объединенный
институт
ядерных
исследований
дубна

E2-92-439

M.N.Tentyukov*

GRAVITATIONAL THEORY WITH THE LOCAL QUADRATIC LAGRANGIAN

Submitted to "Hadronic Journal"

*e-mail: tentukov@theor.jinrc.dubna.su

1992

1. INTRODUCTION

Contemporary theory of gravitation is the Einstein General Relativity (GR). This theory describes the gravitation as a space-time metric g_{mn} which satisfies the Einstein equations

$$G_{mn} = \kappa T_{mn}, \tag{1}$$

where $G_{mn} = R_{mn} - \frac{1}{2} R\, g_{mn}$ is the Einstein tensor.

These equations are very difficult and nonlinear. For investigation of these equations it would be useful to find the action functional. The action is necessary for any development of the theory, for instance, for the problem of quantization.

Usually, the vacuum Einstein equations

$$G_{mn} = 0 \tag{2}$$

are derived from the Hilbert action

$$S_H = \int \sqrt{-g}\, R\, d^4x. \tag{3}$$

These equations are of the second order, and the Hilbert Lagrangian

$$L_H = \sqrt{-g}\, R \tag{4}$$

contains the second - order derivatives too. This leads to the known difficulties [1]. For avoiding them Hibbons and Hawking have suggested the surface term [1], but due to this term local gravitational invariants such as energy-momentum density became *quasi*local [2].

Another way consist in finding a suitable Lagrangian which will be local and contains only first-order derivatives. For a Lagrangian like that to exist, it is necessary to introduce the background object in the theory [3-5]. It should be mentioned that the well-known Einstein Lagrangian $L_E = \sqrt{-g}\, g^{mn}\left(\Gamma^a_{mb}\Gamma^b_{an} - \Gamma^a_{ba}\Gamma^b_{mn}\right)$ contains the background affine connection [4] whose coefficients $\check{\Gamma}^a_{mn}$ are zero in a chosen coordinate map (see sect.2).

Introduction of the background connection permits our to expand the GR by admitting a more general (nonflat) background connection. By comparing such an expanded theory with GR some interesting specific features of the Einstein equations can be found. But it is clear that the main interest to introduce the background geometrical object is connected with the problem of localization of the energy-momentum characteristics of the gravitational field.

Many attempts were made to solve the problem of localization of the gravitational energy by introducing the nondynamical (background) object [6-8]. Usually, it was a background metric (bimetric theories) and the gravitation was considered as a conventional matter field alongside with other fields [8]. The theory remained generally covariant but the dynamical invariance under the group of diffeomorphisms was violated. In the general case, when the background object is arbitrary, the invariance is completely violated, i.e., any residual symmetry is absent. However, if the background object permits the group of motions, the theory is invariant under this group. Usually, the background object is a metric permitting the Poincaré group, and thus the energy-momentum problem seems to be solved. But a more careful analysis shows that it is still unsolved. Although we can construct a gravitational energy-momentum tensor (see sect.4), conserved quantities become trivial because conserved currents have a specific structure (improper current, [9]). It seems that the gravitational field does not carry energy like the electromagnetic field transferring the interaction between the electric charges does not carry charge [4].

In studying non-Einstein generalization of the gravitational equations, we shall be interested mainly in degeneracy of these equations. It is well-known that the Einstein equations (2) are degenerated in the sense that for ten unknown components of the metric tensor there are only six independent equations due to the Bianchi identities. We will see that for equations to have this properties it is necessary that the symmetric part of the Ricci tensor of the background connection $\check{R}_{(mn)}$ should have such a null vector ξ

$$\xi^{i}\check{R}_{(ij)} = 0 \tag{5}$$

that the Lie derivative of $\check{R}_{(ij)}$ should vanish

$$\underset{\xi}{\pounds}\check{R}_{(ij)} = 0. \tag{6}$$

Also we discuss correlations between generalized integrability conditions and the "harmonicity conditions"

$$\check{\nabla}_{i}(\sqrt{-g}\; g^{ij}) = 0. \tag{7}$$

Finally, we investigate one exact solution of non-Einstein equations. We shall see that in this case there is no the Birkhoff theorem but Schwarzschild-like peculiarity can be present.

2. The gravitational action functional and equations of motion

Usually, equations (2) are derived from the Hilbert action (3) with the Hilbert Lagrangian (4). As it has been remarked above, this Lagrangian leads to the known difficulties.

Instead of (4) the noncovariant Einstein Lagrangian is often used

$$L_{E} = \sqrt{-g}\; g^{mn}(\Gamma^{a}_{mb}\Gamma^{b}_{an} - \Gamma^{a}_{sa}\Gamma^{s}_{mn}), \tag{8}$$

which differs from L_{H} by the divergence term

$$L_{H} - L_{E} = \partial_{i}\omega^{i} \tag{9}$$

where

$$\omega^{i} = \sqrt{-g}\; (g^{in}\Gamma^{m}_{mn} - g^{mn}\Gamma^{i}_{mn}). \tag{10}$$

Now let us prove that noncovariance of L_{E} in fact means that the background object is present in the theory [4,5]. It is the affine connection without torsion. We shall denote the background connection coefficients by $\check{\Gamma}^{k}_{mn}$.

The difference between the connection coefficients

$$P^{k}_{mn} = \check{\Gamma}^{k}_{mn} - \Gamma^{k}_{mn} \tag{11}$$

is a tensor. It is named the affine - deformation tensor. Let us consider the Lagrangian

$$\tilde{L} = \sqrt{-g}\; g^{mn}(P^{a}_{mb}P^{b}_{an} - P^{a}_{sa}P^{s}_{mn}). \tag{12}$$

For the action functional

$$\tilde{S} = \int \tilde{L}\, d^{4}x \tag{13}$$

the variational derivative

$$\tilde{\Psi}^{mn} = 2\,\frac{\delta\tilde{S}}{\delta g_{mn}}$$

has been calculated in [5]

$$\tilde{\Psi}^{mn} = \sqrt{-g}\; g^{ma}g^{nb}(\check{R}_{ab} + \check{R}_{ba} - \check{R}_{ij}g^{ij}g_{ab} - 2G_{ab}), \tag{14}$$

where $\check{R}_{ik} = \check{R}^{p}_{pik}$ is the Ricci tensor; $\check{R}^{p}_{lik} = \partial_{l}\check{\Gamma}^{p}_{ik} - \partial_{i}\check{\Gamma}^{p}_{lk} + \check{\Gamma}^{p}_{ls}\check{\Gamma}^{s}_{ik} - \check{\Gamma}^{p}_{is}\check{\Gamma}^{s}_{lk}$ is the Riemann tensor for the background connection. If $\check{R}_{(ik)}=0$, then the equations

$$\tilde{\Psi}^{mn} = 0 \tag{15}$$

coincide with the Einstein equations (2), and

$$L_{H} - \tilde{L} = \check{\nabla}_{i}F^{i}, \tag{16}$$

where $\check{\nabla}_{i}$ is a covariant derivative with respect to the background connection and

$$F^{i} = \sqrt{-g}\; (g^{mn}P^{i}_{mn} - g^{in}P^{m}_{mn}) \tag{17}$$

is the vector density of weight one.

If $\check{R}^{i}_{klm} = 0$ one can choose the coordinate map in which all $\check{\Gamma}^{i}_{km} = 0$. Then, P^{i}_{km} turns into $-\Gamma^{i}_{km}$, $\tilde{L}$ turns into L_{E}, F^{i} into ω^{i} and (16) is transformed into (9).

Since L_{E} is noncovariant, converting L_{H} into L_{E} can be possible only after fixation of the coordinate map. Converting L_{H} into L_{E} by formula (9) is in fact converting L_H into $\tilde{L}$ with the fixation of the background connection whose coefficients in this map are assumed to be zero. Hence, it follows that in this theory it is necessary to use the Lagrangian $\tilde{L}$.

The situation is similar to the appearing of the not unity components of the metric tensor in Maxwell equations when they are written in the curvilinear coordinate map. The generally covariant Maxwell equations

$$g^{mn}\nabla_m \mathcal{F}_{nl} = g_{lm} j^m, \tag{18}$$

where $\mathcal{F}_{mn} = \partial_m \mathcal{A}_n - \partial_n \mathcal{A}_m$ is the electromagnetic strength tensor, contain the metric field g_{mn}. In the Cartesian frame of reference equations (18) turns into

$$\partial_0 \mathcal{F}_{0l} - \partial_1 \mathcal{F}_{1l} - \partial_2 \mathcal{F}_{2l} - \partial_3 \mathcal{F}_{3l} = j_l.$$

The components of the metric tensor become the set of units as though they "disappear".

The same situation takes place if after converting L_H into L_E by formula (9) we want to use (8) in an arbitrary coordinate map. L_E transforms into (12) and nonzero $\check{\Gamma}^l_{km}$ appears. Then, if we want to return to the original map, the background connection coefficients convert into the set of nulls as the metric components in (18) convert into the units.

As it is clear from (14), the conditions $\check{R}^l_{klm} = 0$ are stronger than the necessary conditions for deriving the Einstein equations. If we put

$$\check{R}_{(ij)} = 0 \tag{19}$$

it would be enough. Although we cannot choose a coordinate map with nonzero $\check{\Gamma}^l_{km}$, the latter is not contained in gravitational equations that coincide with the Einstein ones.

Now we will briefly dwell upon the possibility of further generalization of the theory.

Up to a divergence term the Lagrangian (12) can be presented in the form

$$\tilde{L} = \sqrt{-g}\, R - \sqrt{-g}\, g^{mn}\check{R}_{mn} + \text{div}. \tag{20}$$

The first term can be interpreted as the term corresponding to the pure gravitational field, the second is the "cross" $g - \check{\Gamma}$ term describing the interaction between the fields g_{mn} and $\check{\Gamma}^k_{mn}$. For the background connection be dynamical, the Lagrangian needs a "kinetic" term for pure $\check{\Gamma}^k_{mn}$. The simplest term like that has been proposed by A. Eddington [10] (see

also [11]):

$$L_C = \frac{2}{\Lambda_1}\sqrt{|\det(\check{R}_{(ij)})|}, \tag{21}$$

where $\frac{2}{\Lambda_1}$ is the coupling constant. Then, the full Lagrangian can be written as

$$L_D = \tilde{L} + L_C + 2\Lambda_2\sqrt{-g}. \tag{22}$$

In this Lagrangian we have added the cosmological term.

Being varied with respect to g_{mn}, the action

$$S_D = \int L_D d^4x \tag{23}$$

gets the same variational derivatives as (13) but with the cosmological constant. Equations for g_{mn} are

$$R_{ab} - \frac{1}{2} R\, g_{ab} - \Lambda_2 g_{ab} = \check{R}_{(ab)} - \frac{1}{2}\check{R}_{ij} g^{ij} g_{ab}. \tag{24}$$

For $\check{\Gamma}^k_{mn}$ we obtain the equations

$$\check{\nabla}_j\left(\frac{1}{\Lambda_1}\sqrt{|\det(\check{R}_{(ik)})|}\, \bar{R}^{(ik)} - \sqrt{-g}\, g^{ik}\right) = 0. \tag{25}$$

Here $\bar{R}^{(ik)}$ is the inverse matrix for $\check{R}_{(ik)}$: $\bar{R}^{(ik)}\check{R}_{(kj)} = \delta^i_j$. An evident (but not single!) solution of (25) is

$$\check{R}_{(ik)} = \Lambda_1 g_{ik}. \tag{26}$$

Substituting (26) into (24) we come to the Einstein equations with the cosmological constant $(\Lambda_1 - \Lambda_2)$:

$$R_{ij} - \frac{1}{2} R\, g_{ij} + (\Lambda_1 - \Lambda_2)\, g_{ik} = 0. \tag{27}$$

It means that the system described by (22) locally contains all vacuum solutions of the Einstein equations with the cosmological constant $(\Lambda_1 - \Lambda_2)$. If $\Lambda_1 = \Lambda_2$, then the cosmological constant is zero. As we can see, in this theory there is a mechanism which allows our to renormalize the cosmological constant.

Further we shall assume the affine connection $\check{\Gamma}^k_{mn}$ is background and restrict our consideration only to the Lagrangian (12).

3. The integrability conditions and the harmonicity conditions

In this section we consider the full action

$$S_F = \tilde{S} + S_M \tag{28}$$

where

$$S_M = \kappa \int L_M d^4x \tag{29}$$

is the action of the external (nongravity) matter.

Let us suppose that L_M is independent of $\check{\Gamma}^k_{mn}$. Then, the conventional energy-momentum tensor of the external matter

$$\sqrt{-g}\, T^{mn} = \frac{1}{\kappa}\frac{\delta S_M}{\delta g_{mn}} \tag{30}$$

satisfies the ordinary conditions

$$\nabla_m T^{mn} = 0. \tag{31}$$

Here ∇ is the covariant derivative with respect to Christoffel's symbols Γ^i_{jk}.

By varying S_F with respect to g_{mn} we get the equations

$$\Psi^{mn}_F = 2\frac{\delta S_M}{\delta g_{mn}} = \tag{32}$$

$$= \sqrt{-g}\; g^{ma} g^{nb}(\breve{R}_{ab} + \breve{R}_{ba} - \breve{R}_{ik} g^{ik} g_{ab} - 2G_{ab} + 2\kappa T_{ab}) =$$

Let us consider the covariant derivative of (32)

$$\nabla_m \Psi^{mn}_F = 0. \tag{33}$$

As $\nabla^a(G_{ab} - \kappa\, T_{ab}) \equiv 0$, we get

$$g^{ab}\nabla_p S_{ab} = 0, \tag{34}$$

where $S_{ab} = \breve{R}_{ab} + \breve{R}_{ba} - \breve{R}_{pq} g^{pq} g_{ab}$.

The conditions (33) are necessary for equations (32) to be integrable. We will call these conditions the integrability conditions.

It is well-known that the Einstein equations (2) are degenerated in the sense that for ten unknown components of the metric tensor there are only six independent equations due to the Bianchi identities. It leads to the existence of the functional arbitrariness in their solutions. For avoiding this arbitrariness the suitable noncovariant conditions are used. Very popular are the "harmonicity conditions" $\partial_i(\sqrt{-g}\, g^{ij}) = 0$. In the covariant form they can be written as (7):

$$\breve{\nabla}_i \mathfrak{g}^{ij} = 0. \tag{35}$$

Here $\mathfrak{g}^{ij} = \sqrt{-g}\, g^{ij}$ is the contravariant metric density. It is (35) that we will call the harmonicity conditions.

If the background space is flat, then equations coincide with the Einstein one. The harmonicity conditions are not connected with the equations and must be postulated as external conditions. But if the background space is curved, then in general case the gravitational equations differ from the Einstein one. If these equations are not degenerated, then (35) must be either consequences of the (32) or inconsistent with them.

Now we show that the harmonicity conditions (35) are equivalent to the integrability equations (33) and, consequently, follow from equations (32) only if the background space is the Einstein space. More exactly, the following statements are true:

1. For (35) to be consequences of (32),

a) it is necessary that the symmetric part of the Ricci tensor should satisfy the condition $\breve{\nabla}_j \breve{R}_{(ik)} = 0$;

b) it is necessary and sufficient that $\breve{\nabla}_j \breve{R}_{(ik)} = 0$ and $\det(\breve{R}_{(ik)}) \neq 0$. In this case, (35) and (33) are equivalent.

2. If $\breve{\nabla}_j \breve{R}_{(ik)} = 0$ and $\det(\breve{R}_{(ik)}) = 0$, then (35) agrees with (32), but (32) without (35) remains degenerated. It means that

- the functional arbitrariness remains in the solutions;
- (35) can be postulated as external, but they are not consequences of (32).

The contents of 1 b) is equivalent to the statement that the background connection can be in agreement with a certain metric of the Einstein space. In other words, $\breve{R}_{(ij)} = \lambda\, \breve{g}_{ij}$ where $\lambda \neq 0$ and $\breve{g}_{ij}$ is a metric tensor with an *arbitrary* signature.

In [12] it has been shown that if the background space has a constant curvature then (33) coincides with (35). In [13] it has been found that the coincidence of (33) with (35) takes place if $\breve{\nabla}_j \breve{R}_{(ik)} = 0$ and $\det(\breve{R}_{(ik)}) \neq 0$. Let us investigate this question in the general case.

Now we consider the following term:

$$\tilde{\theta}^{mn}_a = \frac{\delta S_F}{\delta \breve{\Gamma}^a_{mn}}. \tag{36}$$

Let us denote

$$\tilde{\theta}_a = \breve{\nabla}_m \breve{\nabla}_n \tilde{\theta}^{mn}_a + \breve{R}^s_{amn} \tilde{\theta}^{mn}_s. \tag{37}$$

It can be shown that

$$\tilde{\theta}_a \equiv \nabla_m \Psi^m_a, \tag{38}$$

where $\Psi^m_a = g_{an}\Psi^{mn}$. Then, the integrability conditions can be written as

$$\tilde{\theta}_a = 0. \tag{39}$$

Let $\breve{\nabla}_i \mathfrak{g}^{ij}$ be denoted by the term Φ^j:

$$\Phi^j = \breve{\nabla}_i \mathfrak{g}^{ij}. \tag{40}$$

As it was shown in [13]

$$\tilde{\theta}_a = 2\breve{\nabla}_n(\mathfrak{g}^{ns}\breve{R}_{(sa)}) - \mathfrak{g}^{ns}\breve{\nabla}_a \breve{R}_{ns}. \tag{41}$$

Let us demand that the harmonicity conditions (35) be valid. Then, $\mathfrak{g}^{ns}$ in the first term in the right-hand side of (41) can be transferred through $\breve{\nabla}_n$ and (41) becomes

$$\mathfrak{g}^{ns}(\breve{\nabla}_n \breve{R}_{(as)} + \breve{\nabla}_s \breve{R}_{(an)} - \breve{\nabla}_a \breve{R}_{(ns)}) = 0. \tag{42}$$

This condition is a constraint on the components of $\mathfrak{g}^{ns}$. The harmonicity conditions (35) are the constraints on derivatives of $\mathfrak{g}^{ns}$. For the integrability conditions (33), not limited $\mathfrak{g}^{ns}$ by any other restrictions besides the harmonicity conditions (35) it is necessary that (42) be true for *arbitrary* g^{ns}. Hence, the necessary condition for (35) and (33) to be not contradictory is

$$\check{\nabla}_n \check{R}_{(as)} + \check{\nabla}_s \check{R}_{(an)} - \check{\nabla}_a \check{R}_{(ns)} = 0. \qquad (43)$$

It is evident that (43) is equivalent to

$$\check{\nabla}_a \check{R}_{(ns)} = 0. \qquad (44)$$

Then, the statement 1 a) is proved.

The right-hand side of (41) can be transformed by means of the following number of identities:

$$2\check{\nabla}_n (g^{ns} \check{R}_{(sa)}) - g^{ns} \check{\nabla}_a \check{R}_{ns} =$$

$$= 2g^{ns} \check{\nabla}_n \check{R}_{(sa)} + 2\check{R}_{(sa)} \check{\nabla}_n g^{ns} - g^{ns} \check{\nabla}_a \check{R}_{ns} =$$

$$= 2\check{R}_{(sa)} \check{\nabla}_n g^{ns} + g^{ns} \check{\nabla}_n \check{R}_{(sa)} + g^{ns} \check{\nabla}_s \check{R}_{(na)} - g^{ns} \check{\nabla}_a \check{R}_{ns} =$$

$$= 2\check{R}_{(sa)} \check{\nabla}_n g^{ns} + g^{ns} (\check{\nabla}_n \check{R}_{(sa)} + \check{\nabla}_s \check{R}_{(na)} - \check{\nabla}_a \check{R}_{(ns)}).$$

By solving the covariant derivatives we can rewrite (39) as

$$\check{R}_{(as)} \check{\nabla}_n g^{ns} = g^{ns} (\check{R}_{(ap)} \check{\Gamma}^p_{ns} - \check{R}_{a,ns}), \qquad (45)$$

where

$$\check{R}_{a,ns} = \frac{1}{2} (\partial_n \check{R}_{(as)} + \partial_s \check{R}_{(an)} - \partial_a \check{R}_{(ns)}). \qquad (46)$$

If (44) holds, then

$$\check{R}_{a,ns} = \check{R}_{(ap)} \check{\Gamma}^p_{ns}, \qquad (47)$$

and, as it is clear from (45), the integrability conditions take the form

$$\check{R}_{(as)} \Phi^s = 0. \qquad (48)$$

Now we can see that if $\det(\check{R}_{(as)}) = 0$, then in fact the number of constrains on g^{ns} in (45) and, consequently, in (33) is smaller than the number of constraints in conditions (35). But (35) does not contradict (33).

In sect.7 we shall show that in this case equations remain degenerated and it is necessary just $4 - \mathrm{rank}(\check{R}_{(as)})$ conditions on g_{ns} for avoiding this degenerating. Hence, it is the harmonicity conditions (35) that can be assumed as the conditions for fixation of the solution of (32).

If $\det(\check{R}_{(as)}) \neq 0$, then from (47) we obtain

$$\check{\Gamma}^p_{ns} = \check{R}^p_{ns}, \qquad (49)$$

where $\check{R}^p_{ns} = \bar{R}^{(pa)} \check{R}_{a,ns}$. Here $\bar{R}^{(pa)} \check{R}_{(aq)} = \delta^p_q$.

It is clear that if we put $\check{R}'_{(pa)} = \alpha \check{R}_{(pa)}$, where $\alpha \neq 0$ then

$$\check{R}^{p\prime}_{(ns)} = \frac{1}{\alpha} \bar{R}^{(pa)} \alpha \check{R}_{a,ns} = \check{R}^p_{(ns)}.$$

Let us consider background metric $\check{g}_{mn} = \frac{1}{\lambda} \check{R}_{(mn)}$. Since $\det(\check{R}_{(mn)}) \neq 0$, $\check{g}^{mn} = \lambda \bar{R}^{(mn)}$, and $\check{g}_{mn}$ is in agreement with $\check{\Gamma}^i_{jk}$ so that $\check{R}_{(mn)} = \lambda \check{g}_{mn}$; $\check{R}_{[mn]} = 0$. It means that the background space is the Einstein space with the metric $\check{g}_{mn}$. But its signature is indefinite and may be arbitrary.

From (45) we can observe the specific role of the harmonicity conditions: It is such term that presents derivatives of g_{ij} in the integrability conditions.

4. The generally covariant representation of the energy-momentum pseudotensor of the gravitational field

A number of pseudotensor objects have been proposed for determining the gravitational energy [14, 15]. Their tensor representations can be found by this method [5, 16]. The geometrical meaning of pseudotensors is that these objects are the tensor functionals of the background connection.

For constructing local gravitational characteristics we can use the method of Lagrangians with covariant derivatives [17] This method consist in changing partial derivatives ∂_i by covariant $\check{\nabla}_i$ ones. But we shall use a more general approach.

Let us consider a general Lagrangian

$$L = L(g_{mn};\ \partial_k g_{mn};\ \check{\Gamma}^k_{mn}). \qquad (50)$$

It is supposed that the background connection is symmetric. Let the following terms be defined as

$$t^k_a = \frac{\partial L}{\partial g_{mn,k}} \check{\nabla}_a g_{mn} - L\, \delta^k_a; \qquad (51)$$

$$\sigma^{jk}_a = \frac{\partial L}{\partial g_{mn,j}} (g_{ma} \delta^k_n + g_{na} \delta^k_m); \qquad (52)$$

$$\Psi^{mn} = 2 \frac{\delta S}{\delta g_{mn}} = 2\left(\frac{\partial L}{\partial g_{mn}} - \partial_j \frac{\partial L}{\partial g_{mn,j}}\right); \qquad (53)$$

$$\Theta^{mn}_k = \frac{\delta S}{\delta \check{\Gamma}^k_{mn}} = \frac{\partial L}{\partial \check{\Gamma}^k_{mn}}\ ; \qquad (54)$$

where

$$S = \int L\, d^4x \qquad (55)$$

is the action functional; comma before index means the partial derivative.

All the terms introduced are the tensor densities of weight one. The tensor $(1/\sqrt{-g})\, t^k_a$ may be assumed as a canonical energy-momentum one,

σ^{jk}_a is connected with the gravitational spin density, the term $(1/\sqrt{-g})\Theta^{mn}_k$ also appeared in literature [18].

The number of identities can be proved by the variational method [19], in particular,

$$t^k_a + \check{\nabla}_j \sigma^{jk}_a + \Psi^{km} g_{ma} = 0, \tag{56}$$

$$-\check{\nabla}_k t^k_a = \Theta^{mn}_k \check{R}^k_{amn} + (1/2)\sigma^{mn}_k \check{R}^k_{mna} + (1/2)\Psi^{mn}\check{\nabla}_a g_{mn}, \tag{57}$$

$$\sigma^{pk}_m = -\Theta^{pk}_m + g^{pa} g_{lm} \Theta_a - g^{ka} g_{lm} \Theta_a \tag{58}$$

and already known expression (37) - (38)

$$\nabla_m \Psi^m_a = \check{\nabla}_m \check{\nabla}_n \Theta^{mn}_a + \check{R}^s_{amn} \Theta^{mn}_s. \tag{59}$$

The terms (51) - (54) are defined for the general Lagrangian (50). Let the terms corresponding to the concrete Lagrangian (12) be marked by the tilde ~ above the letter. After simple calculations we get the known relation (14)

$$\tilde{\Psi}^{mn} = \sqrt{-g}\, g^{ma} g^{nb} (\check{R}_{ab} + \check{R}_{ba} - \check{R}_{ij} g^{ij} g_{ab} - 2G_{ab})$$

and

$$\tilde{\Theta}^{pk}_m = \check{\nabla}_s \left(\sqrt{-g}\,(g^{k[p}\delta^{s]}_m + g^{p[k}\delta^{s]}_m)\right). \tag{60}$$

Substituting (60) into (58) one can get

$$\tilde{\sigma}^{pk}_m = (g_{am}/\sqrt{-g})\check{\nabla}_s U^{sapk} - 2\check{\nabla}_s(\sqrt{-g}\, g^{k[p}\delta^{s]}_m). \tag{61}$$

Here $U^{sapk} = (-g)(g^{sp}g^{ak} - g^{sk}g^{ap})$.

Substituting (61) into (56) we get

$$-(\tilde{t}^k_m + \tilde{\Psi}^{ka} g_{am}) = \check{\nabla}_p \left(\frac{g_{am}}{\sqrt{-g}} \check{\nabla}_s U^{sapk} \right) - 2\check{\nabla}_p \check{\nabla}_s (\sqrt{-g}\, g^{k[p}\delta^{s]}_m). \tag{62}$$

If $\check{R}^i_{klm} = 0$, then the coordinate map, in which $\check{\Gamma}^k_{mn} = 0$, can be chosen. In this map we have $\tilde{\Psi}^{mk} = -2\sqrt{-g}\, G^{mk}$. The covariant derivatives convert into partial derivatives, and the last term in (62) disappears. Then, (62) takes the form

$$\sqrt{-g}\left(\frac{1}{\sqrt{-g}} \tilde{t}^k_m - 2G^k_m \right) = \partial_p \left(\frac{g_{am}}{\sqrt{-g}} \partial_s \left((-g)(g^{pa}g^{ks} - g^{sp}g^{ka}) \right) \right).$$

In the right-hand side we see the derivative of the well-known Einstein superpotential. Hence, we can conclude that (62) is the covariant generalization of usual splitting of G^k_m which is often used in pseudotensor approach for determining the Einstein pseudotensor. Consequently, it is $(1/\sqrt{-g})\tilde{t}^k_a$ that corresponds to the Einstein pseudotensor.

5. The Conservation Laws

If the background object is a metric, then the solution of the problem of integral conservation laws is well known. The metric energy-

momentum tensor satisfies the local conservation law, and if the Killing vector of the background metric is present, then the integral conservation law can be obtained by integrating the local conservation law. In the present case, the metric energy-momentum tensor cannot be defined because the background metric is absent. Moreover, in the general case the divergence of the canonical energy-momentum tensor is not equal to zero (see (57)). Therefore, we use the Noether algorithm.

The general form of the action variation can be written as [19]

$$\delta S = \int \left[\frac{\partial L}{\partial \check{\Gamma}^k_{mn}} \delta\check{\Gamma}^k_{mn} + \frac{\delta S}{\delta g_{mn}} \delta g_{mn} + \partial_j \left(\frac{\partial L}{\partial g_{mn,j}} \delta g_{mn} + L\delta x^j \right) \right] d^4x. \tag{63}$$

Now we substitute the definitions (51) - (54) into (63) and demand that δS should vanish under Lie variations which can be written in the form

$$\delta x^j = \varepsilon\xi^j; \tag{64}$$

$$\delta\check{\Gamma}^k_{mn} = -(\check{\nabla}_m \check{\nabla}_n (\varepsilon\xi^k) + \check{R}^k_{amn} \varepsilon\xi^a); \tag{65}$$

$$\delta g_{mn} = -(g_{ma}\check{\nabla}_n(\varepsilon\xi^a) + g_{na}\check{\nabla}_m(\varepsilon\xi^a) + \varepsilon\xi^a \check{\nabla}_a g_{mn}) \tag{66}$$

where ε is an infinitesimal parameter, ξ is an arbitrary vector field. Then, we obtain

$$\int \Big[\Theta^{mn}_k \check{\nabla}_m \check{\nabla}_n (\varepsilon\xi^k) + \Theta^{mn}_k \check{R}^k_{amn} \varepsilon\xi^a + (1/2)\varepsilon\xi^a \Psi^{mn}\check{\nabla}_a g_{mn} + \Psi^{mn} g_{ma} \check{\nabla}_n(\varepsilon\xi^a) + \partial_j(\sigma^{jk}_a \check{\nabla}_k(\varepsilon\xi^a) + t^j_a \varepsilon\xi^a) \Big] d^4x = 0. \tag{67}$$

Since the range of integration is arbitrary, it follows from (67) that the integrand is zero. By reduction of ε one can obtain

$$\partial_j(\sigma^{jk}_a \check{\nabla}_k \xi^a + t^j_a \xi^a) + \Psi^{mn}((1/2)\xi^a \check{\nabla}_a g_{mn} + g_{ma}\check{\nabla}_n \xi^a) = -\Theta^{mn}_k (\check{\nabla}_m \check{\nabla}_n \xi^k + \check{R}^k_{amn} \xi^a). \tag{68}$$

The action invariance under (64) - (66) is a consequence of the general covariance. But it is not the dynamic invariance because (65) is the transformation of the nondynamical object.

Let the background connection permit the r-parameter group of motion G_r, and let $\xi^j_{(\lambda)}$, $\lambda = 1, \dots, r$, generate this group, i.e., the equations

$$\check{\nabla}_m \check{\nabla}_n \xi^k_{(\lambda)} + \check{R}^k_{amn} \xi^a_{(\lambda)} = 0 \tag{69}$$

be satisfied. Then, infinitesimal transformations of G_r are

$$\delta x^j = \varepsilon^{(\lambda)} \xi^j_{(\lambda)}; \tag{70}$$

$$\delta g_{mn} = -(g_{ma}\check{\nabla}_n(\varepsilon^{(\lambda)}\xi^a_{(\lambda)}) + g_{na}\check{\nabla}_m(\varepsilon^{(\lambda)}\xi^a_{(\lambda)}) + \varepsilon^{(\lambda)}\xi^a_{(\lambda)}\check{\nabla}_a g_{mn}). \tag{71}$$

Under this condition the right-hand side of (68) vanishes. Using (69) we write down (68) in the form corresponding to the first Noether theorem [20]:

$$\partial_j J^j_{(\lambda)} = X_{mn(\lambda)} \Psi^{mn}, \qquad (72)$$

where the Noether currents

$$J^j_{(\lambda)} = \sigma^{jk}_a \breve{\nabla}_k \xi^a_{(\lambda)} + t^j_a \xi^a_{(\lambda)}, \qquad (73)$$

and generators

$$X_{mn(\lambda)} = - \frac{1}{2} \xi^a_{(\lambda)} \breve{\nabla}_a g_{mn} - g_{ma} \breve{\nabla}_n \xi^a_{(\lambda)}. \qquad (74)$$

Then, the energy-momentum problem seems to be solved. Indeed, let us consider the integral

$$A = \int J^j dS_j, \qquad (75)$$

where the integration is over any infinite hypersurface including the whole three-space. Relation (72) means that A is conserved if the equation of motion holds. Formula (72) is invariant because J^j is a vector density of weight one.

However, a more careful analysis shows that the problem is still unsolved. For the concrete Lagrangian $\breve{L}$ defined by (12) the conservation laws following from (72) appear to be trivial if (19) holds.

6. Degeneration of Equations and the Structure of Conserved Currents

Let us consider a system described by the fields φ^A where A is the collective index. Let equations for φ^A follow from the condition of the action functional

$$S = \int L \, d^4x, \qquad (76)$$

where L is the Lagrangian, being stationary.

The statement known as the first Noether theorem was formulated in the first section of the famous Noether paper [20]: If the action is invariant under the r-parameter Lie group G_r, then r linearly independent combinations of the variational derivatives turn into divergences, i.e.

$$\partial_j J^j_{(\lambda)} = \sum_A \Psi_A X^A_{(\lambda)}, \quad \lambda = 1,\dots,r, \qquad (77)$$

where $J^j_{(\lambda)}$ are expressions named the Noether currents, $\Psi_A = \dfrac{\delta S}{\delta \varphi^A}$ are variational derivatives, $X^A_{(\lambda)}$ are the representation generators corresponding to the transformations of φ^A under G_r.

Let the action (76) be invariant under a continuous group which may be parameterized by p arbitrary function of the coordinates. We shall denote this group by $G_{p\infty}$. If one singles out a subgroup G_r from the group $G_{p\infty}$, then according to the first Noether theorem, r local conservation laws will takes place.

In Sect.6 of paper [20] it has been formulated and proved that if G_r is a subgroup of the group $G_{r\infty}$, all currents $J^j_{(\lambda)}$ may be represented in the form

$$J^j_{(\lambda)} = A^j_{(\lambda)} + B^j_{(\lambda)}, \qquad (78)$$

where $A^j_{(\lambda)} = 0$ if $\Psi_A = 0$, and $B^j_{(\lambda)}$ satisfies the condition $\partial_j B^j_{(\lambda)} \equiv 0$. Such currents were named the improper currents [9, 20].

And vice versa, if the Noether current is improper, then the corresponding group G_r must be a subgroup of a certain group $G_{p\infty}$. According to the second Noether theorem [9, 20], there are p identities among the equations $\Psi_A = 0$ and these equations are degenerated. This fact was known since Noether. But if we want to extend G_r to $G_{p\infty}$, it may appear that such an extension is not simple. In sect.8, we shall construct such an extension for the gravitational action.

Now we return to (72). It is easy to verify that the right-hand side of (72) can be presented if the form

$$\Psi^{mn} X_{mn(\lambda)} = \xi^a_{(\lambda)} \nabla_k \Psi^k_a - \breve{\nabla}_k (\Psi^k_a \xi^a_{(\lambda)}). \qquad (79)$$

But if the equations

$$\Psi^{mn} = 0 \qquad (80)$$

coincide with the Einstein equations, then

$$\nabla_k \Psi^k_a \equiv 0 \qquad (81)$$

because of the Bianchi identities. If $\breve{L}$ is used as L and (19) holds, then (72) is

$$\partial_j \breve{J}^j_{(\lambda)} = - \breve{\nabla}_k (\breve{\Psi}^k_a \xi^a_{(\lambda)}) = \partial_j (- \breve{\Psi}^j \xi^a_{(\lambda)}). \qquad (82)$$

It means that $\breve{J}^j_{(\lambda)} = - \breve{\Psi}^j_a \xi^a_{(\lambda)} + B^j$ where $\partial_j B^j \equiv 0$. Hence, it follows that the current $J^j_{(\lambda)}$ can be presented in the form of (78) and, therefore, it is improper.

It is a well-known fact. The Einstein equations are invariant under the group of diffeomorphisms and degenerated. Of more interest is the case when $\breve{R}_{(ij)} \neq 0$. What conditions are to be imposed on the background connection for equations to be degenerated?

7. Conditions for the Equations to be Degenerate

Let the action be invariant under the group generated by the following infinitesimal transformations of the dynamical fields

$g_{mn} \longrightarrow g_{mn} + \delta_K g_{mn}$

$$\delta_K g_{mn} = -\left(g_{ma} \breve{\nabla}_n (\delta\nu^{(\lambda)} \xi^a_{(\lambda)}) + g_{na} \breve{\nabla}_m (\delta\nu^{(\lambda)} \xi^a_{(\lambda)}) + \delta\nu^{(\lambda)} \xi^a_{(\lambda)} \breve{\nabla}_a g_{mn}\right) \qquad (83)$$
$$\lambda = 1,\dots,r,$$

where $\delta\nu^{(\lambda)}$ are arbitrary infinitesimal functions of the coordinates vanishing at the integration limits. These variations are similar to the gauge transformations [8]. As $\delta\nu^{(\lambda)}$ are demanded to be vanishing at the boundary of the range of integration, no surface terms are produced and (83) corresponds to the infinitesimal diffeomorphism of g_{mn} under the vector field $\delta\nu^{(\lambda)}\xi^a_{(\lambda)}$. Then,

$$\delta_K S = \int \frac{\delta S}{\delta g_{mn}} \delta_K g_{mn} \, d^4x = 0. \qquad (84)$$

Now we substitute the terms of $\delta_K g_{mn}$ into this formula. Then, we obtain

$$\int \Psi^{mn}(-\frac{1}{2} \delta\nu^{(\lambda)}\xi^s_{(\lambda)}\check\nabla_s g_{mn} - g_{ms}\check\nabla_n(\delta\nu^{(\lambda)}\xi^s_{(\lambda)})) \, d^4x = 0. \qquad (85)$$

To transform the integrand we shall use the identity that can be easily verified

$$\Psi^{mn}(-\frac{1}{2} \delta\nu^{(\lambda)}\xi^s_{(\lambda)}\check\nabla_s g_{mn} - g_{ms}\check\nabla_n(\delta\nu^{(\lambda)}\xi^s_{(\lambda)})) = \delta\nu^{(\lambda)}\xi^n_{(\lambda)}\nabla_m\Psi^m_n - \check\nabla_m(\Psi^m_n\,\delta\nu^{(\lambda)}\xi^n_{(\lambda)}), \qquad (86)$$

Since Ψ^k_a is a tensor density of weight one, the last term in the right - hand side of (86) is an ordinary divergence and it may be discarded because $\delta\nu^{(\lambda)} = 0$ at the integration limits. Then we are left only with

$$\int \delta\nu^{(\lambda)}\xi^j_{(\lambda)}\nabla_a\Psi^a_j \, d^4x = 0. \qquad (87)$$

Since $\delta\nu^{(\lambda)}$ is arbitrary, it follows from (87) that

$$\xi^j_{(\lambda)}\nabla_a\Psi^a_j = 0. \qquad (88)$$

It means that there are r identities among the equations (80). These identities can be symbolically written down as

$$\int \Psi^{mn}(x') \, \Lambda_{mn(\lambda)}(x'x) \, d^4x' = 0, \qquad (89)$$

where $\Lambda_{mn(\lambda)}$ are generators

$$\Lambda_{mn(\lambda)}(x',x) = - \xi_{(\lambda)(m}(x) \, \nabla_{n)}\delta(x' - x). \qquad (90)$$

Here $\xi_{(\lambda)m} = g_{ma}\,\xi^a_{(\lambda)}$; $\nabla_n\delta(x' - x)$ is a covariant derivative of the four - dimensional δ - function with respect to x'.

As we can see, the conditions of degenerating of equations coincide with the conditions for the covector (38) to have a null vector η:

$$\eta^j\Theta_j = \eta^j\nabla_a\Psi^a_j = 0. \qquad (91)$$

The question concerning the degree of degeneration will be clear from the next section.

For the investigation of the Lagrangian (12) let us recall sect.3. Let us write (45) in the form

$$\frac{1}{2}\Theta_a = \check R_{(as)}\check\nabla_n g^{ns} - \frac{1}{2} g^{ns}(\check\nabla_n\check R_{(sa)} + \check\nabla_s\check R_{(na)} - \check\nabla_a\check R_{(ns)}) = 0. \qquad (92)$$

Using (92) we can rewrite (91) as

$$\eta^a\check R_{(as)}\check\nabla_n g^{ns} = \frac{1}{2}\eta^a g^{ns}(\check\nabla_n\check R_{(sa)} + \check\nabla_s\check R_{(na)} - \check\nabla_a\check R_{(ns)}). \qquad (93)$$

For (93) to hold for arbitrary g^{ns}, the coefficients of g^{ns} and $\check\nabla_n g^{ns}$ must be zero independently of one another. We have

$$\eta^a(\check\nabla_n\check R_{(sa)} + \check\nabla_s\check R_{(na)} - \check\nabla_a\check R_{(ns)}) = 0, \qquad (94)$$

$$\eta^a\check R_{(as)} = 0. \qquad (95)$$

By virtue of (95) we can conclude from (94) that

$$- (\check R_{(sa)}\check\nabla_n\eta^a + \check R_{(na)}\check\nabla_s\eta^a + \eta^a\check\nabla_a\check R_{(ns)}) = 0.$$

This expression can easily be recognized as the Lie derivative:

$$\underset{\eta}{\pounds}\,\check R_{(sa)} = 0. \qquad (96)$$

Thus, the conditions for equations (15) to be degenerate are that the symmetric part of the Ricci tensor of the background connection $\check R_{(ij)}$ should have such a null vector η (95) that the Lie derivative of $\check R_{(ij)}$ should vanish (96).

Notice that if η satisfies (65), then (96) is true automatically.

It is clear that the property (81) $\nabla_m\tilde\Psi^m_a \equiv 0$ singles out the Einstein equations among others derived from (12) because (81) means that (95) must be true for arbitrary η^a and, therefore, $\check R_{(ij)} = 0$.

8. CONSTRUCTING OF EXTENSION OF THE SYMMETRY GROUP WHEN THE CONSERVED CURRENT IS IMPROPER

Let the action be invariant under $G_{p\infty}$. We can always single out the subgroup G_r from $G_{p\infty}$ by splitting the $G_{p\infty}$ group parameters into factors independent of the coordinates and functions remained fixed. For example, if the group parameter of $G_{p\infty}$ is $\delta\nu(x)$, we can consider only such $\delta\nu$ that $\delta\nu(x) = \varepsilon f(x)$, where ε is the G_r group parameter and f(x) is fixed. According to the first Noether theorem there are r conserved currents and since G_r is a subgroup of $G_{p\infty}$, these currents are improper. A more complicated problem consists in building the group $G_{p\infty}$ from G_r if the conserved currents corresponding to G_r are improper. It may occur that the group $G_{p\infty}$ derived acts on the action functional in a somewhat different way than G_r. It is necessary that the action of $G_{p\infty}$ on *the dynamical variables only* would be the same as G_r, but the action of the group $G_{p\infty}$ on independent variables x^j may differ from G_r. As we shall see, in our case $G_{p\infty}$ will not act on x^j.

In other words, there is a group G_Σ, including groups the G_r and $G_{p\infty}$ as subgroups, such that G_r transformations of the dynamical variables must be derived from $G_{p\infty}$ ones. In more detail, this group is the following.

When the group G_r acts on the action, the dynamical fields φ^A and coordinates are transformed by a certain rule:

$$\varphi^A \longrightarrow \bar{\varphi}^A(\bar{x}) = \omega_{\varphi}[\varepsilon^{(\lambda)}]\left(\varphi^A(x)\right); \qquad (97)$$

$$x^j \longrightarrow \bar{x}^j = \omega_x[\varepsilon^{(\lambda)}]\left(x\right); \qquad \lambda = 1,\dots,r,$$

where $\varepsilon^{(\lambda)}$ are the G_r-group parameters independent of the coordinates. It may happen that there is a group $G_{p\infty}$ transforming the dynamical fields $\bar{\varphi}^A$ according to

$$\varphi^A \longrightarrow \bar{\varphi}^A = \Omega[\xi^{(\gamma)}]\left(\varphi^A(x)\right); \ \gamma = 1,\dots,p, \qquad (98)$$

where $\xi^{(\gamma)}(x)$ are the group parameters and

$$x^j \longrightarrow \bar{x}^j = x^j.$$

We are interested in the case when $\xi^{(\gamma)}$ can be presented as

$$\xi^{(\gamma)}(x) = \varepsilon^{(\lambda)}\eta^{(\gamma)}_{(\lambda)}(x) \equiv \sum_{\lambda=1}^{r}\varepsilon^{(\lambda)}\eta^{(\gamma)}_{(\lambda)}(x) \qquad (99)$$

where $\varepsilon^{(\lambda)}$ are independent of the coordinates and $\eta^{(\gamma)}_{(\lambda)}(x)$ is the set of functions *fixed* so that

$$\Omega[\varepsilon^{(\lambda)}\eta^{(\gamma)}_{(\lambda)}]\left(\varphi^A(x)\right) = \omega_{\varphi}[\varepsilon^{(\lambda)}]\left(\varphi^A(x)\right). \qquad (100)$$

If the action is invariant under both G_r and $G_{p\infty}$, then there must be a group G_Σ that preserved the action invariant and includes G_r and $G_{p\infty}$ as a subgroups. Further, we shall see that G_Σ is a semidirect product $G_r \underset{\alpha}{\times} G_{p\infty}$ where α is the set of internal automorphisms of $G_{p\infty}$.

A general structure of the Noether current is as follows:

$$J^j_{(\lambda)} = A^j_{(\lambda)} + B^j_{(\lambda)} + C^j_{(\lambda)}, \qquad (101)$$

where $A^j_{(\lambda)}$ and $B^j_{(\lambda)}$ have been defined earlier and $C^j_{(\lambda)}$ is a proper component of the current. There are certain difficulties in determining $C^j_{(\lambda)}$ because in principle $A^j_{(\lambda)}$ and $B^j_{(\lambda)}$ can be included in $C^j_{(\lambda)}$, but $C^j_{(\lambda)}$ may be characterized by the property that it cannot be decomposed only into $A^j_{(\lambda)}$ and $B^j_{(\lambda)}$.

The field charge is a first integral of equations of motion corresponding to a one - parameter group of the action invariance. As it appears from (101), if $C^j_{(\lambda)} = 0$, i.e., if the current is improper, then the formally calculated charge becomes trivial since it does not depend on equations of motion.

For the nontrivial charge to exist it is necessary for the term $C^j_{(\lambda)}$ to be present. It is a term that does not permit the group of the dynamical invariance to be extended to the infinite - parameter one. More exactly, the following statement can be valid:

If $C^j_{(\lambda)} = 0$, then the group G_r can be included in the group G_Σ as a subgroup of the action invariance. If $C^j_{(\lambda)} \neq 0$, then the group G_r cannot be included in the group G_Σ as a subgroup of the action invariance.

For our case, the last statement has been proved in the previous section. Now we will investigate the consequences of the currents being improper.

From the previous section we can conclude that the C-term is the term $\xi^a_{(\lambda)}\check{\nabla}_m\Psi^m_n$. If the current is improper, then (88) holds and (89) takes place.

Let us consider the infinitesimal transformations

$$\delta_\Lambda g_{mn}(x) = \int \Lambda_{mn(\lambda)}(x,x')\ \delta\nu^{(\lambda)}(x')\ d^4x', \qquad (102)$$

where $\delta\nu^{(\lambda)}$ are arbitrary infinitesimal functions of coordinates vanishing at the boundary of the range of integration and $\Lambda_{mn(\lambda)}$ have been defined in the previous section. Let us substitute (90) into (102) and perform the integration. Then, we obtain

$$\delta_\Lambda g_{mn} = -\frac{1}{2}\left(g_{ma}\nabla_n(\delta\nu^{(\lambda)}\xi^a_{(\lambda)}) + g_{na}\nabla_m(\delta\nu^{(\lambda)}\xi^a_{(\lambda)})\right). \qquad (103)$$

Notice that (103) up to a factor coincides with (83).

Now we shall find the action variation

$$\delta_\Lambda S = \int \frac{1}{2}\ \Psi^{mn}\ \delta_\Lambda g_{mn} d^4x. \qquad (104)$$

If we substitute (102) into (104) and recall that $\delta\nu$ vanishes at the integrating limits, then using (89) we get

$$\delta_\Lambda S = \int \delta\nu^{(\lambda)}(x')\ d^4x' \int \frac{1}{2}\ \Psi^{mn}(x)\ \Lambda_{mn(\lambda)}(x,x')\ d^4x\ = 0.$$

It means that the action is invariant with respect to the group generated by (102). But generators (90) are not independent, and not all the parameters $\delta\nu^{(\lambda)}$ are essential. For the generators to be independent, the system of equations

$$\int \Lambda_{mn(\lambda)}(x,x')\ \delta\nu^{(\lambda)}(x')\ d^4x' = 0 \qquad (105)$$

must have a single solution $\delta\nu^{(\lambda)} = 0$ for *arbitrary* g_{mn}.

If we substitute the definition (90) into (105) and perform integration, we obtain

$$\nabla_{(n}(\xi_{(\lambda)m)}\delta\nu^{(\lambda)}) = 0. \qquad (106)$$

This is the Killing equation. For *arbitrary* g_{ij} the single solution of (106) is $\delta\nu^{(\lambda)}\xi_{(\lambda)m} = 0$. But because $\xi_{(\lambda)m}$ are not independent *at every point*, we cannot conclude that $\delta\nu^{(\lambda)} = 0$ is a single solution of (106).

It is clear that the left - hand side of (106) up to a sign coincides with the right - hand side of (103). Consequently, the condition that all parameters in (103) are essential coincides with the condition that the solution $\delta\nu^{(\lambda)} = 0$ of the system (106) is single.

Let us consider an arbitrary point M within the range of integration. Let the orbit of the point M, i.e. the set of the points of the area which can be transferred to the point M by the transformations of the group G_r, be denoted by the term Q_M. Let among the vector fields $\xi_{(\lambda)}$ there be exactly m fields which are zero fields at M. It can be assumed without loss of generality that the zero fields are $\xi_{(\rho)}$, $\rho = 1, \dots, m$. It means that $\xi_{(\rho)}$ form the Lie algebra of the stability subgroup of the point M. In differential geometry the stability subgroup is more often called the group of isotropy of M. Let us denote this group by the symbol H_M.

So, the vector fields $\xi_{(\gamma)}$, $\gamma = m+1, \dots, r$, are not equal to zero in M. It should be remarked that these fields do not generally form the Lie algebra. Let us prove that in a certain neighborhood of the point M they form a set of basis fields of the orbit Q_M.

Indeed, according to the Frobenious theorem [21], the integral curves of the fields $\xi_{(\lambda)}$ compose a family of submanifolds of the initial manifold because they form the Lie algebra. Each of the points of the initial manifold belongs to one of these submanifolds which are the orbits of these points. A linear envelope spanned over $\xi_{(\lambda)}$ at the point M is a tangent space for the Q_M. Let it be denoted by T_M. Since $\xi_{(\rho)}|_M = 0$, then T_M coincides with the linear envelope of $\xi_{(\gamma)}|_M$. Then, Q_M is homogeneous under the action of the G_r by definition. Therefore, Q_M is isomorphic to G_r/H_M which is a factor space of the group of motion to the group of isotropy. Hence, $dim\, Q_M = dim\, G_r - dim\, H_M = r - m$. Let us denote $r - m = p$. Consequently, the dimension of the linear envelope $\xi_{(\gamma)}|_M$ is equal to the number of the vectors $\xi_{(\gamma)}$, therefore, these vectors in M form a basis set of T_M.

The vector fields $\xi_{(\gamma)}$ are assumed to be differentiable, therefore there is a certain neighborhood U_M of the point M in which these fields remain linearly independent, and since their integral curves completely belong to Q_M, then in U_M the vector fields $\xi_{(\gamma)}$ form a basis set of Q_M.

Let us consider a vector field $\eta^m = \xi^m_{(\gamma)}\delta\nu^{(\gamma)}$. In U_M an arbitrary vector field tangent to Q_M can be decomposed over the fields $\xi_{(\gamma)}$ with a certain variable coefficients. Consequently, in a neighborhood of M any

a priori given tangent to the Q_M vector field can be obtained from η^m by a suitable choice of $\delta\nu^{(\gamma)}$. It means, the generator of an arbitrary diffeomorphism of Q_M at the point M has the form $\xi_{(\gamma)}\delta\nu^{(\gamma)}$.

Now we return to (106). It has been shown that in the neighborhood of M an arbitrary, tangent to Q_M, vector field can be decomposed over the fields $\xi_{(\gamma)}$. The field $\xi_{(\lambda)}\delta\nu^{(\lambda)}$ for arbitrary $\delta\nu^{(\lambda)}$ is tangent to Q_M since all $\xi_{(\lambda)}$ are tangent to Q_M. Therefore, for any $\delta\nu^{(\lambda)}$, $\delta\overset{*}{\nu}{}^{(\gamma)}$ can be picked out such that in some neighborhood of M, $\xi_{(\lambda)}\delta\nu^{(\lambda)} = \xi_{(\gamma)}\delta\overset{*}{\nu}{}^{(\gamma)}$. Then (106) transforms to

$$\nabla_{(n}(\xi_{(\gamma)m)}\delta\overset{*}{\nu}{}^{(\gamma)}) = 0. \tag{107}$$

But (107) is the Killing equation for the covector field $\overset{*}{\eta}_m = \xi_{(\gamma)m}\delta\overset{*}{\nu}{}^{(\gamma)}$. As an arbitrary metric tensor has no Killing vectors, then the only solution of (107) is $\overset{*}{\eta} = 0$, and since now $\xi_{(\gamma)m}$ are linearly independent, we obtain that $\delta\overset{*}{\nu}{}^{(\gamma)} = 0$ is the single solution of (106).

Summarizing, we conclude that the group generated by the infinitesimal transformations (103) has $p = dim\, G_r - dim\, H_M$ essential parameters depending on coordinates. It is the group of metric transformations corresponding to arbitrary diffeomorphisms of the orbits. The diffeomorphism of the orbits is such a diffeomorphism of the whole manifold that the integral curves of the generating vector fields do not leave the orbits. It must be emphasized that the elements of the group $G_{p\infty}$ are not diffeomorphisms because they do not act on the coordinates. It is exactly the metric maps.

So, if $C^j_{(\lambda)} = 0$, then the group G_r can be extended to the group G_Σ including G_r as a subgroup. One can easily be convinced that $G_\Sigma = G_r \underset{\alpha}{\times} G_{p\infty}$ where α is the system of internal automorphisms of $G_{p\infty}$. The symbol $\underset{\alpha}{\times}$ means the semidirect product. Really, $G_{p\infty}$ is invariant in G_Σ, the Cartesian product $G_r \times G_{p\infty}$ is supplied by the multiplication with system of the automorphisms of $G_{p\infty}$ depending on G_r, and the groups G_r and $G_{p\infty}$ as subgroups of G_Σ can be intersected only in the unit element because $G_{p\infty}$ does not act on the coordinates.

The Cartesian product of G_r and $G_{p\infty}$ is a set of all ordered pairs (g, g_∞), where g is an element of G_r, and g_∞ is an element of $G_{p\infty}$. Let g_1 and g_2 be elements of G_r, and $g_{1\infty}$ and $g_{2\infty}$ be elements of $G_{p\infty}$. Action of the pair (g, g_∞) on $\{x^j;\ g_{mn}(x)\}$ consists in successive application of the operations g_∞ and g to $\{x^j;\ g_{mn}(x)\}$. Therefore, to obtain the group structure the Cartesian product $G_r \times G_{p\infty}$ must be supplied by the multiplication via the scheme

$$(g_1; g_{1\infty}) \underset{\alpha}{\times} (g_2;\ g_{2\infty}) = (g_1 g_2;\ g_{1\infty}^{g_2}\ g_{2\infty}), \tag{108}$$

where $g_{1\infty}^{g_2} = \alpha(g_{1\infty}) = g_2^{-1}\ g_{1\infty}\ g_2$ is the automorphism of $G_{p\infty}$ depending on g_2. It is just (108) that means the Cartesian product being a semidirect product.

The property of $G_{p\infty}$ expressed by formula (100) leads to that automorphism α is internal.

9. EXAMPLE OF THE EXACT SOLUTION

As an example let us consider the background connection which is the Christoffel connection derived from the background metric

$$d\check{s}^2 = \check{g}_{ij} dx^i dx^j = c^2 dt^2 - h_{\alpha\beta} dx^\alpha dx^\beta. \tag{109}$$

Latin indices run from 0 to 3, Greek indices denote spatial components 1, 2, 3. Let $h_{\alpha\beta}$ be static:

$$\frac{\partial h_{\alpha\beta}}{\partial t} \equiv 0. \tag{110}$$

Tanks to (110) there is a Killing vector

$$\xi = \frac{\partial}{\partial t} \tag{111}$$

and, consequently, ξ forms the Ricci collineation:

$$\underset{\xi}{\pounds}\ \check{R}_{ij} = 0. \tag{112}$$

In the component form

$$\xi^i = \{1,\ 0,\ 0,\ 0\}. \tag{113}$$

The background Ricci tensor is diagonal and

$$\check{R}_{00} = 0. \tag{114}$$

Hence, $\xi^i\ \check{R}_{ij} = 0$ and equations (15) are degenerated. The solutions of equations (15) must be invariant under the transformations

$$g_{mn}(x) \longrightarrow \bar{g}_{mn}(x) \tag{115}$$

generated by the diffeomorphism

$$x^j = x^j(\bar{x}) \tag{116}$$

where

$$x^\alpha = \bar{x}^\alpha;\quad x^0 = f(\bar{x}^i). \tag{117}$$

Here $f(\bar{x})$ is an arbitrary function of the coordinates $\bar{x}^i$ which satisfy $f \equiv \bar{x}^0$ outside of the compact area.

Let us find these transformations. From the general tensor transformation law we have

$$\bar{g}_{mn}(\bar{x}) = \frac{\partial x^a}{\partial \bar{x}^m}\ \frac{\partial x^b}{\partial \bar{x}^n}\ g_{ab}(x(\bar{x})) \tag{118}$$

where derivatives $\dfrac{\partial x^a}{\partial \bar{x}^m}$ are determined from (116). To obtain (115) we must replace $\bar{x}$ by x in the left-hand side of (118).

Now let us find $h_{\alpha\beta}$ such that never new degenerations would appear.

It is clear that $\det(\check{R}_{\alpha\beta})$ must be nonzero. Consequently, a suitable choice is the Einstein space. Since $h_{\alpha\beta}$ is a three-dimensional metric, this space is a space of constant curvature. A very simple space like that is the Lobachevsky space. That is why we consider the background metric

$$d\check{s}^2 = c^2 dt^2 - dr^2 - k^2 \sinh^2 \frac{r}{k}\ (d\theta^2 + \sin^2\theta\ d\varphi^2) \tag{119}$$

where k is a constant, and will search spherically symmetric static metric

$$ds^2 = V^2 dt^2 - F^2 dr^2 - H^2 (d\theta^2 + \sin^2\theta\ d\varphi^2) \tag{120}$$

satisfying equations (15) which now can be written in the form

$$R_{ij} = \check{R}_{ij}. \tag{121}$$

In (120) V, F, and H are the functions of r. We assume

$$x^0 = t,\ x^1 = r,\ x^2 = \theta,\ x^3 = \varphi.$$

The metric (119) has been considered in [22].

In our case, the alternative (2) from sect.3 takes place. But when we claim the metric be (120), the single "free" harmonicity condition $\Phi^0 = 0$ is satisfied automatically and the functional arbitrariness is absent. Indeed, from (118) we have the diagonal components

$$\bar{g}_{pp} = \frac{\partial x^a}{\partial \bar{x}^p}\ \frac{\partial x^b}{\partial \bar{x}^p}\ g_{ab} \tag{122}$$

For these components be independent on t it is necessary that

$$x^0 = A\bar{x}^0 + f(\bar{r}) \tag{123}$$

where A is a constant. But the nondiagonal term appears

$$\bar{g}_{01} = \bar{g}_{10} = \frac{\partial x^0}{\partial \bar{x}^1}\ \frac{\partial x^0}{\partial \bar{x}^0}\ g_{00}. \tag{124}$$

This term contradict (120). For it vanish, we should have $\dfrac{\partial x^0}{\partial \bar{x}^1} = 0$ and (123) should take the form

$$x^0 = A\bar{x}^0 + B \tag{125}$$

where B is a constant. Thus, the functional arbitrariness disappears and only a trivial linear (125) remains.

All nondiagonal components of both R_{ij} and $\check{R}_{ij}$ are zero, $R_{33} = R_{22}\sin^2\theta$ and $\check{R}_{33} = \check{R}_{22}\sin^2\theta$, and we are only left with three equations for three unknown functions V, F and H:

$$R_{00} = \frac{VV'}{F^2}\left(\frac{F'}{F} + \frac{2H'}{H} - \frac{V'}{V}\right) + \frac{d}{dr}\left(\frac{VV'}{F^2}\right) = 0, \tag{126}$$

$$R_{11} = -\frac{V''}{V} - \frac{2H''}{H} + \frac{F'}{F}\left(\frac{V'}{V} + \frac{2H'}{H}\right) = -\frac{2}{k^2}, \tag{127}$$

$$R_{22} = 1 - \frac{1}{VF}\frac{d}{dr}\left(V\frac{HH'}{F}\right) = -2\sinh^2(r/k). \tag{128}$$

Here $'$ means $\frac{d}{dr}$.

Now the harmonicity condition $\Phi^1 = 0$

$$\frac{d}{dr}\left(\frac{VH^2}{F}\right) = VF\,k\sinh(2r/k) \tag{129}$$

is a consequence of all equations (126)-(128). We use (129) and the combination

$$\frac{1}{2}H\left(R_{11} + \frac{F^2}{V^2}R_{00}\right) = H'\frac{(VF)'}{VF} - H''. \tag{130}$$

Instead of (126)-(128) we obtain

$$H'' - H/k^2 = H'\frac{(VF)'}{VF}, \tag{131}$$

$$\frac{d}{dr}\left(\frac{VH^2}{F}\right) = VF\,k\sinh(2r/k), \tag{132}$$

$$\frac{d}{dr}\left(V\frac{HH'}{F}\right) = VF\cosh(2r/k). \tag{133}$$

Let us denote

$$\frac{HH'}{F^2} = \alpha, \quad \frac{H^2}{F^2} = \beta, \quad \frac{H'}{H} = \gamma, \quad \frac{(VF)'}{VF} = w. \tag{134}$$

Using (134) we write down (131)-(133) in the form

$$\gamma' + \gamma^2 - \gamma w = 1/k^2, \tag{135}$$

$$\beta' + \beta w = k\sinh(2r/k), \tag{136}$$

$$\alpha' + \alpha w = \cosh(2r/k), \tag{137}$$

$$\alpha = \beta\gamma. \tag{138}$$

Let w be considered as a parameter. Equations (136) and (137) are the linear equations, and therefore, can be easily solved. But equation (135) is the Riccati equation. We can't find its general solution. If we substitute α and β into (138) and then substitute γ in terms w in (135), we obtain nonlinear integro-differential equation. This equation cannot be solved too.

But we are able to find a particular solution of our system corresponding to $w = 0$. Indeed, if $w = 0$, then (135) becomes the equation with separated variables. From (136), (137) we have

$$\alpha = (k/2)\left(\sinh(2r/k) + a/2\right), \tag{139}$$

$$\beta = (k^2/2)\left(\cosh(2r/k) + b/2\right), \tag{140}$$

where a and b are the constants of integration. By separating variables in (135) we get

$$\frac{d\gamma}{1/k^2 - \gamma^2} = dr. \tag{141}$$

By integrating (141) we obtain

$$(k/2)\,\ln\left|\frac{1+\gamma k}{1-\gamma k}\right| = r + r_0 \tag{142}$$

where r_0 is the constant of integration.

Let us denote

$$D = \exp(r_0/k). \tag{143}$$

From (142) we have two branches. The first is

$$\frac{\gamma k + 1}{\gamma k - 1} = D^2\exp(2r/k); \tag{144}$$

$$\gamma < -(1/k);\ \gamma > (1/k) \tag{145}$$

and the second is

$$\frac{1+\gamma k}{1-\gamma k} = D^2\exp(2r/k); \tag{146}$$

$$-(1/k) < \gamma < 1/k. \tag{147}$$

In terms of hyperbolic functions we have from (144)

$$\gamma_1 = (1/k)\tanh^{-1}\left((r + r_0)/k\right) \tag{148}$$

and from (146)

$$\gamma_2 = (1/k)\tanh\left((r + r_0)/k\right). \tag{149}$$

As we can see, (145) and (147) are satisfied. But we must satisfy (138). Substituting γ, α and β into (138) we obtain

$$\begin{aligned}\alpha_1 &= (k/2)\left(\sinh(2r/k) - \sinh(2r_0/k)\right),\\ \beta_1 &= (k/2)\left(\cosh(2r/k) - \cosh(2r_0/k)\right),\\ \gamma_1 &= (1/k)\tanh^{-1}\left((r + r_0)/k\right)\end{aligned} \tag{150}$$

for the first branch, and

$$\begin{aligned}\alpha_2 &= (k/2)\left(\sinh(2r/k) + \sinh(2r_0/k)\right),\\ \beta_2 &= (k/2)\left(\cosh(2r/k) + \cosh(2r_0/k)\right),\\ \gamma_2 &= (1/k)\tanh\left((r + r_0)/k\right)\end{aligned} \tag{151}$$

for the second. Using definitions (134) we can obtain

$$H_1 = P\sinh\frac{r+r_0}{k};\quad F_1^2 = \frac{P^2}{k^2}\,\frac{\sinh\frac{r+r_0}{k}}{\sinh\frac{r-r_0}{k}};\quad V_1^2 = \frac{Q^2}{F_1^2}, \tag{152}$$

$$H_2 = P\cosh\frac{r+r_0}{k};\quad F_2^2 = \frac{P^2}{k^2}\,\frac{\cosh\frac{r+r_0}{k}}{\cosh\frac{r-r_0}{k}};\quad V_2^2 = \frac{Q^2}{F_2^2} \tag{153}$$

where P and Q are the constants of integration. Therefore, the two branches of the solution are

$$ds_1^2 = \frac{k^2\sinh\frac{r-r_0}{k}}{P^2\sinh\frac{r+r_0}{k}}Q^2dt^2 - \frac{P^2\sinh\frac{r+r_0}{k}}{k^2\sinh\frac{r-r_0}{k}}dr^2 - P^2k^2\sinh^2\frac{r+r_0}{k}d\Omega^2,$$

$$ds_2^2 = \frac{k^2\cosh\frac{r-r_0}{k}}{P^2\cosh\frac{r+r_0}{k}}Q^2dt^2 - \frac{P^2\cosh\frac{r+r_0}{k}}{k^2\cosh\frac{r-r_0}{k}}dr^2 - P^2k^2\cosh^2\frac{r+r_0}{k}d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$. By demanding the metric g_{mn} to have the same asymptotic as $\breve{g}_{mn}$, we get $P^2 = k^2\exp(-2r_0/k)$, $Q^2 = c^2$. If we denote

$$\Lambda_1 = \exp(2r_0/k)\,\frac{\sinh\frac{r-r_0}{k}}{\sinh\frac{r+r_0}{k}},$$

$$\Lambda_2 = \exp(2r_0/k)\,\frac{\cosh\frac{r-r_0}{k}}{\cosh\frac{r+r_0}{k}},$$

two solutions of (131)-(133) can be written

$$ds_1^2 = \Lambda_1c^2dt^2 - \Lambda_1^{-1}dr^2 - \exp(-2r_0/k)\,k^2\sinh^2\frac{r+r_0}{k}d\Omega^2, \tag{154}$$

$$ds_2^2 = \Lambda_2c^2dt^2 - \Lambda_2^{-1}dr^2 - \exp(-2r_0/k)\,k^2\cosh^2\frac{r+r_0}{k}d\Omega^2. \tag{155}$$

The metric (154) has at first been found in [22].

For determining r_0 we postulate that asymptotic behavior of g_{mn} must get the Newton gravitational law in the Lobachevsky space [23]. It leads to

$$\sinh\frac{2r_0}{k} = \frac{2\gamma M}{kc^2}. \tag{156}$$

Here γ is the Newton constant, M is the mass of the central source. If $r_0 \ll k$, then

$$r_0 \approx \frac{\gamma M}{c^2} \tag{157}$$

is the ordinary Schwarzschild radius.

The metric (154) is similar to the Fock metric

$$ds^2 = \frac{r-r_0}{r+r_0}c^2dt^2 - \frac{r+r_0}{r-r_0}dr^2 - (r+r_0)^2d\Omega^2 \tag{158}$$

and turns into (158) when $k \longrightarrow \infty$. As (158), (154) is singular when $r = r_0$.

But the second solution (155) doesn't have the Einstein limit if $k \longrightarrow \infty$. Corresponding (155) solution (149) has the limit $\lim_{k\to\infty}\gamma_2 = 0$ which is inconsistent with (131). On the other hand, it is (155) that violates the analogy of the Birkhoff theorem.

Notice that if $r_0 \longrightarrow 0$ then (154) is turns into (119), but (155) remains

$$ds^2 = c^2dt^2 - dr^2 - k^2\cosh^2(r/k)\,d\Omega^2 \tag{159}$$

as a solution of (121) in an absolutely empty space.

Radial movement of a photon may be determined by $dS = d\Omega = 0$. For (154) we have

The radial photon velocity

$$v_1 = dr/dt = \pm c\exp(2r_0/k)\,\frac{\sinh\frac{r-r_0}{k}}{\sinh\frac{r+r_0}{k}}; \tag{160}$$

the time of radial motion from R_1 to $r < R_1$

$$\tau_1 = A\ln\frac{\exp(2R_1/k) - \exp(2r_0/k)}{\exp(2r\ /k) - \exp(2r_0/k)} + B\,\frac{R_1 - r}{c} + \tau_1 \tag{161}$$

where $B = \exp(-4r_0/k)$, $A = k(1-B)/(2c)$, τ_1 is the time corresponding to $r = R_1$.

For (155) the same values are:

The radial photon velocity

$$v_2 = dr/dt = \pm c \exp(2r_0/k) \frac{\cosh \frac{r - r_0}{k}}{\cosh \frac{r + r_0}{k}}; \qquad (162)$$

the time of radial motion from R_1 to $r < R_1$

$$\tau_2 = A \ln \frac{\exp(2R_1/k) + \exp(2r_0/k)}{\exp(2r/k) + \exp(2r_0/k)} + B \frac{R_1 - r}{c} + \tau_1. \qquad (163)$$

If $R_1 \sim r >> r_0$, both τ_1 and τ_2 give the ordinary expression

$$\tau = (1 - B)\frac{R_1 - r}{c} + B\frac{R_1 - r}{c} = \frac{R_1 - r}{c}.$$

As we can see, for arriving at $r = r_0$ the photon in (154) needs infinite time, but in (155) this time is finite. If $r \longrightarrow r_0$, then $v_1 \longrightarrow 0$, but from (155) we have monotonous increasing from $\lim_{r\longrightarrow\infty} v_2 = c$ to $v_2 = c \exp(2r_0/k)$ if $r = 0$.

The physical time t_r which is at the point with the radial coordinate r is determined by the relations

$$dt_r = \frac{\sqrt{g_{00}}}{c} dt = \begin{cases} \exp(-r_0/k) \sqrt{\dfrac{\sinh \frac{r - r_0}{k}}{\sinh \frac{r + r_0}{k}}}\, dt & \text{for (154)}, \\ \exp(-r_0/k) \sqrt{\dfrac{\cosh \frac{r - r_0}{k}}{\cosh \frac{r + r_0}{k}}}\, dt & \text{for (155)}. \end{cases}$$

If $r \longrightarrow \infty$ we have $dt_r = dt$ for both the metrics, but if $r \longrightarrow r_0$ then the physical time in (154) stops.

The physical speed of light is

$$\frac{dx_r}{dt_r} = \pm \frac{\sqrt{-g_{11}}\, dr}{\left(\sqrt{-g_{00}} / c\right) dt} = \pm c$$

for both the metrics.

On the contrary, we can assume that the gravitation is an optical medium with the refraction coefficient different from 1 and the physical time interval dp is

$$dp_r = \frac{\sqrt{\breve{g}_{00}}}{c} dt = dt$$

in the spirit of the bimetric theories.

Notice that such an interpretation is possible only if we consider the background metric. But as has been shown, both equations and the Lagrangian contain only the background connection. That is why the last interpretation may contain certain arbitrariness. Just in our case the constant c is not contained in $\breve{\Gamma}^{i}_{jk}$, and for different c we get the same $\breve{\Gamma}^{i}_{jk}$. The result Q = c was obtained only due to demands of the asymptotic behavior of g_{mn}. The background metric in our case is not observable and, hence, not important.

REFERENCES

[1] C.W Gibbons, S.W.Hawking, *Phys. Rew.* **D15**, 2752 (1977).
[2] R.F.Polishcuk, *Positivity of Total Energy and Initial Value Problem in Gravitation* in: *On Relativity Theory*, World Scientific, Singapore 1985, v.2, pp. 153-160.
[3] N.A.Chernikov, Preprint JINR E2-92-394, Dubna 1992, in Russian.
[4] M.M.Tentyukov, *Acta Phys. Pol.* **B20**, 911 (1989).
[5] N.A.Chernikov, Preprint JINR P2-88-778, Dubna 1988, in Russian.
[6] N.Rosen, *Phys. Rev.* **57**, 147, 150 (1940).
[7] N.Rosen, *Found. Phys.* **10**, 673 (1980).
[8] L.P.Grishchuk, A.N.Petrov, A.D.Popova, *Comm. Math. Phys.* **94**, 379 (1984).
[9] B.M.Barbashov, V.V.Nesterenko, *Fortsh. Phys.* **31**, 10, 535 (1983).
[10] A.S.Eddington, *The mathematical theory of relativity*, Cambridge Univ. Press (1924).
[11] E.Schrödinger, *Space-time structure.* Cambridge Univ. Press (1950).
[12] N.Rosen, *Found. Phys.* 15, 997 (1985).
[13] N.A.Chernikov, Comm. JINR P2-90-399, Dubna 1990, in Russian.
[14] J.N.Goldberg, *Phys. Rev.* **111**, 315 (1958).
[15] A.Papapetrou, *Proc. Roy. Irish. Acad.* **A52**, 11 (1948).
[16] M.N.Tentyukov, *Acta Phys. Pol.* **B20**, 659 (1989) in Russian.
[17] S.Manoff, *Proceedings of the second Marcel Grossmann meeting on General Relativity.* North-Holland Publ. Co., 1982, p.641.
[18] F.W.Hehl, Kerlic G.D. *GRG*, **9**, 8, 691 (1978).
[19] M.N.Tentyukov, Comm. JINR P2-88-182, Dubna 1988, in Russian.
[20] E.Noether, Nachr. Konigl. Ges. Wiss. Gottingen, math-phys. **K1**, Heft 2, pp. 235-258 (1918).
[21] B.F.Shutz, *Geometrical Methods of Mathematical Physics*, Cambridge Univ. Press, 1982.
[22] N.A.Chernikov, Comm. JINR P2-92-108, Dubna 1992, in Russian.
[23] N.A.Chernikov, *Acta Phys. Pol.* **B23**, 115 (1992).

Received by Publishing Department
on October 29, 1992.