^2 \sim 1$.

So, we would search a solution in the form of relation (10) with conditions (12), (14), (15) and(17). Let us integrate the left and right parts of the equation (10) with the plain wave and obtain expressions of Heisenberg operators $a_\alpha(\mathrm{k})$ via creation $A^+_\alpha(\mathrm{k})$and annihilation $A_\alpha(\mathrm{k})$operators of quasi-particle:

$$a_\alpha(\mathrm{k}) = \int d^3k' g(\mathrm{k}')u(\mathrm{k}-\mathrm{k}')A_\alpha(\mathrm{k}') + \int d^3k' g(\mathrm{k}')v(\mathrm{k}-\mathrm{k}')\epsilon_{\alpha\beta}A^+_\beta(\mathrm{k}'). \tag{18}$$

Hence we can see that transformations (10), in contrast to the Bogolubov transformations, are nonlocal over momentum. Therefore, vacuum state $| 0 >$ defined as $A_\alpha(\mathrm{k})| 0 >= 0$, should essentially differ from the ground state defined by the Bogolubov transformations. The question is, whether the transformations (10) are "compatible" (consistent) with the coupling dynamics of the Hamiltonian (5)? It will be shown later on that at a particular choice of $u(x)$ and $v(x)$ such a compatibility takes place.

2. One-particle excitation spectrum

Let us choose the parameters $u(x)$ and $v(x)$ as follows

$$\begin{aligned} u(x) = u_0 = const, \qquad & v(x) = v_0 e^{i\alpha(x)}, v_0 = const \\ u_0^2 + v_0^2 &= 1\,. \end{aligned} \tag{19}$$

Then the dynamical mapping will be of the form:

$$\begin{aligned} \psi_\alpha(x) &= u_0\phi_\alpha(x) + v_0 e^{i\alpha(x)}\epsilon_{\alpha\beta}\phi^+_\beta(x) \\ \psi^+_\alpha(x) &= u_0\phi^+_\alpha(x) + v_0 e^{-i\alpha(x)}\epsilon_{\alpha\beta}\phi_\beta(x)\,. \end{aligned} \tag{20}$$

Our aim is to make the state $A^+_\alpha(\mathrm{k})| 0\rangle$ to be an eigenstate of the Hamiltonian (5), with the vacuum being defined with respect to the operator $A_\alpha(\mathrm{k})$

$$A_\alpha(\mathrm{k}) \mid 0\rangle = 0\,. \tag{21}$$

Acting by the Hamiltonian (5) to the one-particle excitation we have:

$$HA^+_\alpha(\mathrm{k}) \mid 0\rangle = A^+_\alpha(\mathrm{k})H \mid 0\rangle + [H, A^+_\alpha(\mathrm{k})] \mid 0\rangle\,. \tag{22}$$

If the vacuum is the Hamiltonian's eigenstate, then for the one-particle excitation to be stationary it is necessary to satisfy the condition:

$$[H, A^+_\alpha(\mathrm{k})] \mid 0\rangle = E(\mathrm{k})A^+_\alpha(\mathrm{k}) \mid 0\rangle\,. \tag{23}$$

where $E(\mathrm{k})$ is the energy of this excitation above the vacuum $| 0\rangle$. This relation can be thought either as a definition of the excitation spectrum or as the Heisenberg's stationary equation.

Leaving for the moment aside the problem of stationarity of the vacuum, let us consider only the equation (23). The commutator in the l.h.s. of (23) splits to the commutators with the kinetic term and the coupling term of the Hamiltonian independently, leading to the relations:

$$\begin{aligned} [\chi^+(x)\chi(x), A^+_\alpha(\mathrm{k})] &= \chi^+(x)\,[\chi(x), A^+_\alpha(\mathrm{k})] - [A^+_\alpha(\mathrm{k}), \chi^+(x)]\,\chi(x), \\ [\chi(x), A^+_\alpha(\mathrm{k})] &= \frac{2u_0}{(2\pi)^{\frac{3}{2}}} g(\mathrm{k})e^{i\mathrm{kx}}\epsilon_{\beta\alpha}\psi_\beta(x) \\ [A^+_\alpha(\mathrm{k}), \chi^+(x)] &= -\frac{2\overline{v}(x)}{(2\pi)^{\frac{3}{2}}} g(\mathrm{k})e^{i\mathrm{kx}}\psi^+_\alpha(x). \end{aligned} \tag{24}$$

To derive (24) we have used the dynamical mapping (20). Putting together all three relations of (24) we obtain:

$$\begin{aligned} [\chi^+(x)\chi(x), A^+_\alpha(\mathrm{k})] &= \frac{2u_0}{(2\pi)^{\frac{3}{2}}} g(\mathrm{k})e^{i\mathrm{kx}}\chi^+(x)\epsilon_{\beta\alpha}\psi_\beta(x) + \\ &+ \frac{2\overline{v}(x)}{(2\pi)^{\frac{3}{2}}} g(\mathrm{k})e^{i\mathrm{kx}}\psi^+_\alpha(x)\chi(x). \end{aligned} \tag{25}$$

So, the problem is now reduced to the calculation of two terms in the r.h.s. of (25). For the first one we get the following:

$$\begin{aligned} &\chi^+(x)\psi_\gamma(x) \mid 0\rangle = v(x)\epsilon_{\gamma\gamma'}\chi^+(x)\phi^+_{\gamma'}(x) \mid 0\rangle = \\ &= v(x)\epsilon_{\gamma\gamma'}\epsilon_{\alpha\beta}\psi^+_\beta(x)\psi^+_\alpha(x)\phi^+_{\gamma'}(x) \mid 0\rangle = \\ &= v(x)\epsilon_{\gamma\gamma'}\epsilon_{\alpha\beta}\left(u_0\phi^+_\beta(x) + \overline{v}(x)\epsilon_{\beta\beta'}\phi_{\beta'}(x)\right)\left(u_0\phi^+_\alpha(x) + \right. \\ &\left. +\overline{v}(x)\epsilon_{\alpha\alpha'}\phi_{\alpha'}(x)\right)\phi^+_{\gamma'}(x) \mid 0\rangle = \\ &= v(x)\epsilon_{\gamma\gamma'}\epsilon_{\alpha\beta}\left(u_0\overline{v}(x)\epsilon_{\alpha\alpha'}\phi^+_\beta(x)\left\{\phi_{\alpha'}(x), \phi^+_{\gamma'}(x)\right\} + \right. \\ &\left. +u_0\overline{v}(x)\epsilon_{\beta\beta'}\phi_{\beta'}(x)\phi^+_\alpha(x)\,\phi^+_{\gamma'}(x)\right) \mid 0\rangle = \\ &= \frac{u_0 v_0^2}{V^*}\epsilon_{\gamma\gamma'}\epsilon_{\alpha\beta}\left(\epsilon_{\alpha\gamma'}\phi^+_\beta(x) - \epsilon_{\beta\gamma'}\phi^+_\alpha(x) + \epsilon_{\beta\alpha}\phi^+_{\gamma'}(x)\right) \mid 0\rangle = 0, \end{aligned} \tag{26}$$

since the last expression in brackets is fully antisymmetric by indices $\alpha\beta\gamma$. Consider now the second term of (25).

$$\psi^+_\alpha\chi(x) \mid 0\rangle = \epsilon_{\beta\beta'}\psi^+_\alpha(x)\psi_\beta(x)\psi_{\beta'}(x) \mid 0\rangle =$$

$$= -v(x)\psi^+_\alpha(x)\psi_\beta(x)\phi^+_\beta(x) \mid 0\rangle = -v(x)\left(u_0\phi^+_\alpha(x) + \bar v(x)\epsilon_{\alpha\alpha'}\phi_{\alpha'}(x)\right) \times$$
$$\times \left(u_0\phi_\beta(x)\phi^+_\beta(x) + v(x)\epsilon_{\beta\beta'}\phi^+_{\beta'}(x)\phi^+_\beta(x)\right) \mid 0\rangle =$$
$$= -v(x)\left(2\frac{u_0^2}{V^*}\phi^+_\alpha(x)\right) - \left(v_0^2\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}\phi^+_{\beta'}(x)\phi_{\alpha'}(x)\phi^+_\beta(x)\right) \mid 0\rangle =$$
$$= -\frac{2v(x)}{V^*}\phi^+_\alpha(x) \mid 0\rangle\,. \tag{27}$$

Here we have used the constraint $u_0^2 + v_0^2 = 1$ With account of the results (26), (27) and the expression (25) we will have:

$$\left[\frac{\lambda}{4}\chi^+(x)\chi(x), A^+_\alpha(\mathbf{k})\right] \mid 0\rangle = -\lambda v_0^2\frac{g(\mathbf{k})}{V^*(2\pi)^{\frac{3}{2}}}e^{i\mathbf{kx}}\phi^+_\alpha(x) \mid 0\rangle\,. \tag{28}$$

Integrating this relation over the whole configuration space the contribution into the energy spectrum of the Hamiltonian reads

$$\left[\frac{\lambda}{4}\int d^3x\chi^+(x)\chi(x), A^+_\alpha(\mathbf{k})\right] \mid 0\rangle = -\lambda\frac{v_o^2}{V^*} \mid g(\mathbf{k}) \mid^2 A^+_\alpha(\mathbf{k}) \mid 0\rangle\,. \tag{29}$$

Now we derive the kinetic term contribution:

$$\left[\psi^+_{\alpha'}(x)\epsilon(\nabla)\psi_{\alpha'}(x), A^+_\alpha(\mathbf{k})\right] \mid 0\rangle = -\psi^+_{\alpha'}(x)A^+_\alpha(\mathbf{k})\epsilon(\nabla)\psi_{\alpha'}(x) \mid 0\rangle +$$
$$\psi^+_{\alpha'}(x)\epsilon(\nabla)\left\{\psi_{\alpha'}(x), A^+_\alpha(\mathbf{k})\right\} \mid 0\rangle - A^+_\alpha(\mathbf{k})\psi^+_{\alpha'}(x)\epsilon(\nabla)\psi_{\alpha'}(x) \mid 0\rangle =$$
$$= -\left\{\psi^+_{\alpha'}(x), A^+_\alpha(\mathbf{k})\right\}\epsilon(\nabla)\psi_{\alpha'}(x) \mid 0\rangle + \psi^+_{\alpha'}(x)\epsilon(\nabla)\left\{\psi_{\alpha'}(x), A^+_\alpha(\mathbf{k})\right\} \mid 0\rangle =$$
$$= -\bar v(x)\left\{\phi_\beta(x), A^+_\alpha(\mathbf{k})\right\}\epsilon(\nabla)v(x)\phi^+_\beta(x) \mid 0\rangle +$$
$$+u_0^2\phi^+_{\alpha'}(x)\epsilon(\nabla)\left\{\phi_{\alpha'}(x), A^+_\alpha(\mathbf{k})\right\} \mid 0\rangle = \tag{30}$$
$$= -\bar v(x)\frac{g(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{kx}}\epsilon(\nabla)v(x)\phi^+_\alpha(x) \mid 0\rangle + u_0^2\frac{g(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{kx}}\epsilon(\mathbf{k})\phi^+_\alpha(x) \mid 0\rangle.$$

The last term in (30) when being integrated over the space variables is proportional to the operator $A^+_\alpha(\mathbf{k})$,and gives therefore normal c-value contribution into the energy spectrum. Rather complicated situation arises after integrating the first term. Evidently, the diagonalization of this term cannot be done with an arbitrary function $v(x)$. So, we need to find the conditions under which the diagonalization is possible. Taking into account that $v(x) = v_0e^{i\alpha(x)}$ we can write:

$$\bar v(x)\epsilon(\nabla)v(x)\phi^+_\alpha(x) = v_0^2\epsilon(\nabla)\phi^+_\alpha(x) + \bar v(x)\left[\epsilon(\nabla), v(x)\right]\phi^+_\alpha(x)\,. \tag{31}$$

Here square brackets mean commutator we need to calculate.

Let $f(x)$ be an arbitrary function , then one may write

$$\frac{1}{v_0}\left[\epsilon(\nabla), v(x)\right]f(x) = \left[\epsilon(\nabla), e^{i\alpha(x)}\right]f(x) = -\frac{1}{2m}\left[(\nabla)^2, e^{i\alpha(x)}\right]f(x) =$$
$$= -\frac{1}{2m}\left(2\nabla e^{i\alpha(x)}\nabla f(x) + f(x)(\nabla)^2e^{i\alpha(x)}\right).$$

Further we have

$$(\nabla)^2e^{i\alpha(x)} = e^{i\alpha(x)}\left(i(\nabla)^2\alpha(x) - (\nabla\alpha(x))^2\right).$$

therefore

$$\left[\epsilon(\nabla), v(x)\right] = \frac{v(x)}{2m}\left(2(\nabla\alpha(x))\hat{\mathbf{k}} - i\nabla^2\alpha(x) + (\nabla\alpha(x))^2\right), \tag{32}$$

where $\hat{\mathbf{k}} = -i\nabla$.

From the considerations above it can be seen that a diagonalization is possible only at $\nabla\alpha(x) = const$. In this case the coefficient at $\phi^+_\alpha(x)$will be homogeneous and the space integration leads to the term proportional to $A^+_\alpha(\mathbf{k})$. Keeping in mind all this we take $v(x)$ in the form

$$\alpha(x) = \mathbf{k}_0\cdot\mathbf{x} \tag{33}$$

where $\mathbf{k}_0$ is some constant vector, and its physical meaning we will explain below. For the commutator we finally get

$$\left[\epsilon(\nabla), v(x)\right] = \frac{v(x)}{2m}\left(2\mathbf{k}_0\hat{\mathbf{k}} + \mathbf{k}_0^2\right). \tag{34}$$

Now we are able to write out the kinetic term contribution into the energy spectrum

$$\left[\psi^+_{\alpha'}(x)\epsilon(\nabla)\psi_{\alpha'}(x), A^+_\alpha(\mathbf{k})\right] \mid 0\rangle = u_0^2\epsilon(\mathbf{k})\frac{g(\mathbf{k})}{(2\pi)_{\frac{3}{2}}}e^{i\mathbf{kx}}\phi^+_\alpha(x) \mid 0\rangle -$$
$$-v_0^2\epsilon(\mathbf{k})\frac{g(\mathbf{k})}{(2\pi)_{\frac{3}{2}}}e^{i\mathbf{kx}}\left(\epsilon(\nabla) + \frac{1}{2m}(2\mathbf{k}_0\cdot\hat{\mathbf{k}} + \mathbf{k}_0^2)\right)\phi^+_\alpha(x) \mid 0\rangle\,. \tag{35}$$

Integrating this relation along the whole space and accounting that

$$\left(\epsilon(\nabla) + \frac{1}{2m}(2\mathbf{k}_0\cdot\hat{\mathbf{k}} + \mathbf{k}_0^2)\right)e^{-i\mathbf{kx}} = \epsilon(\mathbf{k} - \mathbf{k}_0)e^{-i\mathbf{kx}} \tag{36}$$

we have

$$\left[\int d^3x\psi^+_{\alpha'}(x)\epsilon(\nabla)\psi_{\alpha'}(x), A^+_\alpha(\mathbf{k})\right] \mid 0\rangle =$$
$$= \mid g(\mathbf{k}) \mid^2 \left(u_0^2\epsilon(\mathbf{k}) - v_0^2\epsilon(\mathbf{k} - \mathbf{k}_0)\right)A^+_\alpha(\mathbf{k}) \mid 0\rangle\,. \tag{37}$$

Combining (37) and (29) we obtain the expression for the energy spectrum of excitation

$$E(\mathbf{k}) = \mid g(\mathbf{k}) \mid^2 \left(u_0^2\epsilon(\mathbf{k}) - v_0^2\epsilon(\mathbf{k} - \mathbf{k}_0) - \lambda\frac{v_0^2}{V^*}\right). \tag{38}$$

Thus, transformation (20) under condition (33) realizes, indeed, the dynamical mapping in a sense that the fields $\phi_\alpha(x)$, entering the mapping, describe the Hamiltonian (5) eigenstates. However, all this is true due to the fact that the state without excitations $\mid 0\rangle$ is the Hamiltonian eigenstate as well. Let us dwell on this question.

3. Vacuum stability and conterterm

Let us compute the action of the Hamiltonian on the state $\mid 0\rangle$ without excitations

$$H \mid 0\rangle = \int d^3x \left[\psi^+_\alpha(x)\epsilon(\nabla)\psi_\alpha(x) + \frac{\lambda}{4}\chi^+(x)\chi(x)\right] \mid 0\rangle . \qquad (39)$$

For the kinetic term we have

$$\psi^+_\alpha(x)\epsilon(\nabla)\psi_\alpha(x) \mid 0\rangle =$$

$$= \epsilon_{\alpha\beta}\left(u_0\phi^+_\alpha(x) + \bar{v}(x)\epsilon_{\alpha\beta'}\phi_{\beta'}(x)\right)\epsilon(\nabla)v(x)\phi^+_\beta(x) \mid 0\rangle = \qquad (40)$$

$$= \bar{v}(x)\phi_\alpha(x)\epsilon(\nabla)v(x)\phi^+_\alpha(x) \mid 0\rangle + u_0\epsilon_{\alpha\beta}\phi^+_\alpha(x)\epsilon(\nabla)v(x)\phi^+_\beta(x) \mid 0\rangle .$$

The corresponding expression for the interaction term brings the following result

$$\begin{aligned} \chi^+(x)\chi(x) \mid 0\rangle &= -v(x)\chi^+(x)\psi_\alpha(x)\phi^+_\alpha(x) \quad \mid 0\rangle = \\ &= -v(x)\chi^+(x)\left(u_0\phi_\alpha(x) + v(x)\epsilon_{\alpha\beta}\phi^+_\beta(x)\right)\phi^+_\alpha(x) \quad \mid 0\rangle = \\ &= \frac{2u_0v(x)}{V^*}\chi^+(x) \mid 0\rangle - v^2(x)\epsilon_{\alpha\beta}\chi^+(x)\phi^+_\beta(x)\phi^+_\alpha(x) \quad \mid 0\rangle . \end{aligned} \qquad (41)$$

For the first term in this sum we have

$$\begin{aligned} \chi^+(x) \mid 0\rangle &= \epsilon_{\alpha\beta}\psi^+_\beta(x)\psi^+_\alpha(x) \quad \mid 0\rangle = \\ &= \epsilon_{\alpha\beta}\psi^+_\beta(x)\left(u_0\phi^+_\alpha(x) + \bar{v}(x)\epsilon_{\alpha\beta'}\phi_{\beta'}(x)\right) \quad \mid 0\rangle = \\ &= u_0\epsilon_{\alpha\beta}\left(u_0\phi^+_\beta + \bar{v}(x)\epsilon_{\beta\beta'}\phi_{\beta'}(x)\right)\phi^+_\alpha(x) \quad \mid 0\rangle = \\ &= -u_0^2\epsilon_{\alpha\beta}\phi^+_\alpha(x)\phi^+_\beta(x) \mid 0\rangle - \frac{2\bar{v}(x)u_0}{V^*} \quad \mid 0\rangle . \end{aligned} \qquad (42)$$

For the second one we obtain

$$\begin{aligned} &\epsilon_{\alpha\beta}\chi^+(x)\phi^+_\beta(x)\phi^+_\alpha(x) \mid 0\rangle = \epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}\psi^+_{\beta'}(x)\psi^+_{\alpha'}(x)\phi^+_\beta(x)\phi^+_\alpha(x) \mid 0\rangle = \\ &= \epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}\psi^+_{\beta'}(x)\left(u_0\phi^+_{\alpha'}(x) + \bar{v}(x)\epsilon_{\alpha'\alpha''}\phi_{\alpha''}(x)\right)\phi^+_\beta(x)\phi^+_\alpha(x) \mid 0\rangle = \\ &= \bar{v}(x)\epsilon_{\alpha\beta}\psi^+_{\beta'}\phi_{\beta'}(x)\phi^+_\beta(x)\phi^+_\alpha(x) \mid 0\rangle = \\ &= -4\frac{\bar{v}^2(x)}{V^{*2}} \mid 0\rangle - \frac{2}{V^*}u_0\bar{v}(x)\epsilon_{\alpha\beta}\phi^+_\alpha(x)\phi^+_\beta(x) \mid 0\rangle . \end{aligned} \qquad (43)$$

Putting together (41), (42) and (43) we get

$$\chi^+(x)\chi(x) \mid 0\rangle = 4\frac{v_0^2}{V^{*2}} \mid 0\rangle + \frac{2}{V^*}u_0v(x)\epsilon_{\alpha\beta}\phi^+_\alpha(x)\phi^+_\beta(x) \mid 0\rangle . \qquad (44)$$

Thus, for the action of the Hamiltonian density on the vacuum we would have

$$\begin{aligned} &\left(\psi^+_\alpha(x)\epsilon(\nabla)\psi_\alpha(x) + \frac{\lambda}{4}\chi^+(x)\chi(x)\right) \mid 0\rangle = \\ &= \bar{v}(x)\phi_\alpha(x)\left(\epsilon(\nabla) + \frac{\lambda}{2V^*}\right)v(x)\phi^+_\alpha(x) \mid 0\rangle + \\ &+ u_0\epsilon_{\alpha\beta}\phi^+_\alpha(x)\left(\epsilon(\nabla) + \frac{\lambda}{2V^*}\right)v(x)\phi^+_\beta(x) \mid 0\rangle . \end{aligned} \qquad (45)$$

Integrating this relation over the whole space we finally derive

$$H \mid 0\rangle = const_0 \mid 0\rangle + \Delta H(2) \mid 0\rangle \qquad (46)$$

where

$$const_0 = \int d^3x\bar{v}(x)\left\{\phi_\alpha(x), \left(\epsilon(\nabla) + \frac{\lambda}{2V^*}\right)v(x)\phi^+_\alpha(x)\right\} \qquad (47)$$

and $\Delta H(2)$ is two-particle contribution equal to

$$\Delta H(2) = \int d^3x u_0\epsilon_{\alpha\beta}\phi^+_\alpha(x)\left(\epsilon(\nabla) + \frac{\lambda}{2V^*}\right)v(x)\phi^+_\beta(x) . \qquad (48)$$

The braces in (47) stand for the anticommutator. Computing now $const_0$ by passing in (47) to the momentum representation and taking into account that $\epsilon(\mathbf{k})$ is quadratic function of the momentum we can see

$$\begin{aligned} const_0 &= \frac{2}{V^*}\int d^3x\bar{v}(x)\left(\epsilon(\nabla) + \frac{<\mathbf{k}^2>}{2m} + \frac{\lambda}{2V^*}\right) = \\ &= \frac{2v_0^2}{V^*}\int d^3x e^{-i\mathbf{k}_0\mathbf{x}}\left(\epsilon(\nabla) + \frac{<\mathbf{k}^2>}{2m} + \frac{\lambda}{2V^*}\right)e^{i\mathbf{k}_0\mathbf{x}} = \\ &= 2v_0^2\frac{V}{V^*}\left(\epsilon(\mathbf{k}_0) + \frac{<\mathbf{k}^2>}{2m} + \frac{\lambda}{2V^*}\right). \end{aligned} \qquad (49)$$

Here V - is space volume, and

$$<\mathbf{k}^2> = \frac{\int d^3k\mathbf{k}^2 \mid g(\mathbf{k}) \mid^2}{\int d^3k \mid g(\mathbf{k}) \mid^2} \qquad (50)$$

$\Delta H(2)$ being expressed in terms of the creation operators $A^+_\alpha(\mathbf{k})$of quasiparticles has the form

$$\begin{aligned} \Delta H(2) &= u_0v_0\epsilon_{\alpha\beta}\int d^3x\phi^+_\alpha(x)\left(\epsilon(\nabla) + \frac{\lambda}{2V^*}\right)e^{i\mathbf{k}_0\mathbf{x}}\phi^+_\beta(x) = \\ &= \int d^3kD(\mathbf{k})\epsilon_{\alpha\beta}A^+_\alpha(\frac{\mathbf{k}_0}{2} - \mathbf{k})A^+_\beta(\frac{\mathbf{k}_0}{2} + \mathbf{k}) \end{aligned} \qquad (51)$$

where

$$D(\mathbf{k}) = u_0v_0\bar{g}(\mathbf{k} - \frac{\mathbf{k}_0}{2})\bar{g}(\mathbf{k} + \frac{\mathbf{k}_0}{2})\left(\epsilon(\mathbf{k} - \frac{\mathbf{k}_0}{2}) + \frac{\lambda}{2V^*}\right).$$

Thus, as it follows from (46), the non-excited state $\mid 0\rangle$ is not an eigenstate of the Hamiltonian (5) and the $\Delta H(2)$ term is the source of the nonstationarity of $\mid 0\rangle$. This term describes correlated fermionic couple of excitations, moving with the momentum $\mathbf{k}_0$. The physical meaning of the relation (46) is that it points to the existence of energy exchange between the couple and the system of fermions. In order to account this exchange one must input into initial Hamiltonian the term describing the couple. So, in the relation

(46) we transfer $\Delta H(2)$ to the left part, thus, redefining the Hamiltonian and taking as a physical Hamiltonian the quantity H_P equal to

$$H_p =: H - \Delta H(2) : \tag{52}$$

where the normal ordering is referred to the Hiesenberg fields. If

$$\Delta H(2) =: \Delta H(2) : +const,$$

then

$$H_p = H - \Delta H(2) + const. \quad H_p \mid 0\rangle = H \mid 0\rangle \;-\; \Delta H(2) \mid 0\rangle + const \mid 0\rangle\,,$$
$$H_p \mid 0\rangle = W_0 \mid 0\rangle\,, W_0 = const_0 \;+\; const. \tag{53}$$

Consequently, after substitution of the conterterm $\Delta H(2)$ into the Hamiltonian (5) the non-excited state $\mid 0\rangle$ becomes stationary, with the energy equal to W_0. Will this redefinition of the Hamiltonian change the one-particle excitation spectrum $E(\mathbf{k})$? To answer the question let us write $\Delta H(2)$ in the following form:

$$\begin{aligned}\Delta H(2) &= u_0\epsilon_{\alpha\beta}\int d^3x\phi^+_\alpha(x)\left(\epsilon(\nabla)+\frac{\lambda}{2V^*}\right)v(x)\phi^+_\beta(x) - \\ &- u_0\epsilon_{\alpha\beta}\int d^3x\bar{v}(x)\phi_\alpha(x)\left(\epsilon(\nabla)+\frac{\lambda}{2V^*}\right)\phi_\beta(x).\end{aligned} \tag{54}$$

The second term, we have added to the sum (54), does not contribute to the relation (46), however, providing the hermiticity of $\Delta H(2)$. The contribution of $\Delta H(2)$ to the energy spectrum is defined by the commutator (23). It reads:

$$[\Delta H(2), A^+_\alpha(\mathbf{k})] \mid 0\rangle \sim A_\alpha \mid 0\rangle = 0\,. \tag{55}$$

Thus, the addition of the conterterm $\Delta H(2)$ to the initial Hamiltonian does not affect the spectrum.

It is worth noting that the Hamiltonian of H_P type has been studied in the papers ([9]), where some exact results for a set of physical quantities have been obtained. Our case has two essential differences. Firstly, $\Delta H(2)$ is added to the initial Hamiltonian not with an arbitrary distribution function $D(\mathbf{k})$ but in the form "provoked" by the Hamiltonian H itself. Secondly, $\Delta H(2)$ describes correlated pairs not of the Heisenberg fields, but of the excitations $A^+_\alpha(\mathbf{k})$. We will show below that just these excitations form a bound state which is the ground state of the interacting fermions.

At first sight $\Delta H(2)$ seems to break global gauge invariance. However, this is no true. The fact is that coefficient at $D(\mathbf{k})$, connected with the rotation parameters u_0 and $v(x)$, is expressed in terms of abnormal vacuum averages of the Heisenberg fields. So, under the gauge transformations of the fields these parameters are transformed as well by the way by which $\Delta H(2)$ remains invariant. In other words, condensate of correlated fermionic pairs restores the guage symmetry of the Hamiltonian. The constant entering (53) and defining the energy of vacuum state $\mid 0\rangle$ we will calculate later on, but now we consider two particle excitations of fermionic system with the Hamiltonian H_p.

4. Wave function and Bound State Spectrum of Fermionic Excitations

Let us find how the Hamiltonian H_p acts on a state composed of two excitations with momenta $\mathbf{k}$ and $\mathbf{q}$

$$\begin{aligned}H_pA^+_\alpha(\mathbf{k})A^+_\beta(\mathbf{q}) \mid 0\rangle &= (W_0 + E(\mathbf{k}) + E(\mathbf{q}))\,A^+_\alpha(\mathbf{k})A^+_\beta(\mathbf{q}) \mid 0\rangle + \\ &+ \left\{[H_p, A^+_\alpha(\mathbf{k})]\,, A^+_\beta\right\} \mid 0\rangle\,.\end{aligned} \tag{56}$$

If the excitations do not interact with each other, the last term in the sum (56) should be equal to zero. Then this two-particle excitation will be an eigenstate of the Hamiltonian H_P with the energy equal to the sum of energies of both excitations. Now we will demonstrate that this term does not vanish, reducing instead to the none-diagonal two-particle form. In this case, however, it will turn out that there exists a correlated combination of excitations which is stationary and defines the bound states of these excitations. The problem now comes to the computation of the last term in (56). Earlier above we have calculated the action of the commutator in (56) on the $\mid 0\rangle$ state defining, thus, the energy spectrum $E(\mathbf{k})$. It is necessary here to take into consideration those terms of the commutator as well, that vanished previously on the state $\mid 0\rangle$. All contributions to the commutator rise now from the kinetic term, interaction and conterterm $\Delta H(2)$. We begin treatment with the kinetic term:

$$\begin{aligned}&[\psi^+_\gamma(x)\epsilon(\nabla)\psi_\gamma(x), A^+_\alpha(\mathbf{k})] \\ =\;& \bar{v}(x)\frac{g(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{kx}}\epsilon_{\alpha\beta}\epsilon(\nabla)\psi_\beta(x) + u_0\frac{g(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{kx}}\epsilon(\mathbf{k})\psi^+_\alpha(x)\,.\end{aligned} \tag{57}$$

From which follows:

$$\begin{aligned}&\left\{[\psi^+_\gamma(x)\epsilon(\nabla)\psi_\gamma(x), A^+_\alpha(\mathbf{k})]\,, A^+_\beta(\mathbf{q})\right\} = \\ &= \epsilon_{\alpha\beta}u_0\bar{v}(x)\frac{g(\mathbf{k})g(\mathbf{q})}{(2\pi)^3}\,(\epsilon(\mathbf{q})+\epsilon(\mathbf{k}))\,e^{i(\mathbf{k}+\mathbf{q})}.\end{aligned} \tag{58}$$

Let us note that relations (57) and (58) are absolute, i.e. they are fulfilled on any state. The analogous contribution coming from conterterm reads

$$\begin{aligned}[-\Delta H(2), A^+_\alpha(\mathbf{k})] &= u_0\bar{v}(x)\epsilon_{\gamma\gamma'}\left[\phi_\gamma(x)\left(\epsilon(\nabla)+\frac{\lambda}{2V^*}\right)\phi_{\gamma'}(x), A^+_\alpha(\mathbf{k})\right] = \\ &= u_0\bar{v}(x)\frac{g(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{kx}}\epsilon_{\beta\alpha}\left(\epsilon(\mathbf{k})+\frac{\lambda}{2V^*}\right)\phi_\beta(x) + \\ &+ u_0\bar{v}(x)\frac{g(\mathbf{k})}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{kx}}\epsilon_{\beta\alpha}\left(\epsilon(\nabla)+\frac{\lambda}{2V^*}\right)\phi_\beta(x)\end{aligned} \tag{59}$$

and

$$\begin{aligned}&\left\{[-\Delta H(2), A^+_\alpha(\mathbf{k})]\,, A^+_\beta(\mathbf{q})\right\} = \\ &= \epsilon_{\beta\alpha}u_0\bar{v}(x)\frac{g(\mathbf{k})g(\mathbf{q})}{(2\pi)^3}e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}}\left(\epsilon(\mathbf{q})+\epsilon(\mathbf{k})+\frac{\lambda}{V^*}\right).\end{aligned} \tag{60}$$

Thus, as it follows from the relations (58) and (60) the contributions into anticommutator from the kinetic term and conterterm are just "c"-numbers. These contributions, as we will see later, compensate each other so that the only term proportional to the coupling constant λ is left. Now we will show that this term, in its turn, is canceled by the contribution coming from the interaction term $\frac{\lambda}{4}(\chi^+(x)\chi(x))$ of H_P. As a result the sought anticommutator acting to the vacuum $\mid 0\rangle$ will be proportional to the two-fermionic state. Now the role of the conterterm $\Delta H(2)$ is completely determined, it annihilates two-particle component in the $\mid 0\rangle$ state and vacuum component in the $A^+_\alpha(\mathbf{k})A^+_\beta(\mathbf{q})\mid 0\rangle$ state. Let us find out the contribution of the four-fermionic interaction. For the Heisenberg fields entering the relation (25) one can write:

$$
\begin{aligned}
\psi^+_\alpha(x)\chi(x) &= u_0^3\epsilon_{\gamma\gamma'}\phi^+_\alpha(x)\phi_\gamma(x)\phi_{\gamma'}(x) - \frac{2u_0^2}{V^*}v(x)\phi^+_\alpha(x) + \\
&+ 2u_0^2 v(x)\phi^+_\alpha(x)\phi^+_\beta(x)\phi_\beta(x) - \frac{2}{V^*}u_0 v_0^2\epsilon_{\alpha\beta}\phi_\beta(x) + \\
&+ 2u_0 v_0^2\epsilon_{\alpha\beta}\phi_\beta(x)\phi^+_\gamma(x)\phi_\gamma(x) + v_0^2 v(x)\epsilon_{\alpha\beta}\epsilon_{\gamma\gamma'}\phi_\beta(x)\phi^+_\gamma(x)\phi^+_{\gamma'}(x)\,.
\end{aligned} \tag{61}
$$

It is easy now to calculate the action of the corresponding anticommutator on to the vacuum. Omitting the simple calculations, amounting to the commutation of $A^+_\alpha(\mathbf{k})$with fields (61) we have:

$$
\{\psi^+_\alpha(x)\chi(x), A^+_\beta(\mathbf{q})\}\mid 0\rangle = v(x)\frac{g(\mathbf{q})}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{q}\mathbf{x}}\epsilon_{\alpha\beta}\epsilon_{\gamma\gamma'}\phi^+_\gamma(x)\phi^+_{\gamma'}(x)\mid 0\rangle\,. \tag{62}
$$

Here we have used the fact that

$$
\phi^+_\alpha\phi^+_\beta\mid 0\rangle = \frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\gamma\gamma'}\phi^+_\gamma(x)\phi^+_{\gamma'}(x)\mid 0\rangle\,. \tag{63}
$$

Further,

$$
\begin{aligned}
\{\chi^+(x)\psi_\alpha(x), A^+_\beta(\mathbf{q})\} &= \chi^+(x)\{A^+_\beta(\mathbf{q}),\psi_\alpha(x)\} + [A^+_\beta(\mathbf{q}),\chi^+(x)]\,\psi_\alpha(x) \\
\{A^+_\beta(\mathbf{q}),\psi_\alpha(x)\} &= u_0\frac{g(\mathbf{q})}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{q}\mathbf{x}}\delta_{\alpha\beta}\,.
\end{aligned} \tag{64}
$$

Unification of these relations gives

$$
\begin{aligned}
&\left\{\left[\frac{\lambda}{4}\chi^+(x)\chi(x), A^+_\alpha(\mathbf{k})\right], A^+_\beta(\mathbf{q})\right\}\mid 0\rangle = \\
&= \frac{\lambda}{V^*}u_0\bar{v}(x)\epsilon_{\alpha\beta}\frac{g(\mathbf{k})g(\mathbf{q})}{(2\pi)^3}e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}}\mid 0\rangle + \\
&+ \frac{\lambda}{2}\epsilon_{\alpha\beta}\frac{g(\mathbf{k})g(\mathbf{q})}{(2\pi)^3}e^{i(\mathbf{k}+\mathbf{q})\mathbf{x}}\epsilon_{\gamma\gamma'}\phi^+_\gamma(x)\phi^+_{\gamma'}(x)\mid 0\rangle
\end{aligned} \tag{65}
$$

for the interaction term. As it can be seen from this result the first addendum in (65) really compensates the last addendum in (61) so that the anticommutator acting to the vacuum gives the only two-fermionic state. Integration over the whole space yields:

$$
\begin{aligned}
H_pA^+_\alpha(\mathbf{q}_1)A^+_\beta(\mathbf{q}_2)\mid 0\rangle &= (W_0 + E(\mathbf{q}_1) + E(\mathbf{q}_2))\,A^+_\alpha(\mathbf{q}_1)A^+_\beta(\mathbf{q}_2)\mid 0\rangle + \\
&+ \lambda\frac{g(\mathbf{q}_1)g(\mathbf{q}_2)}{(2\pi)^3}\int d^3x e^{i(\mathbf{q}_1+\mathbf{q}_2)\mathbf{x}}\phi^+_\alpha(x)\phi^+_\beta(x)\mid 0\rangle\,.
\end{aligned} \tag{66}
$$

Here we would like to note that the excitations of two fermions with a total spin 1 (symmetrical over α and β) do not interact with each other, because in this case the last term in the sum (66) is equal to zero identically and the energy of the state presents just the sum of free energies of either excitation. Let us pass to the variables $\mathbf{P} = \mathbf{q}_1 + \mathbf{q}_2$, $\mathbf{k} = \frac{1}{2}(\mathbf{q}_1 - \mathbf{q}_2)$. Then using the expression for the fields $\phi_\alpha(x)$ via operators $A_\alpha(\mathbf{k})$, we can rewrite the equation (66) in the following form:

$$
\begin{aligned}
&H_pA^+_\alpha(\frac{\mathbf{P}}{2}-\mathbf{k})A^+_\beta(\frac{\mathbf{P}}{2}+\mathbf{k})\mid 0\rangle = \\
&= \left(W_0 + E(\frac{\mathbf{P}}{2}-\mathbf{k}) + E(\frac{\mathbf{P}}{2}+\mathbf{k})\right)A^+_\alpha(\frac{\mathbf{P}}{2}-\mathbf{k})A^+_\beta(\frac{\mathbf{P}}{2}+\mathbf{k})\mid 0\rangle + \\
&+\frac{\lambda}{(2\pi)^3}g(\frac{\mathbf{P}}{2}-\mathbf{k})g(\frac{\mathbf{P}}{2}+\mathbf{k})\int d^3q\bar{g}(\frac{\mathbf{P}}{2}-\mathbf{q})\bar{g}(\frac{\mathbf{P}}{2}+\mathbf{q})A^+_\alpha(\frac{\mathbf{P}}{2}-\mathbf{q})A^+_\beta(\frac{\mathbf{P}}{2}+\mathbf{q})\mid 0\rangle\,.
\end{aligned} \tag{67}
$$

Define the wave packet $\hat{A}^+(\mathbf{P})$ as:

$$
\hat{A}^+(\mathbf{P}) = \int G(\mathbf{k},\mathbf{P})\epsilon_{\alpha\beta}A^+_\alpha(\frac{\mathbf{P}}{2}-\mathbf{k})A^+_\beta(\frac{\mathbf{P}}{2}+\mathbf{k})d^3k \tag{68}
$$

and choose the wave function $G(\mathbf{k},\mathbf{P})$ in order to $\hat{A}^+(\mathbf{P})\mid 0\rangle$ be an eigenstate of the Hamiltonian H:

$$
H_p\hat{A}^+(\mathbf{P})\mid 0\rangle = (\mathbf{W_0} + \mu(\mathbf{P}))\,\hat{\mathbf{A}}^+(\mathbf{P})\mid 0\rangle\,. \tag{69}
$$

After the simple calculations we obtain from (67) and (69):

$$
G(\mathbf{k},\mathbf{P}) = \gamma_0\cdot\frac{\bar{g}(\frac{\mathbf{P}}{2}-\mathbf{k})\bar{g}(\frac{\mathbf{P}}{2}+\mathbf{k})}{\mu(\mathbf{P}) - E(\frac{\mathbf{P}}{2}-\mathbf{k}) - E(\frac{\mathbf{P}}{2}+\mathbf{k})}, \tag{70}
$$

where

$$
\gamma_0 = \frac{\lambda}{(2\pi)^3}\int d^3kG(\mathbf{k},\mathbf{P})g(\frac{\mathbf{P}}{2}-\mathbf{k})g(\frac{\mathbf{P}}{2}+\mathbf{k}).
$$

This set of relations determines the wave function $G(\mathbf{k},\mathbf{P})$ and the energy of the two excitations bound state $\mu(\mathbf{P})$, with $\mathbf{P}$ being momentum of the bound state as a whole, so that $\mu(\mathbf{P}=0)$ is the bound state mass. In order to obtain the equation on $\mu(\mathbf{P})$ let us substitute the manifest form of the function $G(\mathbf{k},\mathbf{P})$ (70) into the expression for γ_0. We see that γ_0 is cancelled from the left and right parts and we have the following equation:

$$
1 = \frac{\lambda}{(2\pi)^3}\int d^3k\frac{\mid g(\frac{\mathbf{P}}{2}-\mathbf{k})\mid^2\mid g(\frac{\mathbf{P}}{2}+\mathbf{k})\mid^2}{\mu(\mathbf{P}) - E(\frac{\mathbf{P}}{2}-\mathbf{k}) - E(\frac{\mathbf{P}}{2}+\mathbf{k})}. \tag{71}
$$

This equation is the necessary and sufficient condition for existence of two particle bound state, and it determines the bound state energy $\mu(\mathbf{P})$. It is worth noting that this kind of condition always appears when searching the solution of the BS equation[10], what follows from its homogeneity, and equation (71) corresponds to the zero determinant of the homogeneous system. We will analyse this equation below, but now let us turn to the calculation of the BS amplitude $\Phi_{\alpha\beta}(x,y;\mathbf{P})$. It contains all the information about bound state. As has been pointed out in Introduction, to find it one has to solve the BS equation, which kernel in its turn is determined by the solution of the SD equation. However, if the dynamical mapping is known, it is possible to calculate $\Phi_{\alpha\beta}(x,y;\mathbf{P})$ directly, since, by definition, it is a matrix element between two physical states for the product of the Heisenberg fields:

$$\begin{aligned}\Phi_{\alpha\beta}(x,y;\mathbf{P}) &= \langle 0 \mid T\psi_\alpha(x)\psi_\beta(y) \mid A(\mathbf{P})\rangle\,,\\ \mid A(\mathbf{P})\rangle &\equiv \hat{A}^+(\mathbf{P}) \mid 0\rangle\,.\end{aligned} \tag{72}$$

We can construct four BS amplitudes from the fields $\psi_\alpha(x)$. Introducing the variables $\mathbf{R} = \frac{\mathbf{x}+\mathbf{y}}{2}$, $\mathbf{r} = \mathbf{x}-\mathbf{y}$ after straightforward calculations we can write for these amplitudes:

$$\begin{aligned}\langle 0 \mid T\psi^+_\alpha(x)\psi_\beta(y) \mid A(\mathbf{P})\rangle &= u_0v_0e^{-i\mathbf{k}_0\mathbf{x}}\epsilon_{\alpha\alpha'}\delta_{\beta\beta'}G_{\alpha'\beta'}(x,y;\mathbf{P});\\ \langle 0 \mid T\psi_\alpha(x)\psi^+_\beta(y) \mid A(\mathbf{P})\rangle &= u_0v_0e^{-i\mathbf{k}_0\mathbf{y}}\epsilon_{\beta\beta'}\delta_{\alpha\alpha'}G_{\alpha'\beta'}(x,y;\mathbf{P});\\ \langle 0 \mid T\psi_\alpha(x)\psi_\beta(y) \mid A(\mathbf{P})\rangle &= u_0^2\delta_{\alpha\alpha'}\delta_{\beta\beta'}G_{\alpha'\beta'}(x,y;\mathbf{P});\\ \langle 0 \mid T\psi^+_\alpha(x)\psi^+_\beta(y) \mid A(\mathbf{P})\rangle &= v_0^2e^{-2i\mathbf{k}_0\mathbf{R}}\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}G_{\alpha'\beta'}(x,y;\mathbf{P}),\end{aligned} \tag{73}$$

where the vertex part $G_{\alpha\beta}(x,y;\mathbf{P})$ is defined by the matrix element:

$$G_{\alpha\beta}(x,y;\mathbf{P}) = \langle 0 \mid \phi_\alpha(x)\phi_\beta(y) \mid A(\mathbf{P})\rangle = \tag{74}$$
$$= -\frac{2}{(2\pi)^3}e^{i\mathbf{R}\mathbf{P}}\epsilon_{\alpha\beta}\int d^3k G(\mathbf{k},\mathbf{P})g(\frac{\mathbf{P}}{2}+\mathbf{k})g(\frac{\mathbf{P}}{2}-\mathbf{k})e^{i\mathbf{k}\mathbf{r}}e^{-iE(\frac{\mathbf{P}}{2}+\mathbf{k})t_x-iE(\frac{\mathbf{P}}{2}-\mathbf{k})t_y}.$$

Consider this relation at equal times $t_x = t_y = t$. We have:

$$G_{\alpha\beta}(x,y;\mathbf{P})\mid_{t_x=t_y=t} = -\frac{2\epsilon_{\alpha\beta}}{(2\pi)^{\frac{3}{2}}}e^{i\mathbf{R}\mathbf{P}-i\mu(\mathbf{P})t}\gamma_0 q(\mathbf{r},t;\mathbf{P}), \tag{75}$$
$$q(\mathbf{r},t;\mathbf{P}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\frac{\mid g(\frac{\mathbf{P}}{2}+\mathbf{k})\mid^2\mid g(\frac{\mathbf{P}}{2}-\mathbf{k})\mid^2 e^{i\mathbf{k}\mathbf{r}}}{\mu(\mathbf{P})-E(\frac{\mathbf{P}}{2}-\mathbf{k})-E(\frac{\mathbf{P}}{2}+\mathbf{k})}e^{it\left(\mu(\mathbf{P})-E(\frac{\mathbf{P}}{2}-\mathbf{k})-E(\frac{\mathbf{P}}{2}+\mathbf{k})\right)}.$$

The plain wave appeared in (75) describes a movement of the bound state as a whole with the momentum $\mathbf{P}$ and the energy $\mu(\mathbf{P})$, and form factor $q(\mathbf{r},t;\mathbf{P})$ is related with the bound state internal structure. At $t=0$, $\mathbf{P}=0$ $q(\mathbf{r},0;0) \equiv \Phi(\mathbf{r})$ coinsides with Schredinger wave function in "x" representation. Let us consider it in details:

$$\Phi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int d^3k\frac{\mid g(\mathbf{k})\mid^2}{\mu - E(\mathbf{k}) - E(-\mathbf{k})}e^{i\mathbf{k}\mathbf{x}} \tag{76}$$

is thought to be the bound state wave function in the configuration space. The bound state has the structure of the pair strictly correlated over the momenta. Parameter γ_0 is determined by the normalization of distribution $G(\mathbf{k},\mathbf{P})$. Due to the fact that $\lambda \leq 0$ the following relation

$$\mu - E(\mathbf{k}) - E(-\mathbf{k}) < 0 \tag{77}$$

should be satisfied for all $\mathbf{k}$, since only in this case the signs of the left and right sides of the equation (71) will be conformed and the function under the integral will not have the peculiar points in the whole integration region.

Let us calculate now the wave function $\Phi(\mathbf{x})$, considering the case when $\epsilon(\mathbf{k}) = \frac{k^2}{2m}$ and representing $G(\mathbf{k})$ in the following form:

$$\frac{\mid g(\mathbf{k})\mid^2}{E(\mathbf{k})+E(-\mathbf{k})-\mu} = \frac{M}{\mathbf{k}^2+\delta^2}\,. \tag{78}$$

Then the nonequality (77) corresponds to the conditions:

$$M > 0,\ \delta > 0\,. \tag{79}$$

By the use of the manifest form of the energy spectrum (38) we obtain:

$$\begin{aligned}M &= \frac{m}{1-2v_0^2} > 0\\ \delta^2 &= -\frac{1}{1-2v_0^2}\left(v_0^2\mathbf{k}_0^2 + \frac{2m\lambda}{V^*}v_0^2 + \frac{m\mu}{\mid g(\mathbf{k})\mid^2}\right) > 0.\end{aligned} \tag{80}$$

The approximation can be made by putting $\mid g(\mathbf{k})\mid^2 = 1$ and the integral (76) will remain still convergent. As far as $\mid g(\mathbf{k})\mid^2$ plays the role of a cutting-off factor at the large momenta, this approximation will lead to distortion of $\Phi(\mathbf{x})$ only in the region $\mathbf{x} \sim 0$. Having in mind all mentioned above we can see, that from equation (80) follows:

$$\begin{aligned}\mu < -2v_0^2\left(\frac{\mathbf{k}_0^2}{2m} + \frac{\lambda}{V^*}\right) \simeq 2E(0)\,.\\ 1-2v_0^2 > 0\,.\end{aligned} \tag{81}$$

Evidently, $E(0)$ defines the lower boundary of the energy spectrum $E(\mathbf{k})$. The relations (81) represent the necessary conditions for existing of the bound state of the fermionic excitations. There are two possible different physical situations according to the sign of $E(0)$. In the case when $E(0) \leq 0$ the state $\mid 0\rangle$ with the energy W_0 is located in the excitation spectrum, and the bound state is the lowest, being at least $2E(0)$ distant from the lower boundary of spectrum. Because $E(\mathbf{k})$ at $\mathbf{k}$ large enough becomes positive, there exists such a momentum $\mathbf{p}_0$ for which $E(\mathbf{p}_0) = 0$. Thus the state

$$\mid 0,1 >= a \mid 0\rangle + c_\alpha A^+_\alpha(\mathbf{p}_0) \mid 0\rangle\,, \tag{82}$$

where a and c_α - any complex numbers, is a stationary state with the energy W_0. If $E(0) > 0$, then for the value of μ three variants are possible. Firstly, $E(0) \leq \mu < 2E(0)$. Then the bound state lies within the excitation spectrum and the $\mid 0\rangle$ state with the energy W_0 will be the ground state with the energy gap between it and the excitation spectrum. Secondly, when $0 \leq \mu \leq E(0)$ two levels system arises with the ground state equal to $\mid 0\rangle$. And lastly, if $\mu < 0$, we will have again two levels system, though the bound state will be the ground state of two excitations with the energy equal to μ. In the particular case when $E(0) = 0$ the momentum $\mathbf{k}_0$ is determined by the four-fermionic coupling constant λ. ¿From the expression (81) we obtain:

$$\mathbf{k}_0^2 = -\frac{2m\lambda}{V^*}\,. \tag{83}$$

For the wave function $\Phi(\mathbf{x})$ we have, up to the normalization:

$$\Phi(\mathbf{x}) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \frac{|g(\mathbf{k})|^2}{E(\mathbf{k}) + E(-\mathbf{k}) - \mu} e^{i\mathbf{kx}} \simeq M \int d^3k \frac{e^{i\mathbf{kx}}}{\mathbf{k}^2 + \delta^2} . \tag{84}$$

Integrating it over we get:

$$\Phi(\mathbf{x}) = M \frac{2\pi^2}{r} e^{-r\delta} . \tag{85}$$

As a consequence of the approximation we see the singularity of (85) at $r = 0$. In order to account correction to (85) let us introduce the screening parameter r_0:

$$\Phi(\mathbf{x}) \Rightarrow \Phi(\mathbf{x}) = M \frac{2\pi^2}{r + r_0} e^{-r\delta} . \tag{86}$$

From here we obtain

$$r_0 = M \frac{2\pi^2}{\Phi(0)} .$$

Further, as it follows from (84)

$$\begin{aligned} \Phi(0) &= \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \frac{|g(\mathbf{k})|^2}{E(\mathbf{k}) + E(-\mathbf{k}) - \mu} \simeq \\ &\simeq \frac{1}{(2\pi)^{\frac{3}{2}}} \int d^3k \frac{|g(\mathbf{k})|^4}{E(\mathbf{k}) + E(-\mathbf{k}) - \mu} = -\frac{(2\pi)^3}{\lambda} \end{aligned} \tag{87}$$

where we have used the equation (71). Finally we derive

$$r_0 = -\frac{\lambda M}{4\pi} . \tag{88}$$

It is seen from the solution (86) that quantity $\frac{1}{2\delta}$ determines the radius of the bound state. Note, at last, that the wave function $\Phi(\mathbf{x})$ satisfies the equation

$$\left(E(\nabla) + E(-\nabla) - \mu\right) \Phi(\mathbf{x}) = \frac{1}{2} (2\pi)^3 \left\{ \phi_\alpha(\mathbf{x}), \phi_\alpha^+(0) \right\} , \tag{89}$$

which follows from the manifest form of $\Phi(\mathbf{x})$.

As a next step let us examine the solutions of the equation (71). Multiplication of the numerator and the denominator under the integral on the same quantity $E(\mathbf{k}) + E(-\mathbf{k}) - \mu$ gives for μ:

$$\mu = 2 \langle E(\mathbf{k}) \rangle_G + \frac{(2\pi)^3}{\lambda \int d^3k G^2(\mathbf{k})} \tag{90}$$

where

$$2 \langle E(\mathbf{k}) \rangle_G = \frac{\int d^3k G^2(\mathbf{k}) \left(E(\mathbf{k}) + E(-\mathbf{k})\right)}{\int d^3k G^2(\mathbf{k})} .$$

Substituting the explicit form of the spectrum (38) into $\langle E(\mathbf{k}) \rangle_G$ and supposing that $|g(\mathbf{k})|^2 \simeq 1$ we obtain:

$$\mu = 2 \frac{\langle \mathbf{k}^2 \rangle}{2M} - v_0^2 \frac{\mathbf{k}_0^2}{V^*} + \frac{(2\pi)^3}{\lambda \int d^3k G^2(\mathbf{k})} . \tag{91}$$

Hereupon one can see that if the quantity M is interpreted as an effective mass of the excitation then the first term in (90) represents the sum of the kinetic energies each of the excitations. The last two terms describe the bond energy of the excitations.

The solution of the equation (71) we obtain as an asymptotic series over the coupling constant λ. Using again the approximate expression for δ^2, viz. keeping $|g(\mathbf{k})|^2 \approx 1$ as in (83) we can write equation (71) in the following form:

$$1 = -\frac{\lambda M}{(2\pi)^3} \int d^3k \frac{|g(\mathbf{k})|^2}{\mathbf{k}^2 + \delta^2} . \tag{92}$$

This equation defines δ^2 as a function of λ. We will seek solution of (92) in the form of the asymptotic series

$$\begin{aligned} \delta^2 &= a_0 + a\lambda + \frac{c_1}{\lambda} + \frac{c_2}{\lambda^2} + \cdots \\ \mathbf{k}^2 + \delta^2 &= \lambda \left(a + \frac{\mathbf{k}^2 a_0}{\lambda} + \frac{c_1}{\lambda^2} + \frac{c_2}{\lambda^3} + \cdots \right). \end{aligned} \tag{93}$$

By means of the series conversion formulae we can derive

$$\frac{\lambda}{\mathbf{k}^2 + \delta^2} = \frac{1}{a + b\epsilon + c\epsilon^2 + \cdots} = \frac{1}{a} \left[1 - \frac{b}{a}\epsilon + \left(\frac{b^2}{a^2} - \frac{c}{a} \right) \epsilon^2 + \cdots \right] \tag{94}$$

where $\epsilon = \frac{1}{\lambda}$, $b = a_0 + \mathbf{k}^2$, $c = c_1$, ect. Substituting the series expansion (94) into the equation (92), collecting the terms of the same power over λ and equating them to zero, we will obtain the chain of relations defining the coefficients a_0, a, c_1, ect. Straightforward calculations give the following result:

$$\begin{aligned} a &= -\langle \mathbf{k}^2 \rangle, \quad a = -\frac{M}{V*}, \\ c_1 &= -\frac{M}{V*} \frac{\int d^3k \, |g(\mathbf{k})|^2 \left(\mathbf{k}^2 - \langle \mathbf{k}^2 \rangle\right)^2}{\int d^3k \, |g(\mathbf{k})|^2} \equiv -\frac{V^*}{M}\sigma \\ \langle \mathbf{k}^2 \rangle &= \frac{\int d^3k \mathbf{k}^2 \, |g(\mathbf{k})|^2}{\int d^3k \, |g(\mathbf{k})|^2} \end{aligned} \tag{95}$$

from which we receive the asymptotic series for δ^2 and μ:

$$\begin{aligned} \delta^2 &= -\langle \mathbf{k}^2 \rangle - \frac{M}{V^*}\lambda - \frac{V^*}{M} \cdot \frac{\sigma}{\lambda} + \cdots , \\ \mu &= -\frac{\delta^2}{M} + 2E(0), \quad 2E(0) = \left(\frac{\mathbf{k}_0^2}{2m} + \frac{\lambda}{V^*} \right) \cdot \left(\frac{m}{M} - 1 \right) . \end{aligned} \tag{96}$$

According to the condition (80) $\delta > 0$. This condition the restricts region of the admissible values of M and henceforth v_0^2. The magnitude of M is determined from the vacuum energy W_0 minimization condition.

¿From the relations (95) and (96) it is easy to see that the nonperturbative and singular contributions over the coupling constant λ into the energy are defined by the dispersion σ over momentum distribution of $|g(\mathbf{k})|^2$ inside the excitation.

5. Minimization of the Vacuum Energy W_0 With Respect to the Transformation Parameters

We will go on now calculating the vacuum energy W_0 and minimizing it with respect to the parameters. This procedure corresponds physically to the fact that at the spontaneous transition, described by the transformations (10), the system itself fixes the magnitudes of the parameters, choosing the states with the lowest energy.

As it follows from the relations (53), the vacuum energy W_0 is equal to:

$$W_0 = \frac{2v_0^2}{V^*}\int d^3x e^{-i\alpha(x)}\epsilon(\nabla)e^{i\alpha(x)} + 2v_0^2\frac{V}{V^*}\left(\frac{\langle k^2\rangle}{2m}+\frac{\lambda}{2V^*}\right) + const. \tag{97}$$

where *const.* is defined by result of bringing the conterterm $\Delta H(2)$ to the normal ordering form with respect to the Heisenberg fields. v_0 and $\alpha(x)$ are the transformation parameters. When we have been obtaining the spectrum of one-particle excitations, as a necessary condition of their existence we have found that $\alpha(x) = k_0x$. We retain $\alpha(x)$ in the expression for the vacuum energy W_0 as an arbitrary function determining it later from the conditions of the absolute minimum existing for W_0 over the variables v_0 and $\alpha(x)$. As it will be demonstrated in a moment we will have the same result, what points to the strict correlation between the existence of minimum and the one-particle excitation.

To calculate *const.* in (97) it is convenient to pass in the relation (54) from the fields $\phi_\alpha(x)$ to $\psi_\alpha(x)$ in accordance with the transformations (10). As a result we obtain:

$$\begin{aligned}-\Delta H(2) &= u_0\left(u_0^2-v_0^2\right)\epsilon_{\alpha\beta}\psi_\beta^+(x)\left(\epsilon(\nabla)+\frac{\lambda}{2V^*}\right)v(x)\psi_\alpha^+(x)+\\ &+ u_0\left(u_0^2-v_0^2\right)\bar{v}(x)\epsilon_{\alpha\beta}\psi_\alpha(x)\left(\epsilon(\nabla)+\frac{\lambda}{2V^*}\right)\psi_\beta(x)-\\ &- 2u_0^2v_0^2\psi_\alpha^+(x)\left(\epsilon(\nabla)+\frac{\lambda}{2V^*}\right)\psi_\alpha(x)+\\ &+ 2u_0^2\bar{v}(x)\psi_\alpha(x)\left(\epsilon(\nabla)+\frac{\lambda}{2V^*}\right)v(x)\psi_\alpha^+(x).\end{aligned} \tag{98}$$

We can see that the only term in (98) which does not have the normal ordering form is the last addendum. Therefore

$$\begin{aligned}const. &= -2u_0^2\int d^3x\bar{v}(x)\left\{\psi_\alpha(x),\left(\epsilon(\nabla)+\frac{\lambda}{2V^*}\right)v(x)\psi_\alpha^+(x)\right\} = \\ &= -\frac{4u_0^4v_0^2}{V^*}\int d^3xe^{-i\alpha(x)}\epsilon(\nabla)e^{i\alpha(x)} - 4\frac{V^*}{V}u_0^2v_0^2\left(\frac{\langle k^2\rangle}{2m}+\frac{\lambda}{2V^*}\right).\end{aligned} \tag{99}$$

Together with the representation (97) this leads to the following expression for the energy W_0

$$\frac{V^*}{V}W_0 = \rho(v_0)F(\alpha)+\gamma(v_0)\Lambda,$$

where

$$\begin{aligned}F(\alpha) &= \frac{1}{V}\int d^3xe^{-i\alpha(x)}\epsilon(\nabla)e^{i\alpha(x)}, \quad \Lambda = \frac{\langle k^2\rangle}{2m}+\frac{\lambda}{2V^*},\\ \rho(v_0) &= 2v_0^2\left(1-2(1-v_0^2)^2\right), \quad \gamma(v_0) = -2v_0^2(1-2v_0^2).\end{aligned} \tag{100}$$

The functional $F(\alpha)$ turns out to be essentially positive quantity. Varying it with respect to $\alpha(x)$ and putting the first variation to be equal to zero, it is easy to find that an extremum of $F(\alpha)$ (minimum in fact) is achieved on the functions satisfying the equation $\nabla^2\alpha(x) = 0$. Evidently, the extremum would define a minimal value of the energy W_0 provided that $\rho(v_0) > 0$. Therefore, the minimization of W_0 over v_0^2 should be achieved within the region where this condition is fulfilled. ¿From the equation for $\alpha(x)$ one can easily write out the only solution as $\alpha(x) = k_0x$. Let us study now the behavior of W_0 as a function of parameter $v^2 = -\kappa$. There are three possible cases corresponding to the three different regions of $\frac{\Lambda}{\epsilon(k_0)}$: $1+\frac{\Lambda}{\epsilon(k_0)} < 0$; $1+\frac{\Lambda}{\epsilon(k_0)} = 0$; $1+\frac{\Lambda}{\epsilon(k_0)} > O$.

The value of W_0 at the minimum point should be negative and, besides, in this point $\rho(\kappa_0) > 0$. It is seen from the explicit form of $\rho(\kappa)$ (relation (100)) that this condition is realized within to regions, when $\kappa = -v_0^2 > 0$ and when $-0.5 < \kappa < -1+\frac{1}{\sqrt{2}} \simeq -0.3$. The left boundary of this inequality is determined by the condition (80). The equation on κ_0 is obtained from the condition that the derivative of W_0 with respect to κ is equal to zero and has the form

$$\kappa_0^2+\frac{2}{3}\kappa_0\left(2+\frac{\Lambda}{\epsilon(k_0)}\right)+\frac{1}{6}\left(1+\frac{\Lambda}{\epsilon(k_0)}\right)=0. \tag{101}$$

Hence we have

$$\kappa_0 = -\frac{1}{3}\left(2+\frac{\Lambda}{\epsilon(k_0)}\right)+\frac{1}{3}\left[\left(2+\frac{\Lambda}{\epsilon(k_0)}\right)^2-\frac{2}{3}\left(1+\frac{\Lambda}{\epsilon(k_0)}\right)\right]^{\frac{1}{2}}. \tag{102}$$

At $1+\frac{\Lambda}{\epsilon(k_0)} \to \infty, \kappa_0 \to -\frac{3}{8} \simeq -0.375$; and at $\kappa_0 = -1+\frac{1}{\sqrt{2}}$ from (102) follows that $1+\frac{\Lambda}{\epsilon(k_0)} \simeq 2$. Thus, the vacuum energy W_0 has the absolute minimum with respect to the variables $\alpha(x)$ and v_0, provided that $1+\frac{\Lambda}{\epsilon(k_0)} > 2$, and position of this minimum lies within the region $0.3 < v_0^2 < 0.375$. The excitation mass $M = \frac{m}{1-2v_0^2}$ is larger then the fermionic mass m. There are no solutions inside the region $0 \le 1+\frac{\Lambda}{\epsilon(k_0)} < 2$. Finally, if $1+\frac{\Lambda}{\epsilon(k_0)} < 0$ then minimum W_0 is situated at the region of negative values of v_0^2, and transformation (10) corresponds to the hyperbolic rotation. Using expression for Λ and (100) we get

$$\frac{k_0^2}{2m}+\left(\frac{\langle k^2\rangle}{2m}+\frac{\lambda}{2V^*}\right) < 0\,. \tag{103}$$

Let us rewrite this relation on the form either

$$\frac{k_0^2}{2m}+\frac{\lambda}{V^*}+\left(\frac{\langle k^2\rangle}{2m}-\frac{\lambda}{2V^*}\right) < 0 \tag{104}$$

or

$$\frac{k_0^2}{2m}+\frac{\lambda}{V^*} < \left(\frac{\langle k^2\rangle}{2m}+\frac{\lambda}{2V^*}\right) < 0\,. \tag{105}$$

Hence it follows that $2E(0) < 0$. Thus, the state $\hat{A}^+\,|\,0\rangle$ will be the lowest state with the energy equal to μ. The excitation mass M in this case will be less than m. Further, contribution to the bound state energy μ corresponding to the kinetic energy of the pair moving as a whole (second term in the sum (91)) will have positive sign but the third term, describing interaction of the excitations - negative sign, in agreement with the physical picture.

Conclusion

Finally we want to emphasize some aspects of this paper. The basic and most essential one is postulating the dynamical mapping, but the transformations (9) and (10) have been used just as a motivation sample to find the mapping (20).

The next important step is connected with the introduction of the distribution function $g(\mathbf{k})$, which allows to define the product of the physical fields at the same space-time point. This fact corresponds to the "cut-off" at large momenta, and the free parameter V^* (spatial volume of excitation), appearing in our model, is related with the "cut-off" parameter Λ_{cut} by the expression:

$$V^* = \frac{3}{4\pi}\left(\frac{2\pi}{\Lambda_{cut}}\right)^3 .$$

Such a smooth way of regularization leads to the fact that the anticommutation relations for $\phi_\alpha(x)$ (14) cease to be local, that results, in its turn, in the arising of abnormal nonzero anticommutators between the Heisenberg fields.

As a nontrivial cosequence of "g"-regularization we have the fact that in all dynamical characteristics - energy spectrum, vacuum energy density, bound state mass,- the coupling constant λ is renormalized, such that the effective coupling constant turns out to be $\frac{\lambda}{V^*}$.

The method of dynamical mappings in the form used here can be easily generalized on the relativistic model (NJL), at least in separable approximation. This problem will be considered in the forthcoming paper.

Aknowlegments

The authors are grateful to Profs. D.V.Shirkov, V.N.Pervushin, Yu.L.Kalinovsky and V.A.Naumov for the crucial comments, useful disscusions and support.

Appendix

Here we present the formulas for some physical quantities.

I. Condensates.

$$\begin{aligned}
\langle 0 \mid \psi_\alpha(x)\psi^+_\beta(x) \mid 0\rangle &= \frac{u_0^2}{V^*}\delta_{\alpha\beta};\\
\langle 0 \mid \psi^+_\alpha(x)\psi_\beta(x) \mid 0\rangle &= \frac{v_0^2}{V^*}\delta_{\alpha\beta};\\
\langle 0 \mid \psi^+_\alpha(x)\psi^+_\beta(x) \mid 0\rangle &= \frac{u_0 v_0}{V^*}\epsilon_{\alpha\beta}e^{-i\mathbf{k}_0\mathbf{x}};\\
\langle 0 \mid \psi_\alpha(x)\psi_\beta(x) \mid 0\rangle &= -\frac{u_0 v_0}{V^*}\epsilon_{\alpha\beta}e^{i\mathbf{k}_0\mathbf{x}}.
\end{aligned} \tag{A.1}$$

II. Casual Green functions for the Heisenberg fields.

$$\begin{aligned}
\langle 0 \mid T\psi_\alpha(x)\psi^+_\beta(y) \mid 0\rangle &= u_0^2 G^{(1)}_{\alpha\beta}(x-y) + v_0^2 e^{i\mathbf{k}_0(\mathbf{x}-\mathbf{y})}G^{(2)}_{\alpha\beta}(x-y);\\
\langle 0 \mid T\psi^+_\alpha(x)\psi_\beta(y) \mid 0\rangle &= v_0^2 e^{-i\mathbf{k}_0(\mathbf{x}-\mathbf{y})}G^{(1)}_{\alpha\beta}(x-y) + u_0^2 G^{(2)}_{\alpha\beta}(x-y);\\
\langle 0 \mid T\psi^+_\alpha(x)\psi^+_\beta(y) \mid 0\rangle &= u_0 v_0 \epsilon_{\alpha\beta'}\left(e^{-i\mathbf{k}_0\mathbf{x}}G^{(1)}_{\beta'\beta}(x-y) - e^{-i\mathbf{k}_0\mathbf{y}}G^{(2)}_{\beta'\beta}(x-y)\right);\\
\langle 0 \mid T\psi_\alpha(x)\psi_\beta(y) \mid 0\rangle &= -u_0 v_0 \epsilon_{\alpha\beta'}\left(e^{i\mathbf{k}_0\mathbf{x}}G^{(1)}_{\beta'\beta}(x-y) - e^{i\mathbf{k}_0\mathbf{y}}G^{(2)}_{\beta'\beta}(x-y)\right),
\end{aligned} \tag{A.2}$$

where $G^{(i)}_{\alpha\beta}(x-y)$, $i=1,2$ are the casual Green functions for the physical fields:

$$\begin{aligned}
G^{(1)}_{\alpha\beta}(x-y) &= \langle 0 \mid T\phi_\alpha(x)\phi^+_\beta(y) \mid 0\rangle;\\
G^{(2)}_{\alpha\beta}(x-y) &= \langle 0 \mid T\phi^+_\alpha(x)\phi_\beta(y) \mid 0\rangle.
\end{aligned} \tag{A.3}$$

III. Superpropagators.

Fourier image of the function $G^{(i)}_{\alpha\beta}(x-y)$, by the conventional terminology (see e.g.,[11]) defines the superpropagator presented by the infinite seria of free field propagators with initial spectrum $\epsilon(\mathbf{k})$. Let us find this representation. We have, for example:

$$G^{(1)}_{\alpha\beta}(x-y) = \frac{i\delta_{\alpha\beta}}{(2\pi)^4}\int \frac{\mid g(\mathbf{k})\mid^2}{\omega - E(\mathbf{k}) + i\epsilon} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y}) - i\omega(t_x - t_y)} d^3k\, d\omega. \tag{A.4}$$

Extracting the initial spectrum $\epsilon(\mathbf{k})$ from the spectrum $E(\mathbf{k})$:

$$\begin{aligned}
E(\mathbf{k}) &= \mid g(\mathbf{k})\mid^2 \left(u_0^2\epsilon(\mathbf{k}) - v_0^2\epsilon(\mathbf{k}-\mathbf{k}_0) - \lambda\frac{v_0^2}{V^*}\right) \simeq\\
&\simeq u_0^2\epsilon(\mathbf{k}) - v_0^2\epsilon(\mathbf{k}-\mathbf{k}_0) - \lambda\frac{v_0^2}{V^*} = \epsilon(\mathbf{k}) - \Sigma(\mathbf{k}),
\end{aligned} \tag{A.5}$$

where $\Sigma(\mathbf{k}) = v_0^2\left(\epsilon(\mathbf{k}) + \epsilon(\mathbf{k}-\mathbf{k}_0) + \frac{\lambda}{V^*}\right)$, we come to the expression for the superpropagator:

$$\begin{aligned}
\frac{1}{\omega - E(\mathbf{k}) + i\epsilon} &= \sum_{m=1}^{\infty}(-1)^{m-1}(\Sigma(\mathbf{k}))^{m-1}\Delta^m(\mathbf{k}),\\
\Delta(\mathbf{k}) &= \frac{1}{\omega - \epsilon(\mathbf{k}) + i\epsilon}.
\end{aligned} \tag{A.6}$$

We would like to mention that self-energy part of $\Sigma(\mathbf{k})$ is not an analytical function over the four fermionic coupling constant, this fact is related with non-analyticity of $v_0^2(\lambda)$.

References

[1] Y.Nambu, G.Jona-Lasinio, Phys.Rev.**122**, 1961, p.345; ibid. **124**, 1961, p.246.

[2] F.A.Berezin,- "Method of secondary quantization" (in Russian), Izdatel'stvo "Nauka", Moscow, 1965.

[3] W.Heisenberg, "Introduction to the unified field theory of elementary perticles.", Interscience publishers, London, New York,Sydney, 1966.

[4] H.Umezawa , H.Matsumoto, M.Tachiki "Thermo field dynamics and condensed states", 'North - Holland Publishing Company, Amsterdam - New York - Oxford, 1982.

[5] E.E.Salpether, H.S.Bethe, Phys.Rev.**84**, 1951, p.1232.

[6] M.K.Volkov, Sov.J.Part.and Nuclei **17**, 1986, p.483.

[7] J.Gasser, H.Leutwyler, Ann.of Phys. bf 158, 1984, p.142.

[8] V.N.Pervushin *et al.*- Fortschr.Phys.,**38**, 1990, 5, p.333; Yu.L.Kalinovsky, L.Kaschluhn, V.N.Pervushin,- Fortschr. Phys.,**38**, 1990, 5, 353; Yu.L.Kalinovsky *et al.* Few-Body Sys. **10** 1991 p.87.

[9] N.N.Bogolubov, D.N.Zubarev, J.A.Tserkovnikov, Zh.Exp.Teor.Piz. (in Russian) **39**, 120, 1960.

[10] Yu.L.Kalinovsky, L.Kashluhn,- preprint, DESY 92-071, 1992.

[11] M.K.Volkov, V.N.Pervushin,- "Essentially non-linear quantum theories, dynamical symmetries and meson physics" (in Russian), Atomizdat, Moscow, 1978.

Received by Publishing Department
on November 30, 1992.