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# VECTOR SOLUTIONS OF THE LAPLACE EQUATION AND THE INFLUENCE OF HELICITY ON THE AHARONOV-BOHM SCATTERING

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## I. Introduction

The vector spherioal harmonics (VSH) and elementary vector potentials (EVP) [1,2] closely related to them are the powerful tools for the solving of radiation [1] and scattering [3] problems occuring in optical [4], particle, nuclear [5] and atomic physics. EVP are the vector solutions of the Helmholtz equation $(\Delta + K^2)\vec{A} = 0$. Much less is known about the vector solutions of the Laplace Eq. $\Delta\vec{A} = 0$. At first, it seems to be strange. In fact, as the Helmholtz Eq. in the long-wavelengths limit ($k \to 0$) transforms into the Laplace Eq., one may expect the same for their vector solutions. It turns out, however, that contributions of EVP corresponding to electric (E) and longitudinal (L) multipoles diverge in the $k \to 0$ limit. This has given rise to numerous fallacies and controversies in physical literature. Part of them was discussed in the review article [6]. It is the aim of the present consideration to find the correct limiting procedure to the static case.

For the statio case, there are known configurations of charges and currents which being enolosed to the finite region of space S generate electric and magnetic fields vanishing outside S. The typical representatives are the electric oapacitors and magnetio solenoids. Is the same situation possible for the charge and current densities periodically varying with time? Particular examples of that kind were given in refs. [7]. In the present consideration, we find conditions which satisfy nonradiating oharge and particle densities.

An arbitrary distribution of the magnetic field enclosed into the finite space region S may be charaoterized by the number of topological invariants which are not changed under an arbitrary continuous deformation of S [8]. The simplest one is the magnetic flux. The next (in complexity) invariant is helicity. We oonstruct TS with nonzero helicity and investigate how it affects on the charge partiole scattering.

The plan of our exposition is as follows. The main facts oonoerning VSH and EVP are given in § 2. The orthonormal vector solutions of the Laplace Eq. are obtained in § 3. Their properties are discussed. The simplest physical applications of these vector solutions are treated in § 4. The conditions under which the periodical charge and current densities do not radiate are formulated in § 5. The electromagnetio properties of TS with non-zero helicity and the influence of the latter on the Aharonov-Bohm scattering are investigated in § 6.

## 2. Main facts concerning vector spherical harmonics

The usual method to solve the Helmholtz equation

$$(\Delta + K^2)\vec{A} = -\frac{4\pi}{c}\vec{j} \tag{2.1}$$

is to present its solution in terms of VSH /1,2/ :

$$\vec{A} = \frac{1}{c} 4\pi i k \sum h_\ell(kr)\, \vec{Y}^m_{\lambda\ell}(\theta,\varphi)\cdot \mathcal{J}^m_{\lambda\ell}. \tag{2.2}$$

The VSH $\vec{Y}^m_{\lambda\ell}$ are defined as vectorially coupled quantities of usual scalar spherical harmonics $Y^m_\ell$ and unit spherical vectors $\vec{n}_\mu$ $(\vec{n}_0 = \vec{n}_z,\ \vec{n}_{\pm1} = \mp(\vec{n}_x \pm i n_y)/\sqrt{2})$

$$\vec{Y}^m_{\lambda\ell} = \sum C(1,-\mu,\ell,m+\mu;\lambda m)\, Y_\ell^{m+\mu}\cdot\vec{n}_{-\mu}. \tag{2.3}$$

Further $\mathcal{J}^m_{\lambda\ell} = \int j_\ell(kr)\,\vec{Y}^{m*}_{\lambda\ell}\cdot\vec{j}\,dV$ ; $h_\ell(x) = \sqrt{\frac{\pi}{2x}}\, H^{(1)}_{\ell+\frac12}(x)$ and $j_\ell(x) = \sqrt{\frac{\pi}{2x}}\, J_{\ell+\frac12}(x)$ are the spherical Hankel and Bessel funotions. It is implicitly assumed in (2.2) that the vector potential (VP) is defined at the point P lying outside the source current region. VSH are orthonormal

$$\int \vec{Y}^m_{\lambda\ell}\cdot\vec{Y}^{m'}_{\lambda'\ell'}\, d\Omega = \delta_{\lambda\lambda'}\,\delta_{\ell\ell'}\,\delta_{mm'}.$$

They are the eigenfunctions of the orbital and total angular moments squares and of the third projection of the latter:

$$\vec{L}^2\,\vec{Y}^m_{\lambda\ell} = \ell(\ell+1)\vec{Y}^m_{\lambda\ell},\quad \vec{J}^2\,\vec{Y}^m_{\lambda\ell} = \lambda(\lambda+1)\vec{Y}^m_{\lambda\ell},\quad J_z\cdot\vec{Y}^m_{\lambda\ell} = m\,\vec{Y}^m_{\lambda\ell}.$$

(Here $\vec{L} = -i(\vec{r}\times\vec{\nabla})$ . The definition of $\vec{J}$ may be found, e.g., in /1,2/ ). It is clear that $j_\ell \vec{Y}^m_{\lambda\ell}$ and $h_\ell \vec{Y}^m_{\lambda\ell}$ are the vector solutions of (2.1). Instead of VSH one may equally use the EVP $\vec{A}^m_\ell(\tau)$ which are the linear combinations of VSH

$$\vec{A}^m_\ell(M) = -h_\ell\,\vec{Y}^m_{\ell\ell}$$



$$\vec{A}^m_\ell(E) = \left(\sqrt{\ell+1}\, h_{\ell-1}\vec{Y}^m_{\ell,\ell-1} - \sqrt{\ell}\, h_{\ell+1}\vec{Y}^m_{\ell,\ell+1}\right)/\sqrt{2\ell+1},$$

$$\vec{A}^m_\ell(L) = \left(\sqrt{\ell+1}\, h_{\ell+1}\vec{Y}^m_{\ell,\ell+1} + \sqrt{\ell}\, h_{\ell-1}\vec{Y}^m_{\ell,\ell-1}\right)/\sqrt{2\ell+1}. \tag{2.4}$$

The values $\tau$ = E,M and L correspond to the electric, magnetic and longitudinal multipoles. EVP are orthogonal on the surface of the sphere

$$\int \vec{A}^m_\ell(\tau)\cdot\vec{A}^{m'*}_{\ell'}(\tau')\cdot d\Omega = \text{const}\cdot\delta_{\tau\tau'}\,\delta_{\ell\ell'}\,\delta_{mm'}.$$

The same is valid for the EVP $\vec{B}^m_\ell(\tau)$ which are obtained from $\vec{A}^m_\ell(\tau)$ by changing $h_\ell(kr)$ by $j_\ell(kr)$ . The VP (2.2) being expressed in terms of EVP is given by

$$\vec{A} = \frac{4\pi i k}{c}\sum \vec{A}^m_\ell(\tau)\cdot a^m_\ell(\tau),\quad a^m_\ell(\tau) = \int \vec{B}^{m*}_\ell(\tau)\,\vec{j}\,dV. \tag{2.5}$$

The advantage of EVP over VSH is that EVP may be obtained by the action of the $\vec{\nabla}$ and $\vec{L}$ operators on the solutions of the scalar Helmholtz equation

$$\vec{A}^m_\ell(M) = \frac{1}{\sqrt{\ell(\ell+1)}}\vec{L}\,h_\ell Y^m_\ell,\quad \vec{A}^m_\ell(L) = \frac{1}{k}\vec{\nabla}\,h_\ell Y^m_\ell,$$

$$\vec{A}^m_\ell(E) = \frac{1}{ik}\frac{1}{\sqrt{\ell(\ell+1)}}\,\mathrm{rot}\,(\vec{L}\,h_\ell Y^m_\ell). \tag{2.6}$$

The following differential relations between $\vec{A}^m_\ell(\tau)$ are valid

$$\mathrm{rot}\,\vec{A}^m_\ell(M) = ik\vec{A}^m_\ell(E),\quad \mathrm{rot}\,\vec{A}^m_\ell(E) = -ik\,\vec{A}^m_\ell(E). \tag{2.7}$$

## 3. Vector solutions of the Laplace equation

Now we turn to the vector Poisson equation

$$\Delta\vec{A} = -\frac{1}{c}4\pi\,\vec{j}. \tag{3.1}$$

Although Eq.(3.1) is simpler than Eq.(2.1), much less is known on its vector solutions. This seems at first surprising. In fact, in the static limit ($k \to 0$) Eq.(2.2) is transformed into

$$\vec{A} = \frac{4\pi}{c}\sum \frac{1}{2l+1} r^{-l-1}\vec{Y}^m_{1l}\cdot j^m_{1l}, \quad j^m_{1l} = \int r^l \vec{Y}^{m*}_{1l}\cdot\vec{j}\,dV. \qquad (3.2)$$

Clearly, $r^l\vec{Y}^m_{1l}$ and $r^{-l-1}\vec{Y}^m_{1l}$ are the vector solutions of the Laplace equation. They are eigenfunctions of $\vec{L}^2$, $\vec{J}^2$ and $J_z$. We are interested in those vector solutions which are expressible in the form similar to (2.6). However, we cannot form from $r^l Y^m_{1l}$ and $r^{-l-1}\vec{Y}^m_{1l}$ the linear combinations similar to (2.4) since the terms with different $l$ have different dimensionalities and there is no constant (such as the wave number in the nonstatic case) to make these terms dimensionless. Since Eqs. (2.6) have the form which we seek for it is natural to find their static limit by performing expansion in powers of k

$$\begin{aligned} &\vec{A}^m_l(\tau) = k^{-l-2}\left[\vec{A}^m_{1l}(\tau) + k^2\vec{A}^m_{2l}(\tau)\right], \\ &\vec{B}^m_l(\tau) = k^{l-1}\left[\vec{B}^m_{1l}(\tau) + k^2\vec{B}^m_{2l}(\tau)\right], \quad \tau = E, L \qquad (3.3) \\ &\vec{A}^m_l(M) = k^{-l-1}\vec{A}^m_{1l}(M), \quad \vec{B}^m_l(M) = k^l\vec{B}^m_{1l}(M). \end{aligned}$$

The explicit values of the vector functions entering into the RHS of this Eq. are given in Appendix. They are independent of k. The terms with higher powers of k do not contribute in the long-wave-length limit and they are omitted in the development (3.3). The co-efficients $a^m_l(\tau)$ entering into the definition (2.5) of VP may be also developed in powers of k

$$\begin{aligned} &a^m_l(\tau) = k^{l-1}\left[a^m_{1l}(\tau) + k^2 a^m_{2l}(\tau)\right], \quad \tau = E, L, \\ &a^m_l(M) = k^l a^m_{1l}(M), \quad a^m_{1l}(\tau) = \int \vec{B}^{m*}_{1l}(\tau)\cdot\vec{j}\cdot dV \qquad (3.4) \\ &a^m_{2l}(\tau) = \int \vec{B}^{m*}_{2l}(\tau)\cdot\vec{j}\,dV. \end{aligned}$$



It follows from Eqs. (2.5), (3.3) and (3.4) that the contributions of the E and L multipoles taken separately diverge in the long-wave-length limit like $k^{-2}$. On the other hand, the development (2.2) which is completely equivalent to (2.5) turns in the same limit into (3.2). No singularities arise during this transition. This means that singularities of the E and L multipoles in (2.5) compensate each other. In fact, the singular term appearing in (2.5) is given by

$$k^{-2}\left[a^m_{1l}(E)\vec{A}^m_{1l}(E) + a^m_{1l}(L)\vec{A}^m_{1l}(L)\right].$$

It is easy to check that this Eq. vanishes after substitution of the exact values of $a^m_{1l}(\tau)$ and $\vec{A}^m_{1l}(\tau)$. After these preliminaries we may obtain the static limit of (2.5)

$$\begin{aligned} \vec{A} = &\frac{4\pi}{c}\sum\frac{1}{2l+1}\frac{1}{l(l+1)}\vec{C}^m_l(M)\cdot d^m_l(M) + \qquad (3.5) \\ &+\frac{2\pi}{c}\sum\frac{1}{4l^2-1}\vec{C}^m_l(E)\cdot d^m_l(L) - \frac{2\pi}{c}\sum\frac{1}{(2l+1)(2l+3)}\vec{C}^m_l(L)\,d^m_l(E) \end{aligned}$$

Here

$$\begin{aligned} &\vec{C}^m_l(M) = (\vec{r}\times\vec{\nabla})r^{-l-1}Y^m_l, \quad \vec{\mathcal{D}}^m_l(M) = (\vec{r}\times\vec{\nabla})r^l Y^m_l, \qquad (3.6) \\ &\vec{C}^m_l(E) = \left[\vec{\nabla} - \frac{1}{l}\mathrm{rot}(\vec{r}\times\vec{\nabla})\right]r^{1-l}Y^m_l, \quad \vec{\mathcal{D}}^m_l(E) = \left[\vec{\nabla} + \frac{1}{l+1}\mathrm{rot}(\vec{r}\times\vec{\nabla})\right]r^{l+2}Y^m_l \\ &\vec{C}^m_l(L) = \vec{\nabla}r^{-l-1}Y^m_l, \quad \vec{\mathcal{D}}^m_l(L) = \nabla r^l Y^m_l, \quad d^m_l(\tau) = \int \vec{\mathcal{D}}^m_l(\tau)^*\cdot\vec{j}\,dV. \end{aligned}$$

These are just expressions which we need. The vector functions $\vec{C}$ and $\vec{\mathcal{D}}$ are the vector solutions of the Laplace equation ($\Delta\vec{C}^m_l = \Delta\vec{\mathcal{D}}^m_l = 0$). This is not evident for $\vec{C}^m_l(E)$ and $\mathcal{D}^m_l(E)$. In fact, the particular terms entering into their definitions do not satisfy the Laplace. Eq. Only their linear combination does. This follows at once if we apply the $\Delta$ operator to $C^m_l(E)$ and $\mathcal{D}^m_l(E)$ and use the identity

$$\Delta r^d Y^m_l = (d-l)(d+l+1)r^{d-2}Y^m_l$$

with $d = 1-l$ for $C^m_l(E)$ and $l+2$ for $\mathcal{D}^m_l(E)$. In addition, $C^m_l(\tau)$ and $\mathcal{D}^m_l(\tau)$ satisfy the following Eqs.:

$$\mathrm{div}\,\vec C_l^m(E) = -2(2l-1)r^{-l-1}Y_l^m,\quad \mathrm{div}\,\vec D_l^m(E) = 2(2l+3)r^l Y_l^m,$$

$$\mathrm{rot}\,\vec C_l^m(E) = -\frac{2}{l}(2l-1)\vec C_l^m(M),\quad \mathrm{rot}\,\vec D_l^m(E) = -\frac{2(2l+3)}{l+1}\vec D_l^m(M),$$

$$\mathrm{rot}\,\vec C_l^m(M) = l\,\vec C_l^m(L),\quad \mathrm{rot}\,\vec D_l^m(M) = -(l+1)\vec D_l^m(L),$$

$$\mathrm{div}\,\vec C_l^m(M) = \mathrm{div}\,\vec D_l^m(M) = \mathrm{div}\,\vec C_l^m(L) = \mathrm{div}\,\vec D_l^m(L) =$$

$$= \mathrm{rot}\,\vec C_l^m(L) = \mathrm{rot}\,\vec D_l^m(L) = 0.$$

They are orthogonal on the surface of the sphere

$$\int \vec C_l^m(\tau)\,\vec C_{l'}^{m'}(\tau')^*\,d\Omega = \mathrm{const}\cdot\delta_{ll'}\,\delta_{mm'}\,\delta_{\tau\tau'},$$

$$\int \vec D_l^m(\tau)\cdot\vec D_{l'}^{m'}(\tau')^*\,d\Omega = \mathrm{const}\cdot\delta_{ll'}\,\delta_{mm'}\,\delta_{\tau\tau'}.$$

In the static limit Eqs.(2.4) are transformed into

$$r^{-l-1}\vec Y^m_{l-1,l} = \frac{1}{\sqrt{l(2l-1)}}\vec C^m_{l-1}(L),\quad r^l\vec Y^m_{l+1,l} = \frac{1}{\sqrt{(l+1)(2l+3)}}\vec D^m_{l+1}(L),$$

$$r^{-l-1}\vec Y^m_{l+1,l} = \frac{1}{2}\sqrt{\frac{l+1}{2l+3}}\,\vec C^m_{l+1}(E),\quad r^l\vec Y^m_{l-1,l} = -\frac{1}{2}\sqrt{\frac{l}{2l-1}}\,\vec D^m_{l-1}(E),$$

$$r^{-l-1}\vec Y^m_{ll} = \frac{i}{\sqrt{l(l+1)}}\vec C^m_l(M),\quad r^l\vec Y^m_{ll} = \frac{i}{\sqrt{l(l+1)}}\vec D^m_l(M). \qquad (3.7)$$

It follows from this that $\vec C_l^m(\tau)$ and $\vec D_l^m(\tau)$ are eigenfunctions of $\vec L^2, \vec J^2, J_z$. It is evident that the development (3.5) coincides with (3.2). The novel is that we have succeeded in presenting $r^l\vec Y^m_{\lambda l}$ and $r^{-l-1}\vec Y^m_{\lambda l}$ in terms of differential operators (see Eqs.(3.6) and (3.7).

## 4. Some applications of the Laplace vector spherical harmonics

The VP representation (3.5) may be useful for the solution of different physical problems (especially, magnetostatic). Consider, e.g., stationary current (div $\vec j$=0) enclosed into the finite region of space S. Under what conditions the magnetic field $\vec H$ (=rot $\vec A$) does not go beyond S? For the stationary current the coefficients $d_l^m(L)$ are equal to zero. In fact, integrating by parts one gets



$$d_l^m(L) = \int\vec\nabla(r^lY_l^{m*})\,\vec j\,dV = -\int r^lY_l^{m*}\,\mathrm{div}\,\vec j\,dV = 0.$$

Apply to Eq. (3.5) the rot operation. Then the third term in it disappears and one has

$$\vec H = \frac{4\pi}{c}\sum\frac{1}{2l+1}\,\frac{1}{l+1}\,\vec C_l^m(L)\,d_l^m(M).$$

Instead of the stationary current one may equally use the magnetization ($\vec j$=c rot$\vec M$). Then,

$$d_l^m(M) = c\int(\vec r\times\vec\nabla)r^lY_l^{m*}\cdot\mathrm{rot}\,\vec M\,dV =$$

$$= -c\int r^lY_l^{m*}\cdot\left(2 + r\frac{\partial}{\partial r}\right)\mathrm{div}\,\vec M\,dV.$$

From this it follows at once that the magnetic field disappears outside the finite region of the space filled by the substance with solenoidal (i.e., with div $\vec M$ = 0) magnetization. This result was obtained in refs. /7/ in a quite different way. The corresponding current is restored by the use of Eq. j=c rot M . The simplest example of such a substance is a closed uniformly magnetized filament $C$ of an arbitrary form. The magnetic field remains enclosed inside C when it undergoes an arbitrary continuous deformation. From these filaments the solenoid of an arbitrary form can be constructed. The conditions for the disappearance of H outside the region with nonzero current were obtained earlier in an interesting ref. /9/. Let $j_\mu$ be the spherical components of $\vec j$ ($j_0 = j_z$, $j_{\pm1} = \mp(j_x \pm i j_y)/\sqrt{2}$. Then, the conitions mentioned above are given by

$$R^{l,m}_{-1} = \left(\frac{1}{2}\,\frac{l-m}{l+m+1}\right)^{1/2}R_0^{l,m+1},\quad R_1^{lm} = \left(\frac{1}{2}\,\frac{l+m}{l-m+1}\right)^{1/2}R_0^{l,m-1}. \qquad (4.1)$$

Here $R_\mu^{lm} = \int r^lY_l^{m*}\cdot j_\mu\,dV$. These Eqs. may be used to check the disappearance of H outside S. However, they are not constructive in the sense that they do not give the prescription for the construction of currents satisfying the above conditions. In addition, the evaluation of $R_\mu^{\ell m}$ is not a trivial task even for the simplest toroidal current configurations /10/ . On the other hand, the use of magnetization formalism used here makes the construction of the solenoids with an arbitrary form almost trivial. This is widely used by experimentalists (Cf. /11/ ). The current components corresponding to the chosen solenoidal magnetization satisfy the conditions (4.1) automatically. It follows from (3.5) that if the current distribution has the form $\vec{j}=0$ rot rot $\vec{t}$ where the vector field $\vec{t}$ is the solenoidal one (div $\vec{t}=0$), then VP differs from zero only in those space regions where $\vec{t}$ = 0. The presentation of $t$ in such a form is valid, e.g., for the toroidal moments /6/. Physically, this means that toroidal moments uniformly distributed along the arbitrary closed curve and being tangential to it generate VP that differs from zero only on that curve. From these filaments the finite distributions of toroidal moments may be constructed having the same self--screening property.

## 5. Nonradiating charge and current sources

Now we turn again to the nonstatic case. The question arises: under what conditions the charge and current densities periodically changing with time and confined to the finite space region S generate electromagnetic strengths E,H vanishing outside S? To see this we act by the rot operator on Eq.(2.5). Then

$$\vec{H}=\frac{1}{c}4\pi k^2\sum\left[a_\ell^m(E)\cdot\vec{A}_\ell^m(M)-a_\ell^m(M)\,\vec{A}_\ell^m(E)\right].$$

As $\vec{A}_\ell^m(E)$ and $\vec{A}_\ell^m(M)$ are linear independent, so the conditions for the disappearance of H are

$$a_\ell^m(E)=a_\ell^m(M)=0. \tag{5.1}$$

The corresponding scalar and vector potentials are given by

$$\Phi=4\pi ik\sum h_\ell Y_\ell^m\int j_\ell Y_\ell^{m*}\rho\, dV,$$

$$\vec{A}=\frac{1}{c}4\pi ik\vec{\nabla}\sum h_\ell Y_\ell^m\int\vec{\nabla}(j_\ell Y_\ell^{m*})\,\vec{j}\,dV.$$

(The overall periodical factor exp(−iωt) is dropped in this and other evident cases). $\Phi$ and $\vec{A}$ satisfy the Lorentz gauge condition div $\vec{A}+\frac{1}{c}\dot{\Phi}=0$ . It is easy to check that the electric



field $\vec{E}=-\nabla\Phi-\frac{1}{c}\dot{\vec{A}}$ also disappears outside S. To see explicitly what Eqs.(5.1) mean we present $\vec{j}$ in the form

$$\vec{j}=\sum\left[\vec{r}\,j_{\ell m}^{(1)}+j_{\ell m}^{(2)}\cdot\vec{\nabla}+j_{\ell m}^{(3)}(\vec{r}\times\vec{\nabla})\right]Y_\ell^m. \tag{5.2}$$

(An arbitrary vector function can be presented in this form. See, e.g., /12/ ). The functions $j_{\ell m}^{(i)}$ depend only on the radial coordinate r. Substitution of (5.2) into (5.1) gives

$$\int j_\ell(kr)\,j_{\ell m}^{(1)}(r)\,r^2dr=\int j_\ell(kr)\,j_{\ell m}^{(3)}(r)\,r^2dr=0.$$

The current distribution with $j_{\ell m}^{(i)}$ satisfying these Eqs. do not radiate because $\vec{E}=\vec{H}=0$ outside S. On the other hand, it is possible to construct a nonradiating system with $\vec{E},\vec{H}\neq 0$ outside S. This happens if the Poynting vector $\vec{P}=\frac{1}{4\pi c}\vec{E}\times\vec{H}$ decreases faster than $r^{-2}$. Consider the explicit expressions for the electromagnetic potentials:

$$\Phi=\int G_k(\vec{r},\vec{r}\,')\,\rho(\vec{r}\,')\,dV',\qquad G_k=\frac{\exp(ik|\vec{r}-\vec{r}\,'|)}{|\vec{r}-\vec{r}\,'|},$$

$$\vec{A}=\frac{1}{c}\int G_k(\vec{r},\vec{r}\,')\,\vec{j}(\vec{r}\,')\,dV'.$$

At large distances one has

$$\Phi_0=\frac{1}{r}\exp(ikr)\int\exp(-ik\vec{n}_r\vec{r}\,')\,\rho(\vec{r}\,')\,dV',\qquad \vec{n}_r=\vec{r}/r,$$

$$\vec{A}_0=\frac{1}{cr}\exp(ikr)\int\exp(-ik\vec{n}_r\vec{r}\,')\,\vec{j}(\vec{r}\,')\,dV', \tag{5.3}$$

$$\vec{E}=-ik\vec{n}_r\Phi_0+ik\vec{A}_0,\qquad \vec{H}=ik(\vec{n}_r\times\vec{A}_0).$$

The terms of the order $r^{-2}$ and higher are omitted since they do not contribute to the energy flux. The radial component of the Poynting vector equals

$$S_r=\frac{1}{4\pi c}(\vec{n}_r\cdot(\vec{E}\times\vec{H}^*))=\frac{k^2}{4\pi c}(\vec{n}_r\cdot(\vec{A}_0\times(\vec{A}_0^*\times\vec{n}_r)))=$$

$$=\frac{1}{4\pi c}k^2(|\vec{A}_0|^2-|\vec{n}_r\vec{A}_0|^2)=\frac{k^2}{4\pi c}(|\vec{A}_0|^2-|A_{0r}|^2)=$$

$$=\frac{1}{4\pi c}k^2(|A_\theta^0|^2+|A_\varphi^0|^2).$$

It follows from this that the energy flux into the surrounding space vanishes if $A_\theta = A_\varphi = 0$. These conditions were obtained earlier in refs. [13]. The particular realizations of such nonradiating systems may be found, e.g., in [13,14].

## 6. Magnetic solenoids with non-zero helicity

For the pedagogical purposes we consider first the cylindrical solenoid C of the radius R. Let the current $\vec{j} = j \cdot \vec{n}_\varphi \cdot \delta(\rho - R)$ flow on its surface. The corresponding VP is $\vec{A} = A \cdot \vec{n}_\varphi$ where $A = \frac{\Phi}{2\pi\rho}$ outside C and $\Phi\rho/2\pi R^2$ inside it. The magnetic field differs from zero only inside C: $\vec{H} = \vec{n}_z \cdot \Phi/\pi R^2$. Here $\Phi$ is the magnetic flux inside C: $\Phi = \iint H_z \rho d\rho d\varphi = 4\pi^2 R^2 j$. In the treated case the magnetic field and VP are mutually orthogonal. So,

$$S = \int \vec{A} \cdot \vec{H} dV = 0. \tag{6.1}$$

This quantity is called helicity[15,16]. Thus, the usual cylindrical solenoid has zero helicity. Instead of the current $\vec{J}$ one may equally use the magnetization $\vec{M}$: $\vec{j} = c \operatorname{rot} \vec{M}$. For the treated case $\vec{M} = M \cdot \vec{n}_z \cdot \theta(R - \rho)$, $M = \pi R^2 j$. It is convenient to forget the initial current and to treat the solenoid as a cylinder uniformly magnetized along its symmetry axis. Let the magnetization $\vec{M}$ has the $\varphi$ component (in addition to the existing Z one):

$$\vec{M} = M \cdot \theta(R - \rho) \cdot (\vec{n}_z \cos\alpha + \vec{n}_\varphi \cdot \sin\alpha).$$

The needed components of VP and magnetic induction are

$$A_z = 4\pi M \cdot (R - \rho)\sin\alpha, \quad A_\varphi = 2\pi M \rho \cos\alpha, \quad B_\varphi = 4\pi M \sin\alpha, \quad B_z = 4\pi M \cos\alpha$$

inside the cylinder and $A_z = 0$, $A_\varphi = 2\pi R^2 M \cos\alpha/\rho$, $\vec{B} = 0$ outside it. As a result, the helicity per unit of the cylinder length equals

$$S = \frac{16}{3}\pi^3 M^2 \sin 2\alpha \cdot R^3.$$

We turn to the toroidal solenoid (TS). It may be viewed as the set of magnetized filaments filling the torus $T$: $(\rho - d)^2 + z^2 = R^2$. In the toroidal coordinates ($\rho = \frac{a \operatorname{sh}\mu}{\operatorname{ch}\mu - \cos\theta}$, $z = \frac{a \sin\theta}{\operatorname{ch}\mu - \cos\theta}$) the magnetization and induction are equal to

$$\vec{M} = M \cdot \vec{n}_\varphi, \quad \vec{B} = 4\pi \vec{M}, \quad M = M_0 \frac{\operatorname{ch}\mu - \cos\theta}{\operatorname{sh}\mu} \theta(\mu - \mu_0). \tag{6.2}$$



The values $\mu > \mu_0$ and $\mu < \mu_0$ correspond to the points lying inside and outside the torus T, resp. The VP of TS was obtained in ref.[17], its properties were discussed in [18]. The nonvanishing toroidal components of VP are $A_\mu$ and $A_\theta$. It follows from this that the torus with magnetization (6.2) possesses zero helicity. Let the magnetization M has the $\theta$ component (in addition to the existing $\varphi$ one)

$$\vec{M} = M(\vec{n}_\varphi \cdot \cos\alpha + \vec{n}_\theta \cdot \sin\alpha), \quad \vec{B} = 4\pi \vec{M}. \tag{6.3}$$

The $\theta$ component of $\vec{M}$ generates the $A_\varphi$ component of VP which is different from zero only inside T:

$$A_\varphi = -8\pi M_0 \sin\alpha \frac{\operatorname{ch}\mu - \cos\theta}{\operatorname{sh}\mu \sin\theta} \cdot \operatorname{arctg} \frac{\sin\theta \cdot \operatorname{sh}\frac{\mu - \mu_0}{2}}{\operatorname{ch}\frac{\mu + \mu_0}{2} - \cos\theta \cdot \operatorname{ch}\frac{\mu - \mu_0}{2}}.$$

The $\mu$ and $\theta$ components of the VP are obtained from that of refs.[17,18] by multiplying them by $\cos\alpha$. It follows from this that non-zero helicity corresponds to the magnetization (6.3). Since the $A_\mu$ and $A_\theta$ components are rather complicated for the finite torus T we limit ourselves to the infinitely thin one ($R \ll d$ or $\mu_0 \gg 1$). In this case, the following components of VP and magnetic induction differ from zero inside the TS

$$A_\theta = 4\pi a M_0 [\exp(-\mu) - \mu_0 \cdot \cos\theta \cdot \exp(-\mu_0)] \cdot \cos\alpha,$$

$$A_\varphi = -8\pi a M_0 \cdot \exp(-\mu_0) \cdot \sin\alpha,$$

$$B_\theta = 4\pi M_0 \sin\alpha, \quad B_\varphi = 4\pi M_0 \cos\alpha.$$

As a result, we obtain for the helicity:

$$S = \int \vec{A} \cdot \vec{B} dV = \frac{32}{3}\pi^4 a^4 M_0^2 \sin 2\alpha \cdot \exp(-3\mu_0).$$

The question arises: is it possible to get information on the helicity by performing experiments outside T (which may be surrounded by the impenetrable torus)? We note that the $\theta$ component of magnetization does not contribute to the VP outside T. The wave function describing the scattering of the charged particles on the impenetrable toroidal solenoid depends on the geometrical dimensions (d,R) of impenetrable torus and on the $\varphi$ component of the magnetic flux inside the solenoid (it is just this component that generates non-zero VP outside TS) /19/:

$$\Psi = \exp(ikz) + \Psi_s ,$$

$$\Psi_s = i\,\frac{1+\cos\theta_s}{2}\exp(ikr)\exp\left(ik\frac{d^2+R^2}{2r}\right)\cdot$$

$$\cdot\left[\exp(i\omega)\cdot W_1 - \exp(i\gamma - i\omega)\cdot W_2\right].$$

Here $\theta_s$ and $r$ are the scattering angle and distance from the solenoid to the observation point, $\omega = \frac{kdR}{r}$, $\gamma = \frac{e\Phi}{\hbar c}\cos\alpha$; $W_1$ and $W_2$ are the linear combinations of the Lommel functions of two variables:

$$W_{1,2} = U_1\left[\frac{k(d\pm R)^2}{r}, k(d\pm R)\sin\theta\right] - iU_2\left[\frac{k(d\pm R)^2}{r}, k(d\pm R)\sin\theta\right].$$

It follows from this Eq. that the intensity $I = |\Psi|^2$ is a periodical function of the angle $\alpha$. The helicity and magnetic flux are the simplest representatives of the topological invariants, characterizing the structure of the magnetio field. These invariants remain the same for the arbitrary continuous deformation of the solenoids. There exist topologioal invariants different from $\Phi$ and $S$ /20/ which describe more subtle features of the magnetic field.

## 7. Conclusion

We briefly summarize the main results obtained:

1. The vector solutions of the Laplace Eq. are presented in the differential form. This makes easier the solution of magnetostatic problems. The simplest applications are given.

2. The conditions are formulated for the non-radiation of charge and particle densities periodically changing with time.

3. The electromagnetic properties of the toroidal solenoid with non-zero helicity and the influence of the latter on the Aharonov-Bohm scattering are studied.

This consideration has arisen from the numerous discussions with Prof. J.A.Smorodinsky who passed away so prematurely.



## Appendix

Here are the explicit values of the vector functions occurring in the development (3.3):

$$\vec{A}^m_{1l}(E) = \frac{2d_l}{\sqrt{l(l+1)}}\,\mathrm{rot}\,\vec{L}\,r^{-l-1}Y^m_l,\quad \vec{A}^m_{2l}(E) = \frac{d_l}{\sqrt{l(l+1)}}\frac{1}{2l-1}\,\mathrm{rot}\,\vec{L}\,r^{1-l}Y^m_l,$$

$$\vec{B}^m_{1l}(E) = \frac{i\beta_l}{\sqrt{l(l+1)}}\,\mathrm{rot}\,\vec{L}\,r^{l}Y^m_l,\quad \vec{B}^m_{2l}(E) = \frac{i\beta_l}{\sqrt{l(l+1)}}\frac{1}{2(2l+3)}\,\mathrm{rot}\,\vec{L}\,r^{l+2}Y^m_l,$$

$$\vec{A}^m_{1l}(L) = 2id_l\nabla r^{-l-1}Y^m_l,\quad \vec{A}^m_{2l}(L) = \frac{id_l}{2l-1}\vec{\nabla}\,r^{1-l}Y^m_l,$$

$$\vec{B}^m_{1l}(L) = \beta_l\nabla r^{l}Y^m_l;\quad \vec{B}^m_{2l}(L) = -\frac{\beta_l}{2\cdot(2l+3)}\nabla r^{l+2}Y^m_l,$$

$$\vec{A}^m_{1l}(M) = \frac{2id_l}{\sqrt{l(l+1)}}\vec{L}\,r^{-1-l}Y^m_l,\quad \vec{B}^m_{1l}(M) = \frac{\beta_l}{\sqrt{l(l+1)}}\vec{L}\,r^{l}Y^m_l,$$

$$d_l = (-1)^{l+1}\frac{\sqrt{\pi}\,2^{l-1}}{\Gamma(\frac{1}{2}-l)},\quad \beta_l = \frac{\sqrt{\pi}}{\Gamma(l+\frac{3}{2})}2^{-l-1} \tag{A.1}$$

For $k \to 0$ Eqs. (2.7) are transformed into

$$\mathrm{rot}\,\vec{A}^m_{1l}(M) = i\vec{A}^m_{1l}(E),$$
$$\mathrm{rot}\,[\vec{A}^m_{1l}(E) + k^2\vec{A}^m_{2l}(E)] = -ik^2\vec{A}^m_{1l}(M). \tag{A.2}$$

It follows from (A.1) that first of Eqs.(A.2) is satisfied automatically. In the second of them we equalize the terms at the same power of $k$

$$\mathrm{rot}\,\vec{A}^m_{1l}(E) = 0,\quad \mathrm{rot}\,\vec{A}^m_{2l}(E) = -iA^m_{1l}(M). \tag{A.3}$$

The validity of these Eqs. may be proved without appealing to Eq. (2.7) if we take into account the following relation

$$\mathrm{rot}\,(\vec{r}\times\vec{\nabla})r^{d}Y^m_l = -(d+1)\nabla r^{d}Y^m_l + (d-l)(d+l+1)\vec{r}\,r^{d-2}Y^m_l \tag{A.4}$$

Setting $\alpha=-l-1$ gives

$$\mathrm{rot}\,(\vec{r}\times\vec{\nabla})\,r^{-l-1}Y_l^m = l\,\nabla r^{-l-1}Y_l^m. \qquad (A.5)$$

Since $\vec{A}_{1l}(E)$ is proportional to the LHS of (A.5), the first Eq. (A.3) is satisfied. For $\alpha=1-l$ one obtains from (A.4)

$$\mathrm{rot}\,(\vec{r}\times\vec{\nabla})\,r^{1-l}Y_l^m = (l-2)\,\nabla r^{1-l}Y_l^m - 2\cdot(2l-1)\,\vec{r}\cdot r^{-1-l}Y_l^m.$$

Applying to both sides of this Eq. the $\mathrm{rot}$ operator we arrive at the second Eq. (A.3). For $\alpha=l$ and $l+2$ we find from (A.4) the following relations

$$\mathrm{rot}\,\vec{B}_{1l}^m(E)=0\,, \quad \mathrm{rot}\,\vec{B}_{1l}^m(E)=-i\,\vec{B}_{1l}^m(M).$$

It follows from (A.5) that the same vector function may be simultaneously represented as the rot and grad. A similar relation for the positive powers of r will be

$$\mathrm{rot}\,(\vec{r}\times\vec{\nabla})\,r^{l}Y_l^m = -(l+1)\,\nabla r^{l}Y_l^m.$$

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