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дубна

E2-92-491

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GRAVITATIONAL THEORY WITH THE DYNAMICAL AFFINE CONNECTION

Submitted to «Classical and Quantum Gravity»

1992

Introduction

In a recent cycle of papers [1] - [3] prof. N.A. Chernikov succeeded in showing that in the Einstein General Relativity (GR) the important geometrical object was lost. It was a background affine connection. This object appears when we consider the gravitational action functional [4].

At present, physical meaning of the background connection is unknown. Maybe, this geometrical object is the unusual description of the frame of reference. Another opinion was suggested in [5], [6] where the background connection was used for describing the external gravitational field created by the sources outside of the observable universe.

In present paper the model is suggested in which the background connection become the dynamical one. It means that the second connection coefficients $\check{\Gamma}^i_{jk}$ are the potentials of some real physical field.

1. Gravity as a compensating field

Let us consider the vacuum Maxwell equations

$$\partial_i F^{ij} = 0 \tag{1}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ is the strength tensor of the electromagnetic field A_i. If we write down (1) in curvilinear coordinates, then we obtain

$$g^{ik} \nabla_i F_{kj} = 0 \tag{2}$$

where g^{ik} is the metric tensor, ∇_i is a covariant derivative with respect to the Christoffel symbols Γ^i_{jk}.

Equations (2) are generally covariant and, consequently, they are invariant under the diffeomorphisms group $Diff(M)$ of the space-time manifold M:

$$\begin{aligned} x^i \to x'^i &= x'^i(x^i); & (3)\\ A_i(x) \to A'_i(x') &= \frac{\partial x^k}{\partial x'^i} A_k; & (4)\\ g^{ik}(x) \to g'^{ik}(x') &= \frac{\partial x'^i}{\partial x^a}\frac{\partial x'^k}{\partial x^b} g^{ab} & (5)\end{aligned}$$

where $x'^i(x^k) \equiv x^i$ outside the compact area.

But this invariance is not the dynamical one because transformations (5) are those of the non-dynamic (background) fields g^{ik}. Such invariance must be named a covariance because it can be interpreted as follows:
at first, action of $Diff(M)$ on x^i and A_i damages the equations (2), but then it restores (2) by suitable transformations of g^{ik} (5).

The background field g^{ik} is a set of functional parameters. It is a characteristic of the scene like the Newtonian potential V in the Newton equation

$$m\frac{d^2x}{dt^2} = -\frac{\partial V}{\partial x}.$$

There are no "equations of motion" for a background objects and they must be determined by a supplementary physical considerations, although the latter may have a form of an equation, for example, $\Delta V = -4\pi\rho$.

On the other hand, equations (2) are invariant under the gauge transformations

$$A_i \to A'_i = A_i + \partial_i \chi \qquad (6)$$

with an arbitrary scalar field χ.

Mathematical meaning of (6) is the same as of (3)-(5) but from the physical point of view these transformations are essentially different. Transformations (3)-(5) describe the form–invariant properties of (2) under the general coordinate transformations, and have no physical content. On the contrary, (6) represent gauge transformations and appear as a symmetry group of the electromagnetic interaction.

The simplest $Diff(M)$-invariant generalization of (2) consists in adding to (2) the equations on g^{ik}

$$R_{ik} = \kappa T_{ik} \qquad (7)$$

where T_{ik} is the electromagnetic energy-momentum tensor; the Ricci tensor $R_{ik} = R^p_{pij}$ is derived from the Riemann-Christoffel tensor $R^p_{ijk} = \partial_i \Gamma^p_{jk} - \partial_j \Gamma^p_{ik} + \Gamma^p_{is}\Gamma^s_{jk} - \Gamma^p_{js}\Gamma^s_{ik}$.

Unlike equations (2) describing electromagnetic field, the system

$$\left\{ \begin{aligned} g^{ik}\nabla_i F_{kj} &= 0\\ R_{ij} &= \kappa T_{ij} \end{aligned} \right. \qquad (8)$$

describes the electromagnetic field interacting with the gravitational field. Since (8) is invariant under (3)-(5), the group $Diff(M)$ takes new physical meaning and becomes the symmetry group of the gravitational interaction. But (8) remains generally covariant. The generally coordinate transformation "group" is similar to the group $Diff(M)$ and leads to some difficulties in the analysis of symmetry properties. Luckily, if we turn to the Lagrangian approach, the generally coordinate transformations and $Diff(M)$ group can be separated.

Let us apply the above scheme not to equations, but to Lagrangians. The electromagnetic Lagrangian is

$$L_A = \sqrt{-g} g^{ab} g^{ij} F_{ai} F_{bj}. \qquad (9)$$

It is both generally- and $Diff(M)$- covariant.

If we shall attempt to convert $Diff(M)$-covariance into $Diff(M)$- invariance, i.e. to make g^{ij} the dynamical field, then we must find a suitable Lagrangian producing the Einstein tensor

$$G_{ik} = R_{ik} - \frac{1}{2} R g_{ik} \tag{10}$$

as a variational derivative. Usually, the Hilbert Lagrangian

$$L_H = \sqrt{-g} R \tag{11}$$

is used because it leads to the variational derivative $\sqrt{-g} G_{ik}$. But (10) can be considered as a second-order differential operator acting on g_{ij}, and (11) contains the second-order derivatives too. This leads to the known difficulties [7]. It seems probable that these difficulties are result from the presence of two types, the generally coordinate and $Diff(M)$, of possible transformations which remain (11) invariant.

The high-order derivatives are contained in (11) in a special way. We can write down

$$L_H = L_E + \partial_i \omega^i \tag{12}$$

where

$$\omega^i = \sqrt{-g}(g^{in}\Gamma^m_{mn} - g^{mn}\Gamma^i_{mn}) \tag{13}$$

and the Einstein Lagrangian

$$L_E = \sqrt{-g} g^{mn}(\Gamma^a_{mb}\Gamma^b_{an} - \Gamma^a_{sa}\Gamma^s_{mn}) \tag{14}$$

contains only first-order derivatives. Although L_E not generally covariant, the action

$$S_E = \int L_E d^4x \tag{15}$$

is invariant under $Diff(M)$ since the $Diff(M)$-transformations add only the divergence-like term to L_E. Noncovariance (14) in

fact means that the background affine connection is contained in L_E.

Indeed, let us consider some background affine connection $\check\Gamma^i_{jk}$. We restrict our consideration to the case $\check\Gamma^i_{jk} = \check\Gamma^i_{kj}$. The difference between the connection coefficients

$$P^i_{mn} = \check\Gamma^k_{mn} - \Gamma^k_{mn} \tag{16}$$

forms a tensor named the affine-deformation tensor. Let us consider the Lagrangian

$$\tilde L = \sqrt{-g} g^{mn}(P^a_{mb}P^b_{an} - P^a_{sa}P^s_{mn}). \tag{17}$$

For the action

$$\tilde S = \int \tilde L d^4x \tag{18}$$

the variational derivative

$$\tilde\Psi^{mn} = 2\frac{\delta\tilde S}{\delta g_{mn}} \tag{19}$$

has been calculated in [1]. It is

$$\tilde\Psi^{mn} = \sqrt{-g} g^{ma} g^{nb}(\check R_{ab} + \check R_{ba} - \check R_{ij} g^{ij} g_{ab} - 2G_{ab}). \tag{20}$$

If $\check R_{(ik)} = 0$, then the equations

$$\tilde\Psi^{mn} = 0 \tag{21}$$

coincide with the vacuum Einstein ones

$$G_{ij} = 0 \tag{22}$$

and the Lagrangian (17) differs from L_H by a divergence term

$$L_H - \tilde L = \check\nabla_i F^i \tag{23}$$

where $\check{\nabla}_i$ is a covariant derivative with respect to $\check{\Gamma}^i_{jk}$, and

$$F^i = \sqrt{-g}(g^{mn}P^i_{mn} - g^{in}P^m_{mn}) \tag{24}$$

is a vector density of weight one.

If $\check{R}^i_{jkl} = 0$, the coordinate map can be chosen in which all $\check{\Gamma}^i_{jk} = 0$. Then, P^i_{jk} turns into $-\Gamma^i_{jk}$, $\tilde{L}$ turns into L_E, F^i into ω^i, and (23) transforms into (12). Converting L_H into L_E by (12) in fact means converting L_H into $\tilde{L}$ according to formula (23) with the fixation of the background connection whose coefficients in this map are assumed to be zero.

So, instead of (9), (11) we have obtained the system described by the Lagrangian

$$\begin{aligned} L = \kappa L_A + \tilde{L} = \kappa\sqrt{-g}g^{ab}g^{ij}F_{ai}F_{bj} \\ +\sqrt{-g}g^{mn}(P^a_{mb}P^b_{an} - P^a_{sa}P^s_{mn}) \end{aligned} \tag{25}$$

with the condition on the background connection

$$\check{R}_{ij} = 0. \tag{26}$$

Now the generally coordinate transformations take the form

$$x^i \to x'^i = x'^i(x^i); \tag{27}$$

$$A_i(x) \to A'_i(x') = \frac{\partial x^k}{\partial x'^i}A_k; \tag{28}$$

$$g^{ik}(x) \to g'^{ik}(x') = \frac{\partial x'^i}{\partial x^a}\frac{\partial x'^k}{\partial x^b}g^{ab}; \tag{29}$$

$$\check{\Gamma}^i_{jk}(x) \to \check{\Gamma}'^i_{jk}(x') = \left(\check{\Gamma}^r_{pq}\frac{\partial x^p}{\partial x'^j}\frac{\partial x^q}{\partial x'^k} + \frac{\partial^2 x^r}{\partial x'^j \partial x'^k}\right)\frac{\partial x'^i}{\partial x^r}. \tag{30}$$

But the $Diff(M)$-transformations of the system (25) have another form

$$\left\{ \begin{array}{lcl} g^{ik}(x) & \to & g'^{ik}(x), \\ A_i(x) & \to & A'_i(x), \\ \check{\Gamma}^i_{jk}(x) & \to & \check{\Gamma}'^i_{jk}(x) \equiv \check{\Gamma}^i_{jk}(x) \end{array} \right. \tag{31}$$

where $g'^{ik}(x)$ and $A'_i(x)$ are obtained from (5) and (4), respectively, by changing the arguments x' by x after calculating the right-hand sides of (5), (4).

2. The dynamical affine connection

So, the g^{ik}-dynamics derived from L_H (12) is determined only by L_E but L_E is noncovariant. If we assume the action S_E (15) to be invariant under the general coordinate transformations, we must interpret the appearing divergence term as the result of the $Diff(M)$ transformations. In other words, the general coordinate transformations must be in agreement with the $Diff(M)$ transformations.

Introducing of the background connection permits us to separate the generally coordinate transformations and $Diff(M)$. The Lagrangian $\tilde{L}$ (17), determining the g^{ik} dynamics, is generally covariant and, therefore, we do not need the general coordinates transformations to be in agreement with $Diff(M)$.

Now we may attempt to make that the background affine connection to become a dynamical field. It would seem that this returns us to the situation when the general coordinate transformations were mixed with $Diff(M)$. But it is not right. The mixing takes place only due to the noncovariance of the Lagrangian L_E.

Let us consider the Lagrangian with the cosmological term Λ_2:

$$L_\Lambda = \tilde{L} + 2\Lambda_2\sqrt{-g}. \tag{32}$$

It is easy to see that (32) is equal to

$$L_\Lambda = \sqrt{-g}(R + 2\Lambda_2) - \sqrt{-g}\check{R}_{ij}g^{ij} - \partial_i F^i. \tag{33}$$

Let us denote

$$L_g = \sqrt{-g}(R + 2\Lambda), \tag{34}$$

$$L_{g\check{\Gamma}} = -\sqrt{-g}\check{R}_{ik}g^{ik}. \tag{35}$$

Up to a divergence term the Lagrangian L_Λ can be presented in the form

$$L_\Lambda = L_g + L_{g\check{\Gamma}}. \tag{36}$$

The first term can be interpreted as a term corresponding to the pure gravitational field, the second is the "cross" g–$\check{\Gamma}$ term describing the interaction between the fields g_{ik} and $\check{\Gamma}^i_{jk}$. For the background connection be dynamical, the Lagrangian needs a "kinetic" term for pure $\check{\Gamma}^i_{jk}$. The simplest term like that has been proposed by A.Eddington [8]:

$$L_{\check{\Gamma}} = \frac{2}{\Lambda_1}\sqrt{\left|det(\check{R}_{(ik)})\right|} \tag{37}$$

where Λ_1 is the coupling constant. Then, the full Lagrangian can be written as

$$L_\Sigma = L_g + L_{g\check{\Gamma}} + L_{\check{\Gamma}}. \tag{38}$$

Up to a divergence term

$$L_\Sigma = L_\Lambda + L_{\check{\Gamma}}. \tag{39}$$

Let the corresponding actions be denoted by the same marks as the Lagrangians:

$$\begin{aligned} S_\Lambda &= \int L_\Lambda d^4x; \\ S_{g\check{\Gamma}} &= \int L_{g\check{\Gamma}} d^4x; \\ S_{\check{\Gamma}} &= \int L_{\check{\Gamma}} d^4x; \\ S_\Sigma &= \int L_\Sigma d^4x. \end{aligned} \tag{40}$$

Let us find the variations. From (20) and (32) we obtain

$$\frac{\delta S_\Lambda}{\delta g_{ik}} = \frac{1}{2}\tilde{\Psi}^{ik} + \Lambda_2\sqrt{-g}g^{ik}. \tag{41}$$

Then, we consider $\delta S_{\check{\Gamma}}/\delta\check{\Gamma}^s_{ik}$. Let us denote $\left|det(\check{R}_{(ik)})\right| = r$. We have

$$\delta\frac{2}{\Lambda_1}\int\sqrt{r}d^4x = \int\frac{2}{\Lambda_1}\frac{1}{2}\sqrt{r}\bar{R}^{(ik)}\delta\check{R}_{(ik)}d^4x \tag{42}$$

where $\bar{R}^{(ik)}$ is the inverse matrix for $\check{R}_{(ik)}$. It means

$$\delta S_{\check{\Gamma}} = \int\frac{1}{\Lambda_1}\sqrt{r}\bar{R}^{(ik)}\delta\check{R}_{(ik)}d^4x. \tag{43}$$

Using the known formula [9]

$$\check{R}_{ln} - R_{ln} = \check{\nabla}_a P^a_{ln} - \check{\nabla}_l P^a_{na} + P^a_{ls}P^s_{an} - P^a_{sa}P^s_{ln} \tag{44}$$

we obtain

$$\delta\check{R}_{(ik)} = \check{\nabla}_s\delta\check{\Gamma}^s_{ik} - \check{\nabla}_{(i}\delta\check{\Gamma}^s_{k)s}. \tag{45}$$

Let us denote

$$\frac{1}{\Lambda_1}\sqrt{r}\bar{R}^{(ik)} = T^{ik}. \tag{46}$$

By substituting (45) into (43) and neglecting the divergence terms we get

$$\delta S_{\check{\Gamma}} = \int\delta\check{\Gamma}^s_{ik}(\check{\nabla}_p T^{pk}\delta^i_s - \check{\nabla}_s T^{ik})d^4x. \tag{47}$$

Since $\check{\Gamma}^s_{ik}$ is symmetrical, it follows that

$$\frac{\delta S_{\check{\Gamma}}}{\delta\check{\Gamma}^s_{ik}} = \frac{1}{2}\check{\nabla}_p T^{pk}\delta^i_s + \frac{1}{2}\check{\nabla}_p T^{pi}\delta^k_s - \check{\nabla}_s T^{ik}. \tag{48}$$

Now we begin varying $S_{g\check{\Gamma}}$ with respect to $\check{\Gamma}^s_{ik}$:

$$\delta S_{g\check{\Gamma}} = -\delta\int\sqrt{-g}\check{R}_{ik}g^{ik}d^4x = -\int\sqrt{-g}g^{ik}\delta\check{R}_{ik}d^4x. \tag{49}$$

If we use (45) again and reject the divergence terms, we obtain

$$\delta S_{g\check{\Gamma}} = \int \delta\check{\Gamma}^s_{ik} \left(\check{\nabla}_p(-\sqrt{-g}g^{pk})\delta^i_s - \check{\nabla}_s(-\sqrt{-g}g^{ik})\right) d^4x. \quad (50)$$

Consequently,

$$\frac{\delta S_{g\check{\Gamma}}}{\delta\check{\Gamma}^s_{ik}} = \frac{1}{2}\check{\nabla}_p(-\sqrt{-g}g^{pk})\delta^i_s + \frac{1}{2}\check{\nabla}_p(-\sqrt{-g}g^{pi})\delta^k_s - \check{\nabla}_s(-\sqrt{-g}g^{ik}). \quad (51)$$

Since $\frac{\delta S_g}{\delta\check{\Gamma}^s_{ik}} = 0$, from (38), (48) and (51) we get

$$\frac{\delta S_\Sigma}{\delta\check{\Gamma}^s_{ik}} = \frac{1}{2}\check{\nabla}_p(T^{pk} - \sqrt{-g}g^{pk})\delta^i_s + \frac{1}{2}\check{\nabla}_p(T^{pi} - \sqrt{-g}g^{pi})\delta^k_s - \check{\nabla}_s(T^{ik} - \sqrt{-g}g^{ik}). \quad (52)$$

By contracting s and i in (52) we see that $\check{\nabla}_p(T^{pk} - \sqrt{-g}g^{pk}) = 0$. Finally, the system of equations on the fields g^{ik} and $\check{\Gamma}^s_{ik}$ is

$$\frac{1}{2}\tilde{\Psi}^{ik} + \Lambda_2\sqrt{-g}g^{ik} = 0, \quad (53)$$

$$\check{\nabla}_s(T^{ik} - \sqrt{-g}g^{ik}) = 0. \quad (54)$$

An evident (but not single !) solution of (54) is

$$T^{ik} = \sqrt{-g}g^{ik}. \quad (55)$$

We find $\check{R}_{ik}$ from (55) and then substitute it into (53). From (55) we have

$$\check{R}_{(ik)} = \Lambda_1 g_{ik}. \quad (56)$$

Substituting (56) into (53) one can obtain

$$\frac{1}{2}\sqrt{-g}g^{ia}g^{kb}(2\Lambda_1 g_{ab} - \Lambda_1 g_{mn}g^{mn}g_{ab} - 2G_{ab}) + \Lambda_2\sqrt{-g}g^{ik} = 0. \quad (57)$$

From this formula the equations on g_{ik} are

$$G^{ik} + (\Lambda_1 - \Lambda_2)g^{ik} = 0.$$

As we can see, the vacuum Einstein equations with the cosmological constant $(\Lambda_1 - \Lambda_2)$ are derived.

3. Discussion

It must be emphasized that if we want to consider the gravity as a $Diff(M)$-compensating field, then the second affine connection must be introduced. Without this connection the group $Diff(M)$ becomes mixed with the generally coordinate transformations. Although equations may by invariant under both the $Diff(M)$ and generally covariant transformations, we cannot construct local field invariants since a suitable Lagrangian is absent.

The simplest way is to consider the second connection as a background. The resulting theory was observed in [10]. The present paper concerns the theory with the dynamical affine connection. A theory like that may be interesting from two points.

At first, the set of solutions of (53)-(54) is wider than (22). It would be interesting to investigate possible physical meaning of the non-Einstein solutions of (53)-(54). The second aspect is connected with the opportunity of renormalizing the cosmological constant. This may be useful in the gauge theory of gravitation and in the supergravity.

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Received by Publishing Department
on November 24, 1992.

Тентюков М.Н. E2-92-491
Теория гравитации с
динамической аффинной связностью

Предполагается, что вакуумные уравнения гравитационного поля должны получаться из локального лагранжиана, который содержит производные только первого порядка. Исследуется лагранжиан эйнштейновского типа, содержащий фоновую аффинную связность. Предлагается модель, в которой фоновая связность становится динамической.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1992

Tentyukov M.N. E2-92-491
Gravitational Theory with the
Dynamical Affine Connection

It is suggested that the gravitational equations should be derived from the local Lagrangian containing only first-order derivatives. The Einstein-like Lagrangian containing the background affine connection is investigated. The model is suggested in which the background connection becomes the dynamical one.

The investigation has been performed at the Laboratiry of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1992