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V.V.Nesterenko

ON THE MODEL OF THE RELATIVISTIC PARTICLE WITH CURVATURE AND TORSION

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I. INTRODUCTION

Recently the interest has been revived in constructing and investigating new models of point-like relativistic particles. To a considerable extent this is due to the development of the string approach to the unification of all the fundamental interactions. The straightforward way for obtaining new particle models is pure geometrical. The integrals along the particle world curve of its higher geometrical invariants should be added to the standard action of the relativistic particle [1-6]. The most general model of this kind in the D-dimensional space-time has been investigated in paper [1] (see also ref. [5]). It was the so called relativistic particle with curvature and torsion. Inclusion into the Lagrangian of higher derivatives with respect to time of dynamical variables entails the account of additional degrees of freedom. As a result, it turns out well in this approach to describe particles with a nonzero spin without introducing additional spin variables.

In paper [1] a generalized Hamiltonian dynamics for a relativistic particle with curvature and torsion was constructed at the very beginning in the D-dimensional space-time ($D \geq 3$) and only then it was put $D = 3$ to simplify quantization. In the present paper the same model is formulated at once in the three-dimensional space-time, the possibility to define the torsion of the world curve with a sign for $D = 3$ being used. Quantization of this version of the relativistic particle with curvature and torsion results in the spectrum of states which is a linear counterpart of the spectrum obtained in [1]. The layout of the paper is as follows. In Sec. II the action of a relativistic particle with curvature and torsion is discussed for arbitrary space-time dimension D and for $D = 3$. In the latter case the torsion of a world curve is defined with a sign. Here the basic results of paper[1] are also presented. In Sec. III a generalized Hamiltinian formalism for a new version of a relativistic particle with curvature and torsion is developed ($D = 3$ and the torsion is defined with a sign). In Sec. IV the mass-spin relation is derived and the canonical quantization of the model is briefly discussed. In Sec. V the obtained results are compared with the other treatment of this problem.

II. ACTION OF A RELATIVISTIC PARTICLE WITH CURVATURE AND TORSION

In paper [1] the following action was considered in the D-dimensional space-time

$$S = -m \int ds - \alpha \int k(s)\, ds - \beta \int \kappa(s)\, ds\,, \tag{2.1}$$

where $k(s)$ and $\kappa(s)$ are respectively the first and second curvatures of the world trajectory of the particle. When $D = 3$, $k(s)$ is called usually the curvature of the world curve and $\kappa(s)$ is its torsion. The model defined by the action (2.1) will be for simplicity referred to as a particle with curvature and torsion.

As it is known from elementary differential geometry [7], the torsion is defined at any point of the curve $x^\mu(s)$ where its curvature

$$k^2 = \frac{d^2x_\mu}{ds^2}\frac{d^2x^\mu}{ds^2} \tag{2.2}$$

does not vanish [1]. Here s is the natural parameter along the curve, that is, its length $(dx_\mu/ds)^2 = 1$. For arbitrary $D > 3$ the second curvature (torsion) $\kappa(s)$ is defined as follows

$$\kappa = \frac{\sqrt{\det(d_{\alpha\beta})}}{k^2(s)}\,, \tag{2.3}$$

$$d_{\alpha\beta} = \overset{(\alpha)}{x^\mu}\overset{(\beta)}{x_\mu}, \quad \overset{(\alpha)}{x} \equiv d^\alpha x/ds^\alpha, \quad \alpha, \beta = 1, 2, 3.$$

[1]The Lorentz metric with a signature $(+, -, \ldots, -)$ is used.

Obviously the function $\kappa(s)$ by its definition is positive definite. In the case of plane curves ($D = 2$) torsion equals zero identically. When $D = 3$, formula (2.3) has also meaning but in this case there is a possibility to define the torsion of a curve with a sign

$$\kappa(s) = k^{-2}\varepsilon_{\mu\nu\rho}x'^\mu x''^\nu x'''^\rho, \tag{2.4}$$

where $\varepsilon_{\mu\nu\rho}$ is a completely antisymmetric tensor of the third rank, $\varepsilon_{012} = +1$, the prime denotes the differentiation with respect to s. Thus the torsion of a curve introduced by (2.4) is proportional to a mixed product of three vectors x'_μ, x''_ν, x'''_ρ. Torsion defined by (2.4) may differ from (2.3) only by sign minus at certain points of a curve.

In paper [1] the generalized Hamiltonian description of the action (2.1) was constructed for arbitrary D and the canonical quantization was accomplished for $D = 3$. The torsion of the world curve was defined by (2.3) both in the case of arbitrary $D > 3$ and for $D = 3$. The extension of the Dirac theory of Hamiltonian systems with constraints to the Lagrangians with higher derivatives developed in paper [2] was essentially used there. When quantizing this model, natural requirements for the state vectors $|\psi>$ were introduced. They should be eigenvectors of the Casimir operators P^2 and W of the Poincaré group, where P^μ is the total energy-momentum vector and W is a generalization to the arbitrary D of the Pauli-Lubanski vector squared (spin squared) that is introduced when $D = 4$

$$W = \frac{1}{2}M_{\mu\nu}M^{\mu\nu}P^2 - (M_{\mu\nu}P^\nu)^2. \tag{2.5}$$

Here $M_{\mu\nu}$ are the generators of the Lorentz group. Thus, the state vectors obey the equations

$$P^2\,|\psi> = M^2\,|\psi>, \tag{2.6}$$

$$W\,|\psi> = M^2 j(j+1)\,|\psi>. \tag{2.7}$$

It is essential that the spin of the state j can take arbitrary non-negative values (integer, half-odd-integer and fractional, $D = 3$).

For $M^2 > 0$ the mass-spin relation (the Regge trajectory)

$$\left(\frac{M}{m}\right)^2 j^2 = \left(\alpha\sqrt{1-\left(\frac{M}{m}\right)^2} + |\beta|\right)^2 \tag{2.8}$$

was derived.

The aim of the present paper is the investigation of the model (2.1) in the three-dimensional space-time by making use of the torsion definition given by (2.4). It should be noted at once that in this case space symmetry of the theory is violated due to the pseudoscalar nature of formula (2.4). But it is this model that arises under investigation of charged scalar particles placed in an external abelian Chern-Simons field [8, 9].

III. GENERALIZED HAMILTONIAN FORMALISM

Upon introducing the arbitrary parametrization of the world curve, $x_\mu(\tau)$, $\mu = 0, 1, 2$, the action (2.1) with allowance for (2.4) reads as follows

$$S = -m\int d\tau\sqrt{\dot{x}^2} - \alpha\int d\tau\frac{\sqrt{(\dot{x}\ddot{x})^2 - \dot{x}^2\,\ddot{x}^2}}{\dot{x}^2} -$$

$$-\beta\int d\tau\sqrt{\dot{x}^2}\,\frac{\varepsilon_{\mu\nu\rho}\dot{x}^\mu\,\ddot{x}^\nu\,\dddot{x}^\rho}{\sqrt{(\dot{x}\ddot{x})^2 - \dot{x}^2\,\ddot{x}^2}}, \tag{3.1}$$

$$\dot{x} \equiv dx(\tau)/d\tau, \quad D = 3.$$

The Hamiltonian description of the model (3.1) can be constructed following the papers [1, 2]. Canonical variables are introduced in a standard fashion

$$q_1 = x, \quad q_2 = \dot{x}, \quad q_3 = \ddot{x}$$

$$p_1 = -\frac{\partial L}{\partial \dot{x}} - \dot{p}_2, \quad p_2 = -\frac{\partial L}{\partial \ddot{x}} - \dot{p}_3, \quad p_3 = -\frac{\partial L}{\partial \dddot{x}}, \tag{3.2}$$

where L is the Lagrangian function in (3.1). The equations of motion are reduced to the obvious conclusion p_1 =constant (the conservation law of the energy-momentum). Therefore the dynamics of the model under consideration is determined by the constraints.

By making use of the definition of the canonical momentum p_3 and the exact form of L we derive three primary constraints

$$\overset{(1)}{\varphi}_\mu = p_{3\mu} + \beta\frac{\sqrt{q_2^2}}{g}\,\varepsilon_{\mu\nu\lambda}\,q_2^\nu\,q_3^\lambda = 0, \tag{3.3}$$

$$\mu,\,\nu,\,\lambda = 0,\,1,\,2,$$

where $g = (q_2\,q_3)^2 - q_2^2\,q_3^2$. After squaring this equation and projecting it onto q_2 and q_3 one could get the primary constraints identical with those used in paper [1]. However it is important in the following that the primary constraints are kept in the original form (3.3).

The canonical Hamiltonian has the same form as in paper [1]

$$H = -p_1\,\dot{x} - p_2\,\ddot{x} - p_3\,\dddot{x} - L = -p_1\,q_2 - p_2\,q_3 + m\sqrt{q_2^2} + \alpha\frac{\sqrt{g}}{q_2^2}. \tag{3.4}$$

A complete set of constraints can be determined by the Dirac method [2]. The requirement of stationarity of the primary constraints results in three secondary constraints

$$\overset{(2)}{\varphi}_\mu = p_{2\mu} - \frac{\alpha}{q_2^2\sqrt{g}}\,[\,(q_2\,q_3)\,q_{2\mu} - q_2^2\,q_{3\mu}\,] -$$

$$-\beta\,\varepsilon_{\mu\nu\lambda}\,q_2^\nu\,q_3^\lambda\,\frac{(q_2\,q_3)}{g\,\sqrt{q_2^2}} = 0, \quad \mu = 0,\,1,\,2. \tag{3.5}$$

In its turn the stationarity condition for (3.5) entails three new ternary constraints. They are rather complicated in form, therefore, it is worthwhile to write them at once in the proper time gage

$$q_2^2 = \text{const}, \quad q_2\,q_3 = 0; \tag{3.6}$$

then one gets

$$\overset{(3)}{\varphi}_\mu = p_{1\mu} - m\,\frac{q_{2\mu}}{\sqrt{q_3^2}} - \beta\,\frac{\varepsilon_{\mu\nu\lambda}\,q_2^\nu\,q_3^\lambda}{q_2^2\sqrt{q_2^2}} = 0, \quad \mu = 0,1,2\,. \qquad (3.7)$$

There are no other constraints in this model. The constraints (3.5) and(3.7) with allowance for (3.3) and (3.6) coincide with formulae (3.8) and (3.10) of paper [1].

IV. SPIN-MASS RELATION

The canonical momentum $p_{1\mu}$ represents the total energy-momentum vector. Squaring Eq. (3.7) one obtains

$$p_1^2 \equiv M^2 = m^2 + \beta^2\,\frac{q_3^2}{(q_2^2)^2}\,, \qquad (4.1)$$

where M^2 is the mass of a particle in the model under consideration. Since q_3^μ may be time-like, space-like or an isotropic vector it follows that M^2 is not positive definite. By making use of the definition of the curvature (2.2) Eq. (4.1) can be rewritten as

$$p_1^2 \equiv M^2 = m^2 - \beta^2\,k^2(s). \qquad (4.2)$$

Thus the mass of the state is expressed in terms of the curvature of the world curve in the same way as in the former version of this model [3]. From (4.2) it follows in particular that curvature $k(s)$ is a constant (it is an integral of motion).

If the states are classified under eigenvalues of the quadratic Casimir operators (2.6) and (2.7) as it was done in paper [1], then one arrives at the same mass spectrum (2.8). A somewhat different situation will be in the case when the spin operator S given by

$$S = \frac{1}{2\sqrt{|p_1^2|}}\,\varepsilon_{\mu\nu\lambda}\,p_1^\mu\,M^{\nu\lambda}. \qquad (4.3)$$

is used instead of W. The spin S is a pseudoscalar quantity and it may take both positive and negative values.

To derive the mass-spin relation, one has to calculate the spin S on the submanifold of the phase space defined by the constraint equations (3.3), (3.5), (3.7), and the gauge conditions (3.6). Substituting the Lorentz generators $M_{\mu\nu}$

$$M_{\mu\nu} = \sum_{a=1}^{3}(\,q_{a\mu}\,p_{a\nu} - q_{a\nu}\,p_{a\mu}\,) \qquad (4.4)$$

into (4.2) one obtains

$$S = \frac{1}{\sqrt{|p_1^2|}}\,\varepsilon_{\mu\nu\lambda}\,p_1^\mu\,(\,q_2^\nu\,p_2^\lambda + q_3^\nu\,p_3^\lambda\,)\,. \qquad (4.5)$$

Now the constraints and the gauge conditions should be taken into account. Here the manifest form of the primary constraints is very important. Only in the case when they are taken in the original form (3.3) instead of a squared form (see Eqs. (2.8) – (2.10) in paper [1]) the spin value on the submanifold introduced above can be calculated exactly. Equation (4.5) acquires now the form

$$S = \frac{-\,\varepsilon_{\mu\nu\lambda}}{\sqrt{|p_1^2|}\,q_2^2\,\sqrt{-q_3^2}}\left(m\,q_2^\mu + \frac{\beta}{q_2^2}\,\varepsilon^{\mu\sigma\rho}\,q_{2\sigma}\,q_{3\rho}\right)$$
$$\left(\alpha\,q_2^\nu\,q_3^\lambda + \frac{\beta}{\sqrt{-q_3^2}}\,q_3^\nu\,\varepsilon^{\lambda\rho\sigma}\,q_{2\rho}\,q_{3\sigma}\right). \qquad (4.6)$$

After simplification it reads

$$S = \frac{\beta}{\sqrt{|p_1^2|}}\left(m + \alpha\,\frac{\sqrt{-q_3^2}}{q_2^2}\right). \qquad (4.7)$$

From (4.1) it follows that

$$|\beta|\,\frac{\sqrt{-q_3^2}}{q_2^2} = \sqrt{m^2 - p_1^2}\,. \qquad (4.8)$$

Substitution of Eq. (4.8) into (4.7) gives

$$S = \varepsilon_\beta \left(\alpha \sqrt{\frac{1}{\mu^2} - \varepsilon} + \frac{|\beta|}{\mu} \right), \qquad (4.9)$$

where $\varepsilon_\beta = \operatorname{sign}\beta$, $\varepsilon = \operatorname{sign} p_1^2$, $\mu = \sqrt{|p_1^2|}/m \leq 1$. At certain values of the parameters α and β the right hand side in Eq. (4.9) considered as a function of μ has an extremum. For example, if $\beta^2 > \alpha^2$ and $\varepsilon = 1$ this takes place at the point $\mu = \sqrt{\beta^2 - \alpha^2}/|\beta|$. Near by this point two values of the state mass μ correspond to the same spin value S. Thus, there is a "degeneracy" of the mass spectrum with respect to spin. The analogous property is encountered in the theory of infinite component relativistic wave equations [10].

Squaring the left- and right-hand sides of Eq. (4.9) one gets, as it could be expected, the mass-spin relation (2.8). All, that concerns the double-valued property of the spectrum (4.9), is valid obviously for Eq. (2.8) too. In order to reveal the double-valued behavior of Eq. (2.8) it was resolved in paper [1] with respect to μ but unfortunately not quite correctly. The right formula has rather a cumbersome form

$$\mu = \frac{\left| j\,|\beta| \pm \sqrt{\alpha^2 (j^2 + \alpha^2 - \beta^2)} \right|}{\alpha^2 + j^2}, \quad j \geq 0\,.$$

For each of two values of μ the corresponding conditions

$$\operatorname{sign}\alpha = \operatorname{sign}\left(-|\alpha\,\beta| \pm j\sqrt{j^2 + \alpha^2 - \beta^2} \right), \quad \mu \leq 1$$

should be satisfied.

The quantization of the model retains the mass-spin relation (4.9) for the eigenvalues of the spin operator S and the mass squared operator p_1^2. In the rest frame ($p_1^0 = M$, $\mathbf{p}_1 = 0$) the spin operator S is

$$S = M^{12}. \qquad (4.10)$$

Without loss of generality it can be realized as follows

$$S = -i\frac{\partial}{\partial\varphi} + c, \qquad (4.11)$$

where φ is an angular variable and c is a constant to be determined below. As a wave function we take 2π-periodic functions

$$\psi(\varpi) = \sum_{l \ni Z} e^{il\varphi} a_l\,. \qquad (4.12)$$

The requirement that $\psi(\varphi)$ should be an eigenfunction of S gives

$$l + c = S, \qquad (4.13)$$

where S is determined in Eq. (4.9). Hence l is an integer part of the spin S and c is its fractional part, that is, the spectrum of the spin operator is continuous.

V. CONCLUSIONS

As it should be expected, the allowance for the torsion sign in the action (3.1) results in the mass-spin relation (4.9) that can be treated as the square root of the spectrum (2.8). In paper [5] the action (3.1) has been investigated by eliminating higher derivatives with the aid of the corresponding Lagrange multipliers. The spectrum derived there is not identical with the spectrum (4.9) though they are alike. Instead of the degeneracy with respect to spin, the mass-spin relation obtained in [5] gives the degeneracy with respect to mass of a state, that is, at certain values of the model parameters the states with different values of the spin may have the same mass.

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