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$GL_q(N)$-COVARIANT QUANTUM ALGEBRAS AND COVARIANT DIFFERENTIAL CALCULUS

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1 Introduction

Noncommutative geometry [1] has awakened increasing interest and has started to play a very significant role in mathematical physics for last few years. The attractive field of investigations here is the theory of quantum groups [2]-[6] and especially several differential geometric aspects of this theory such as differential calculus on the quantum groups. A bicovariant version of this calculus has been formulated in the general form by S.L.Woronowicz [7]. Then, an intimate relation of the Woronowicz's bicovariant calculus with R-matrix formalism for the quantum groups [6] has been established in Refs. [8,9]. Quite recently, a systematic realization of the bicovariant differential calculus in the framework of the R-matrix approach has been achieved in [10]. These results give us the promising possibility to use the quantum groups as generalizations of the classical symmetry groups in various physical models.

In this paper we realize the ideas of Refs. [7]-[10] and derive explicit formulas for $GL_q(N)$ ($SL_q(N)$)-bicovariant differential calculus by means of considering quantum algebras which are covariant under the coaction of $Fun(GL_q(N))$[1]. The starting point of our considerations is the observation that right(left)-invariant vector fields E^i_j and differential 1-forms Ω^i_j [2] on $GL_q(N)$ can be treated as elements of the adjoint $GL_q(N)$-comodules or, in other words, they realize the adjoint representations of $GL_q(N)$ in the sense of Ref.[6]. Then, we consider the general associative algebras with unity whose generating elements A^i_j (the unified notation for E^i_j or Ω^i_j) are constrained by certain quadratic polynomial relations. We require these relations to be covariant under the transformations of A^i_j as the adjoint $GL_q(N)$-comodule ($T^i_j \in GL_q(N)$)

$$A^i_j \to T^i_{i'} S(T)^{j'}_j \otimes A^{i'}_{j'} \equiv (TAT^{-1})^i_j \, . \tag{1.1}$$

In the last part of (1.1) the short notation is introduced to be used below. Besides, we demand that the quadratic polynomial relations for A^i_j allow us to make the lexicographic ordering for any monomial of the type $A^{i_1}_{j_1} A^{i_2}_{j_2} \cdots A^{i_n}_{j_n}$. Later on we refer to the algebras with such features as the $GL_q(N)$-covariant quantum algebras.

The quadratic polynomial relations for $GL_q(N)$-covariant quantum algebras can be written in the following general form

$$\langle \alpha |^{jl}_{ik} \rangle A^i_j A^k_l = \langle \alpha |^m_n \rangle A^n_m + C(\alpha), \tag{1.2}$$

where the index α enumerates different relations and the coefficients $\langle \alpha |^{jl}_{ik} \rangle$, $\langle \alpha |^m_n \rangle$ and $C(\alpha)$ are functions of the deformation parameter q. On the condition that Eqs.(1.2) are covariant under transformations (1.1) we obtain that parameters $\langle \alpha |^{jl}_{ik} \rangle$ are q-analogs of the Clebsch-Gordon coefficients coupling two adjoint $GL_q(N)$ representations into the

[1]Further we use the short notation $GL_q(N)$ instead of $Fun(GL_q(N))$.

[2]Here the elements E^i_j or Ω^i_j ($i,j \in 1,\ldots,N$) form the basis in the space of right(left)-invariant vector fields or 1-forms, respectively.

irreducible representations (irreps). Parameters $\langle\alpha|^m_n\rangle$ can be considered as harmonics which are not equal to zero only if $\langle\alpha|^{jl}_{ik}\rangle$ couple $A\otimes A$ into the adjoint $GL_q(N)$-comodule again, while $C(\alpha)\neq 0$ only if combination $\langle\alpha|^{jl}_{ik}\rangle A^i_j A^k_l$ is expressed in terms of Casimir operators. Here we use the idea that arbitrary monomials $A^{i_1}_{j_1}A^{i_2}_{j_2}\cdots A^{i_n}_{j_n}$ (transformed in accordance with (1.1)) can be considered as components of $GL_q(N)$-tensor operators. Some papers have already appeared in which tensor operators for quantum groups are discussed in another context [11].

We find that, up to some arbitrariness discussed in Sect.3, there are only two kinds of $GL_q(N)$-covariant quantum algebras. For the first one the left-hand side of Eq.(1.2) is the q-deformed commutator while for the second one it has the form of q-deformed anticommutator. It is natural to call the algebras of the first and second kind as "bosonic" and "fermionic" $GL_q(N)$-covariant quantum algebras and relate them with the algebras of right(left)-invariant vector fields and 1-forms on $GL_q(N)$, respectively. As we shall see, these conjectures are justified by some explicit construction for the differential calculus on $GL_q(N)$ and are in agreement with the results obtained in Refs. [7]-[10].

2 R-matrix formulation of $GL_q(N)$ and $GL_q(N)$- covariant commutator and anticommutator

This section is a review of some facts about quantum groups needed in the consideration below. We follow the approach by Faddeev, Reshetikhin and Takhtajan [6]. The generators of the quantum group $GL_q(N)$ can be defined as elements of N by N matrix T^i_j ($T\in Mat(N,C)$) with commutation relations

$$R_{12}T_1T_2 = T_2T_1R_{12}\,. \tag{2.1}$$

Here and henceforth we use the notation of Ref.[6]. The R-matrix for $GL_q(N)$ looks like [3]

$$R_{12} = R^{i_1i_2}_{j_1j_2} = \delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\left(1+(q-1)\delta^{i_1i_2}\right)+(q-q^{-1})\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}\theta(i_1-i_2) \tag{2.2}$$

where $\theta(i-j)=\begin{cases}1, & i>j\\ 0, & i\leq j\end{cases}$. The associativity conditions for the relations (2.1) yield the Yang-Baxter equation for the R-matrix

$$R_{12}R_{13}R_{23}=R_{23}R_{13}R_{12}\Longleftrightarrow R_{12}R^{-1}_{31}R^{-1}_{32}=R^{-1}_{32}R^{-1}_{31}R_{12}\,. \tag{2.3}$$

Comparing (2.1) and (2.3) we see that possible matrix realizations of the operators T^i_j are

$$(T^i_j)^k_l = R^{ik}_{jl}\,,\ \ (T^i_j)^k_l=(R^{-1})^{ki}_{lj}\,. \tag{2.4}$$

The R-matrix (2.2) obeys the Hecke relation which can be rewritten as

$$R_{21}=(q-q^{-1})P_{12}+R^{-1}_{12}\,. \tag{2.5}$$

where $(P_{12})^{i_1i_2}_{j_1j_2}=\delta^{i_1}_{j_2}\delta^{i_2}_{j_1}$ is the permutation matrix. According to Eq. (2.5) one can define two projectors

$$P^{\pm}_{12}=\frac{P_{12}}{q+q^{-1}}(R_{12}\pm q^{\pm1}P_{12})\,, \tag{2.6}$$

which are quantum analogs of the symmetrizer $\frac{1}{2}(I+P_{12})$ and antisymmetrizer $\frac{1}{2}(I-P_{12})$. Here $(I)^{i_1i_2}_{j_1j_2}=\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}$ is the identity matrix. As it has been shown in Refs.[12], if q is not a root of unity, the representation theory for $GL_q(N)$ can be constructed in the same way as for $GL(N)$. Indeed, with the help of the projectors (2.6) one can construct q-analogs of the Young operators of symmetrization [3,13] and thus realize the program of extracting irreducible $GL_q(N)$- and $SL_q(N)$-comodules from the direct product of the fundamental comodules.

Let us demonstrate this by decomposing the direct product of two adjoint comodules. The method coincides in principle with the well known prescription for decomposing the direct product of two mesonic representations considered in the framework of the $SU(N)$-quark models of strong interactions (see e.g. remarkable reviews [14,15]). First, we note that the tensor A^i_j has N^2 components and it is possible to decompose it into the scalar $Tr_q(A)$ and the q-traceless tensor $\tilde{A}^i_j$ with (N^2-1) independent components

$$\tilde{A}^i_j = A^i_j-\delta^i_jTr_q(A)/(\sum_{i=1}^N q^{2i})\,. \tag{2.7}$$

Here we have introduced the q-deformed trace [6,9,10,16]

$$Tr_qA\equiv Tr(DA)\equiv\sum_{i=1}^N q^{2i}A^i_i \tag{2.8}$$

satisfying the following invariance property ($T^i_j\in GL_q(N)$)

$$Tr_q(A)\to Tr_q(TAT^{-1})=Tr_qA \tag{2.9}$$

which is true for any matrix representation of T^i_j, in particular, for (2.4). Using the construction of the q-trace one can reproduce the $GL_q(N)$-invariants as

$$C_n=Tr_q(A^n)\,,\qquad n\geq 1\,. \tag{2.10}$$

Now we introduce the basic covariant bilinear combinations $P_{12}A_1R_{21}A_2$ of tensors A^i_j with the transformation rule

$$P_{12}A_1R_{21}A_2\to T_2T_1(P_{12}A_1R_{21}A_2)T^{-1}_1T^{-1}_2\,. \tag{2.11}$$

Using projectors (2.6) it is possible to decompose tensor (2.11) into the four independently transformed tensors

$$X_q^{\pm\pm} = P_{21}^{\pm}(P_{12}A_1R_{21}A_2)P_{21}^{\pm}\ , \quad X_q^{\pm\mp} = P_{21}^{\pm}(P_{12}A_1R_{21}A_2)P_{21}^{\mp}\ . \qquad (2.12)$$

The dimensions of these $GL_q(N)$-comodules are $\frac{N^2(N+1)^2}{2}$ (for X_q^{++}), $\frac{N^2(N-1)^2}{2}$ (for X_q^{--}) and $\frac{N^2(N^2-1)}{2}$ (for $X_q^{\pm\mp}$). Their undeformed (q=1) analogs are nothing but

$$X^{\pm\pm} = \frac{1}{4}(P_{12} \pm I)[A_1, A_2]_+\ , \quad X^{\pm\mp} = \frac{1}{4}(P_{12} \pm I)[A_1, A_2]_-\ . \qquad (2.13)$$

As it is seen from (2.13), $X^{\pm\pm}$ are expressed in terms of the anticommutators, while $X^{\pm\mp}$ yield the combinations of the commutators. On the other hand, one can express the commutator and anticommutator as linear combinations of $X^{\pm\mp}$ and $X^{\pm\pm}$ as given below

$$X^{+-} - X^{-+} = \frac{1}{2}[A_1, A_2]_-\ , \quad X^{++} - X^{--} = \frac{1}{2}[A_1, A_2]_+\ . \qquad (2.14)$$

It is worth noting here that linear combinations of X^{++} with X^{--} or X^{+-} with X^{-+} are the only two possibilities to obtain for any pair of generators A^i_j, A^k_l the bilinear expressions of the type $[A^i_j,\ A^k_l]_\alpha = A^i_jA^k_l - \alpha A^k_lA^i_j$ ($\alpha \neq 0$) which can be used as the left-hand side of (1.2) ($q = 1$). Only such quadratic polynomial relations allow us to reorder any monomial $A^i_j \cdots A^k_l$ in an appropriate way (see Sect.1). Indeed, combining, for example, X^{++} with X^{+-} or X^{--} with X^{-+} we are unable to commute A^i_j and A^k_l when $j = l$, while the combinations of X^{++} and X^{-+} or X^{--} and X^{+-} are unsatisfactory for reordering the pairs A^i_j, A^k_l when $k = i$. So, it seems reasonable to use only X_q^{++} together with X_q^{--} or X_q^{+-} together with X_q^{-+} in defining relations (1.2) in order to solve the ordering problem. For these arguments it is natural to define the q-deformed covariant commutator and anticommutator, respectively, as

$$(q+q^{-1})(X_q^{+-} - X_q^{-+}) = R_{12}A_1R_{21}A_2 - A_2R_{12}A_1R_{21}\ , \qquad (2.15)$$
$$(q+q^{-1})(X_q^{++} - X_q^{--}) = R_{12}A_1R_{21}A_2 + A_2R_{12}A_1R_{12}^{-1}\ . \qquad (2.16)$$

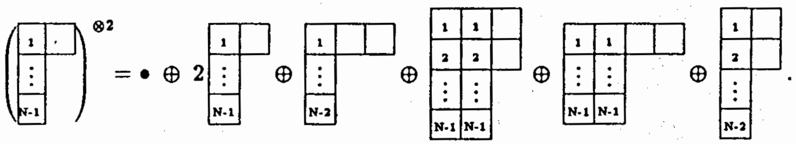
Let us note that the tensors (2.12) do not realize irreps of $GL_q(N)$. Indeed, contracting them over the first or second spaces by means of q-traces (2.7) we obtain tensors transforming as in (1.1). As we have seen above, such tensors are reduced to the 1-dimensional and (N^2-1)-dimensional irreps. Taking into account these remarks we obtain finally the following decomposition (cf. with [14,15])

$$X_q^{++}\ :\ \frac{N^2(N+1)^2}{4} = 1 \oplus (N^2-1) \oplus \frac{N^2(N+3)(N-1)}{4}\ , \qquad (2.17)$$
$$X_q^{--}\ :\ \frac{N^2(N-1)^2}{4} = 1 \oplus (N^2-1) \oplus \frac{N^2(N+1)(N-3)}{4}\ , \qquad (2.18)$$
$$X_q^{\pm\mp}\ :\ \frac{N^2(N^2-1)}{4} = (N^2-1) \oplus \frac{(N^2-1)(N^2-4)}{4}\ . \qquad (2.19)$$

We stress here that (N^2-1)- and $\frac{N^2(N+1)(N-3)}{4}$-dimensional irreps appear in (2.18) only for $N \geq 3$ and $N \geq 4$, respectively, while $\frac{(N^2-1)(N^2-4)}{4}$-dimensional irrep appears in (2.19) only for $N \geq 3$. Using the decomposition (2.17)-(2.19) one can deduce that the direct product of two q-traceless tensors can be decomposed into irreps of the following dimensions (here $N \geq 4$):

$$(N^2-1)^{\otimes 2} = [1] \oplus 2 \cdot \left[N^2-1\right] \oplus \left[\frac{(N^2-1)(N^2-4)}{4}\right] \oplus \left[\frac{(N^2-1)(N^2-4)}{4}\right]^* \oplus \left[\frac{N^2(N+3)(N-1)}{4}\right] \oplus \left[\frac{N^2(N+1)(N-3)}{4}\right] . \qquad (2.20)$$

In terms of the Young tableaux this formula looks like


(2.21)

The dimensions of the irreps related to the Young tableaux listed in (2.21) are given by the Weyl formula [13]. Naturally, they coincide with that expressed in Eq.(2.20).

As it will be seen in the next Section, this information is enough to conclude that "fermionic" (with q-deformed anticommutators (2.16) in the l.h.s. of (1.2)) and "bosonic" (with q-deformed commutators (2.15) in the l.h.s. of (1.2)) quantum algebras are defined uniquely up to some inessential rescaling factors. Moreover, we show that up to some arbitrariness disscussed below there are no other well defined $GL_q(N)$-covariant algebras with quadratic polynomial structure relations (1.2).

3 $GL_q(N)$-covariant quantum algebras

In this Section, using the R-matrix approach [6] we discuss the Jordan-Schwinger (J-S) construction for covariant quantum algebras. This is the most simple way to reproduce explicitly quadratic polynomial relations (1.2) for the generators of these algebras. We start with the formulation of the $GL_q(N)$-covariant differential calculus [18] on a bosonic (fermionic) quantum hyperplane. Commutation relations for hyperplane coordinates and derivatives are identical with the commutation relations for the $GL_q(N)$-covariant q-(super)oscillators [17]–[20]. It is known (see e.g. [19,21] and Refs. therein) that the generators of the quantum algebras $U_q(gl(N))$ can be constructed as bilinear combinations of the bosonic or fermionic q-oscillators (J-S construction). In this Section, following the idea of J-S construction we realize the covariant quantum algebra generators A^i_j as bilinears of the $GL_q(N)$-covariant q-oscillators.

It is known [4,6] that the bosonic (fermionic) hyperplanes with coordinates $\{x^i\} = |x\rangle$ (i=1,2,...,N) can be defined by using the projectors (2.6)

$$(R_{12} - cP_{12})|x\rangle_1|x\rangle_2 = 0\ , \qquad (3.1)$$

Here $c = q$ and $c = -q^{-1}$ for bosonic and fermionic coordinates, respectively. Relations (3.1) are covariant under the left rotations of vectors $|x\rangle$ by the matrix $T^i_j \in GL_q(N)$ ($|x\rangle$ is the space of the fundamental representation of $GL_q(N)$):

$$x^i \to T^i_j x^j \ . \tag{3.2}$$

One can extend the algebra (3.1) introducing the dual vector $\langle\partial| = \partial_i$ with the transformation rule

$$\partial_i \to \partial_j (T^{-1})^j_i \ , \tag{3.3}$$

Then the covariant associative extension of the algebra (3.1) is

$$R_{12}|x\rangle_1|x\rangle_2 = c|x\rangle_2|x\rangle_1 \ , \quad \langle\partial|_1\langle\partial|_2 R_{12} = c\langle\partial|_2\langle\partial|_1 \ , \tag{3.4}$$

$$|x\rangle_1\langle\partial|_2 = \nu\delta_{12} + c\langle\partial|_2 R_{12}|x\rangle_1 \ . \tag{3.5}$$

Here $\delta_{12} = \delta^{i_1}_{j_2}$ is a unit matrix and ν are arbitrary rescaling factors ($\nu = b$ for bosons and $\nu = f$ for fermions). Note that making the replacements $R_{12} \to R_{21}^{-1}, c \to c^{-1}$ in Eqs.(3.4),(3.5) we obtain another (and the last) possible covariant extension of (3.1). Below, we concentrate only on the consideration of the algebra (3.4),(3.5) (the other possibility can be treated analogously).

In the bosonic case ($c = q$) the formulas (3.4) and (3.5) define the covariant q-oscillators [17] or covariant differential calculus on the quantum hyper-plane [18]. This algebra can be interpreted also as differential calculus on the paragrassmann hyperplane [22] or as finite dimensional Zamolodchikov-Faddeev algebra [20,23]. In the fermionic case ($c = -q^{-1}$) the algebra (3.4) and (3.5) defines covariant fermionic q-oscillators or fermionic part of the covariant super q-oscillators [19].

Now, we recall that the coordinates $\{x^i\}$ and the derivatives $\{\partial_i\}$ (as vector spaces) are tensors realizing the fundamental and contragradient representations of $GL_q(N)$ (see (3.2) and (3.3)). The higher order tensors can be constructed as direct products of the vectors $|x\rangle$ and $\langle\partial|$. The simplest tensor of that kind is

$$A^i_j = x^i\partial_j \ . \tag{3.6}$$

The transformation rule for this tensor coincides with (1.1) and, thus, A realizes the adjoint representation of $GL_q(N)$ both for bosonic and fermionic cases. Using formulas (3.5) and (3.6) we obtain equation

$$cA_1R_{21}A_2 + \nu A_1P_{12} = |x\rangle_1|x\rangle_2\langle\partial|_1\langle\partial_2| \ . \tag{3.7}$$

Then, applying (3.4) to the right-hand side of (3.7) we deduce the following two relations for the operators $A^i{}_j$

$$(R_{12} - cP_{12})(cA_1R_{21}A_2 + \nu A_1P_{12}) = 0 \ , \tag{3.8}$$

$$(cA_2R_{12}A_1 + \nu A_2P_{12})(R_{21} - cP_{21}) = 0 \ . \tag{3.9}$$

Difference between (3.8) and (3.9) gives the q-deformed commutation relations (cf. with (2.15))

$$R_{12}A_1R_{21}A_2 - A_2R_{12}A_1R_{21} = \mu(P_{12}A_1R_{21} - R_{12}A_1P_{12}) \ , \quad \mu = \frac{\nu}{c} \ . \tag{3.10}$$

By construction, these relations are covariant under the adjoint $GL_q(N)$-coaction (1.1). Note that the algebra (3.10) is the same for bosonic and fermionic q-oscillators (up to some trivial rescaling of the generators A^i_j). In the classical limit $q = 1$, Eqs.(3.10) coinside with the usual commutation relations for the $gl(N)$-algebra. We call the algebra with the structure relations (3.10) as "bosonic" $GL_q(N)$-covariant quantum algebra. One can check that this algebra is associative. The invariant central elements (Casimir operators) for the algebra (3.10) are represented in the form (2.10). The identities $[C_n, A^i_j] = 0$ can be obtained by using the Hecke relation (2.5), the property of the q-trace (2.9) and the fact that the matrix $Tr_2(D_2P_{12}R_{12})$ is proportional to the unit matrix in the first space.

The q-deformed commutation relations (3.10) can be rewritten in the form

$$R_{12}\tilde{A}_1R_{21}\tilde{A}_2 - \tilde{A}_2R_{12}\tilde{A}_1R_{21} = \kappa(P_{12}\tilde{A}_1R_{21} - R_{12}\tilde{A}_1P_{12}) \ , \quad [H, \tilde{A}^i_j] = 0 \ , \tag{3.11}$$

$$\kappa = \mu + \frac{(q-q^{-1})^2}{(q^N - q^{-N})}H \ .$$

Here $\tilde{A}^i_j$ are the q-traceless generators (see (2.7)) and $H = q^{-N-1}Tr_q(A)$. Thus, the algebra (3.10) is the direct sum of the trivial algebra generated by the central element H and the algebra generated by the operators $\tilde{A}^i_j$. As we will see below, the operators $\tilde{A}^i_j$ and A^i_j can be interpreted as invariant vector fields on the $SL_q(N)$ and $GL_q(N)$, respectively. Finally, we rewrite the relations (3.10) in the form

$$R_{12}Y_1R_{21}Y_2 - Y_2R_{12}Y_1R_{21} = 0 \ , \tag{3.12}$$

where $A^i_j = \frac{-\mu}{(q-q^{-1})}\delta^i_j + Y^i_j$. Eq.(3.12) is well known as reflection equation [23] or as relations for the operator $Y = (L^-)^{-1}L^+$, where the elements of triangular matrices $L^\pm$ are defined by the generators of the Borel subalgebras of $U_q(gl(N))$ (see [6]). In Refs. [7]-[10] the operator Y is interpreted as differential operators (vector fields) of the bicovariant differential calculus on $GL_q(N)$. The algebra (3.12) is known also as the braided algebra [24]. We present here also the commutation relations of Y with $\langle\partial|$ and $|x\rangle$

$$|x\rangle_1Y_2 = R_{21}Y_2R_{12}|x\rangle_1 \ , \quad Y_2\langle\partial|_1 = \langle\partial|_1R_{21}Y_2R_{12} \ .$$

We have considered only part of the relations (3.8) and (3.9), namely the relations (3.10). Now we proceed to the discussion of the rest of Eqs. (3.8), (3.9). First of all we rewrite them in the equivalent form

$$(R_{12} - cP_{12})(cA_1R_{21}A_2 + \nu A_1P_{12})(R_{12} - cP_{12}) = 0 \ , \tag{3.13}$$

$$(R_{12} \mp c^{\pm 1}P_{12})(cA_1R_{21}A_2 + \nu A_1P_{12})(R_{12} \pm c^{\mp 1}P_{12}) = 0 \ . \tag{3.14}$$

The pair of Eqs. (3.14) are equivalent to the commutation relations (3.10) for the "bosonic" $GL_q(N)$-covariant quantum algebra. Indeed, acting on (3.10) by the projectors $(R_{12} \pm c^{\mp 1} P_{12})$ from the left we obtain (3.14). On the other hand, difference between two of Eqs.(3.14) gives (3.10). The remaining relation (3.13) takes the different forms for the bosonic and fermionic oscillators. For the bosonic case we obtain

$$(R_{12} - qP_{12})(A_1 R_{21} A_2 + bq^{-1} A_1 P_{12})(R_{12} - qP_{12}) = 0 \, , \tag{3.15}$$

while for the fermionic case we have

$$(R_{12} + q^{-1} P_{12})(A_1 R_{21} A_2 - fqA_1 P_{12})(R_{12} + q^{-1} P_{12}) = 0 \, . \tag{3.16}$$

The bilinear parts of Eqs.(3.15),(3.16) coinside with $X_q^{--} P_{12}$ and $X_q^{++} P_{12}$, respectively (see (2.12)) and, hence, combining these equations together we shall obtain $GL_q(N)$-covariant relations with the q-deformed anticommutator (2.16). Indeed, subtracting (3.15) from (3.16) we deduce

$$R_{12} A_1 R_{21} A_2 + A_2 R_{12} A_1 R_{12}^{-1} =$$

$$P_{12} \left(q^{-1} b P_{12}^{-} + qf P_{12}^{+}\right) (A_1 R_{21} + R_{12}^{-1} A_2) = \nu (R_{12} A_1 R_{21} + A_2) \, . \tag{3.17}$$

We interpret (3.17) as structure relations for "fermionic" $GL_q(N)$-covariant algebra and we are obliged to put $b = f = \nu$ in order to have the associative algebra. The contraction $b = 0$, $f = 0$ of the algebra (3.17) leads to the relations

$$R_{12} A_1 R_{21} A_2 + A_2 R_{12} A_1 R_{12}^{-1} = 0 \, , \tag{3.18}$$

which, as we will see below, are the q-deformed anticommutation relations for the Cartan's 1-forms on the $GL_q(N)$. Note that the relations (3.17) can be rewritten in the form

$$R_{12} W_1 R_{21} W_2 + W_2 R_{12} W_1 R_{12}^{-1} = \frac{\nu^2}{2} (R_{12} R_{21} + 1) \, , \tag{3.19}$$

where $A^i_j = \frac{\nu}{2} \delta^i_j + W^i_j$.

The logic of J-S construction allows us in principle to change the q-deformed commutation relations (3.10) by mixing them with the additional relations (3.13). The existense of these additional relations has been pointed out in Ref.[21] where J-S construction has been considered in the noncovariant way. But it is natural to demand the covariance of q-commutation relations under the transformation (1.1). This remark and the requirements discussed in the previous Sections impose very strong restrictions on the possible form of q-commutation relations. It seems that the only reasonable choices here are those of (3.10) and (3.17). However, there is remaining abitrariness which now we have to discuss.

Covariant relations (3.10) and (3.17) define the covariant "bosonic" and "fermionic" algebras which are "good" in the sense that they allow to reorder any monomial

$A^i_j \ldots A^k_l$. But these relations are not the only possible covariant relations of the kind (1.2). It is clear that (3.10) and (3.17) are linear combinations of the "irreducible" sets of covariant relations (ISCR) which correspond to the irreps presented in (2.17)-(2.21). Note that among these ISCR there are several independent "adjoint" ISCR, namely a couple of trivial "adjoint" ISCR ($[\tilde{A}, Tr_q(A)]_\pm$) and a couple, for $N \geq 3$ (or one, for $N = 2$), of nontrivial ones (see (2.21)). Some linear combinations of these "adjoint" ISCR are included in both the "bosonic" and "fermionic" covariant algebras. Their presence is evident due to the existence of linear terms in the formulas (3.10) and (3.17). Leaving aside here the problem of the associativity one can use the different combinations of the "adjoint" ISCR instead of original ones in the covariant relations (3.10) and (3.17). The only restriction is that these combinations must contain both the trivial and nontrivial "adjoint" ISCR (to solve the problem of ordering). However, it is rather difficult to write the new algebras in the compact form. So, the covariant algebras (3.10) and (3.17) look preferable.

To conclude this Section, we illustrate our results by considering, in detail, the special case of $N = 2$. For this we introduce the new notation

$$A^i_j = \begin{pmatrix} A^1_1 & A^1_2 \\ A^2_1 & A^2_2 \end{pmatrix} = \begin{pmatrix} \frac{H + qA_0}{q + q^{-1}} & A_+ \\ A_- & \frac{H - q^{-1} A_0}{q + q^{-1}} \end{pmatrix} \tag{3.20}$$

where $H = q^{-3} Tr_q A = (q^{-1} A^1_1 + q A^2_2)$ and $A_0 = A^1_1 - A^2_2$.

The $GL_q(2)$-covariant "bosonic" quantum algebra (3.10) is rewritten as (we change the notation A to E bearing in mind the interpretation of the matrix elements (3.20) as invariant vector fields on $GL_q(2)$)

$$[E_-, E_+] = \frac{q^2 - 1}{q^2 + 1} E_0^2 + \frac{\kappa}{q} E_0 \, , \tag{3.21}$$

$$[E_\pm, E_0]_{(q^{\mp 1}, q^{\pm 1})} \equiv q^{\mp 1} E_\pm E_0 - q^{\pm 1} E_0 E_\pm = \pm (q + q^{-1}) \frac{\kappa}{q} E_\pm \, , \tag{3.22}$$

$$[H, E_\pm] = [H, E_0] = 0 \, , \tag{3.23}$$

where κ is defined in (3.11) for $N = 2$. Performing the transformations (1.1) for the generators (3.20) we may directly convince ourselves that the relations (3.21)-(3.22) define the covariant algebra. The central element $H = q^{-3} Tr_q A$ of the algebra (3.21)-(3.23) can be removed by the following rescaling $E_{\pm,0} = (1 + q^{-2}) \kappa \hat{E}_{\pm,0}$ and finally we obtain

$$(q + q^{-1})[\hat{E}_-, \hat{E}_+] - (q - q^{-1}) \hat{E}_0^2 = \hat{E}_0 \quad , \quad [\hat{E}_\pm, \hat{E}_0]_{(q^{\mp 1}, q^{\pm 1})} = \pm \hat{E}_\pm \, . \tag{3.24}$$

These relations really correspond to the adjoint irrep ($\Box\Box$) of $SL_q(2) \subset GL_q(2)$. As a covariant object, the algebras (3.21)-(3.23) and (3.24) have been considered in [16]. Note that up to some trivial rescalings the commutation relations (3.24) coincide with those for Witten's deformation of the algebra $sl(2)$ (see Eqs.(5.2) of [25]).

The defining relations for the $GL_q(2)$-covariant "fermionic" quantum algebra looks like (we change the notation A^i_j to Ω^i_j bearing in mind the interpretation of this matrix elements as Cartan's 1-forms on $GL_q(2)$):

$$\square\square\square\square : \quad \begin{cases} q^2\Omega_+\Omega_- + q^{-2}\Omega_-\Omega_+ - \Omega_0^2 = 0 \,, \\ q^{\mp 1}\Omega_0\Omega_\pm + q^{\pm 1}\Omega_\pm\Omega_0 = 0 \quad , \quad \Omega_\pm^2 = 0 \,; \end{cases} \tag{3.25}$$

$$\begin{matrix} r\,\square\square + \\ + \{H, \Omega\} \end{matrix} : \begin{cases} r\left((q+\frac{1}{q})[\Omega_-, \Omega_+] - (q-\frac{1}{q})\Omega_0^2\right) + \{H, \Omega_0\} = \lambda\Omega_0, \\ r\,[\Omega_\pm, \Omega_0]_{(q^{\mp 1}, q^{\pm 1})} \pm \{H, \Omega_\pm\} = \pm\lambda\Omega_\pm; \end{cases} \tag{3.26}$$

$$\textit{the scalar ISCR} : \begin{cases} -\frac{1}{q}Tr_q(\Omega^2) + qH^2 = \nu H \,, \\ Tr_q(\Omega^2) + H^2 = q(q^2+1+q^{-2})\nu H \,. \end{cases} \tag{3.27}$$

Here $\lambda = (q+q^{-1})\nu$, $r = (1-q^2)/(q^2+q^{-2})$ and

$$\frac{1}{q^3}Tr_q(\Omega^2) \; = \; \left(q^{-1}\Omega_+\Omega_- + q\Omega_-\Omega_+\right) + \frac{\Omega_0^2 + H^2}{q+q^{-1}} \,. \tag{3.28}$$

In the limit $\nu = 0$, Eqs. (3.25)-(3.27) become the commutation relations for the Cartan's 1-forms on $GL_q(2)$. These relations in another form have been presented in Ref. [9]. Note that just the presence of the ISCR $\square\square$ in these relations prevents us (for $q \neq 1$) to remove H and pass over to the Cartan's 1-forms on $SL_q(2)$ [3]. We believe that the right way to obtain the commutation relations for Cartan's 1-forms on $SL_q(2)$ is simply to ignore the Eqs. (3.26) and use only the Eqs. (3.25) and (3.27) for $H = 0$. These relations define the associative covariant algebra and have the correct classical limit.

Finally, one can check directly that the quadratic Casimir operators for the algebras (3.21),(3.22) and (3.25)-(3.27) are related to the invariant C_2 (see (2.10) and (3.28)).

4 Conclusion

To conclude, we present here an explicite construction for the invariant vector fields and 1-forms on $GL_q(N)$ and thus illustrate the connection between $GL_q(N)$-covariant quantum algebras and the covariant differential calculus on $GL_q(N)$. Let us introduce the quantum group derivatives $\partial^i_j = \partial/\partial T^j_i$ and differentials $d(T^i_j)$ to extend $GL_q(N)$ in the following way

$$\begin{aligned} R_{12}T_1T_2 \; &= \; T_2T_1R_{12} \quad , \quad R_{12}\partial_2\partial_1 \; = \; \partial_1\partial_2R_{12} \,, \\ \partial_2R_{12}T_1 \; &= \; \nu P_{12} + T_1R_{21}^{-1}\partial_2 \quad , \quad R_{21}^{-1}T_1d(T_2) \; = \; d(T_2)T_1R_{12} \,, \\ R_{21}^{-1}d(T_1)d(T_2) \; &= \; -d(T_2)d(T_1)R_{12} \,. \end{aligned} \tag{4.1}$$

This algebra is covariant under the left(right) $GL_q(N)$-coaction on the operators T, ∂ and $d(T)$ which can be considered as bicomodules of $GL_q(N)$: $T \to \tilde{T}TT'$, $\partial \to$

[3]This feature was pointed out in Ref.[9] for $SL_q(2)$ and in Ref.[7] in general.

$T'^{-1}\partial\tilde{T}^{-1}$, $d(T) \to \tilde{T}d(T)T'$, where $\tilde{T}^i_j$, T'^i_j are generators of various examples of $GL_q(N)$. Using the relations (4.1) one can directly check that operators $E = T\partial$ satisfy "bosonic" commutation relations (3.12) and operators $\Omega = d(T)T^{-1}$ satisfy contracted "fermionic" anticommutation relations (3.18) [4]. Thus, we relate the $GL_q(N)$-covariant quantum algebras introduced in the previous Section with bicovariant differential calculus on $GL_q(N)$.

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$GL_q(N)$-ковариантные квантовые алгебры и ковариантное дифференциальное исчисление

Рассматриваются $GL_q(N)$-ковариантные квантовые алгебры, генераторы которых удовлетворяют квадратичным соотношениям. Показано, что, с точностью до некоего несущественного произвола, существует только два типа таких алгебр, а именно: алгебры с q-деформированными коммутационными и q-деформированными антикоммутационными соотношениями. Обсуждается связь этих алгебр с ковариантным дифференциальным исчислением на линейных квантовых группах.

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$GL_q(N)$-Covariant Quantum Algebras and Covariant Differential Calculus

We consider $GL_q(N)$- covariant quantum algebras with generators satisfying quadratic polynomial relations. We show that, up to some inessential arbitrariness, there are only two kinds of such quantum algebras, namely, the algebras with q-deformed commutation and q-deformed anticommutation relations. The connection with the bicovariant differential calculus on the linear quantum groups is disscussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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