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EVEN AND ODD SYMPLECTIC AND KÄHLERIAN STRUCTURES ON PROJECTIVE SUPERSPACES

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1 Introduction

On the supermanifolds it is possible to define not only even, but also odd symplectic structures [1]. Phase space structure corresponded to even symplectic structure and odd one.

Such on the superspace $E^{2N,M}$ with coordinates $z^A = (x^1, ..., x^{2N}, \theta^1, ..., \theta^M)$ one can consider an even symplectic structure with corresponding even canonical Poisson bracket:

$$\{f,g\}_0 = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^{i+N}} - \frac{\partial f}{\partial x^{i+N}} \frac{\partial g}{\partial x^i} \right) + \sum_{\alpha=1}^{M} \epsilon_\alpha \frac{\partial^R f}{\partial \theta^\alpha} \frac{\partial^L g}{\partial \theta^\alpha}, \quad \epsilon_\alpha = \pm 1. \tag{1.1}$$

On $E^{N.N}$ one can consider an odd one with corresponding canonical odd Poisson bracket (Buttin bracket, antibracket):

$$\{f,g\}_1 = \sum_{i=1}^{N} \left(\frac{\partial f}{\partial x^i} \frac{\partial^L g}{\partial \theta^i} + \frac{\partial^R f}{\partial \theta^i} \frac{\partial g}{\partial x^i} \right). \tag{1.2}$$

In the [2, 3] Batalin and Vilkovisky used odd bracket for formulating Lagrangian BRST quantization formalism (BV -formalizm). Its provides a possibility to give covariant and the most elegant formulation of the conditions on all the ghosts . BV-formalism is an effective method for quantization of gauge theories with open Lie algebra. An attempt to consider it as a framework of background independent open-string field theory was made [4, 5].

On other hand the possiblity importance of the odd bracket in twistorial program and supersymmetric mechanics was emphasised [6-9]. The problem of reformulation of supersymmetric mechanics in terms of odd bracket, using the supercharge as a new Hamiltonian and the attempts to quantize it were performed in [8-10].

There is no doubt that odd bracket needs to be geometrically investigated.

It is possible to formulate Hamiltonian mechanics in term of odd brackets as well as in term of even one [11]. Arbitrary even nondegenerate bracket can be reduced (locally) to canonical form (1.1), and arbitrary odd one – to canonical form (1.2) [12]. But in the general case even and odd brackets cannot be simultaneously reduced to form (1.1) and (1.2).

The structure of the supergroup of transformations which preserve both brackets, depends on their mutual position. Anyway this supergroup is finite-dimensional and it is the different grading of the brackets that leads to this fact [13].

There are nontrivial geometrical objects depending on second derivatives which are invariant under transformations preserving odd bracket and the volume form connected with even bracket. It is the "operator Δ" [13] which used in BV -formalism [2, 3] and the semidensity constructed in [14]. These objects have no analogs in a classical case.

These results strongly indicate that nontrivial geometry arises on the supermanifolds which are provided by Poisson brackets of different grading. Geometrical properties of superspaces provided by Poisson brackets of different gradings were investigated in [13, 15-17]. Superspaces, provided simultaneously by even and odd canonical one was investigated in [15]. It was shown in [16] that exists a large class of supermanifolds (the supermanifolds, associated with tangent bundles of Kählerian manifolds) on which one can defined simultaneously even and odd symplectic (and Kählerian) structures. These structures turn out to be lifting of the corresponding structures on the underlying manifolds. Therefore their properties have to be expressible in terms of classical geometrical objects. They are good models for revealing geometrical properties of two -bracket supermanifolds.

But there don't support by nontrivial examples where even and odd symplectic structures have natural geometrical origin.

In this work we construct the example of such supermanifold as reduced phase superspaces of the superoscillator. (The dynamics of the superoscillator in the superspace $E^{2N,2N}$ can be described either in terms of the canonical even bracket (1.1) or in terms of the canonical odd one (1.2). In the second case the role of the Hamiltonian is played by one of its supercharges [15].)

In the *Section 2* we demonstrate reduction procedure on the simple examples constructing phase superspaces reduced by the Hamiltonian of the superoscillator. This procedure performed in terms of even and odd structures leeds to the two different supermanifolds .

In the *Section 3* we perform the reduction procedure by Hamiltonian of the superoscillator and by its supercharges. In terms of both symplectic structures we come to the same supermanifold which naturally inherits its even and odd structures of the initial superspace. Canonical complex structure on the initial superspace $E^{2N,2N}$ provides this supermanifold with the complex structure and with the even and odd Kählerian structures corresponding to them.

It occurs that this supermanifold is associated to the tangent space of the underlying manifold - complex projective space.

This supermanifold obtained by reduction procedure can be naturally included in the family of the supermanifolds (which are associated with tangent bundles of arbitraty Kählerian manifold) with even and odd Kählerian structures lifted from the Kählerian structure of the underlying manifold, which was investigated in [16].

In the *Section 4* we investigate the bi-Hamiltonian mechanics (i.e. the even vector fields preserving both symplectic structures) and "operator Δ " on the constructed supermanifold and discuss theirs connection with the geometrical objects on underlying manifold.

In the *Appendix A* for the general case we briefly mention the method of Hamiltonian reduction in the terms convenient for our purposes.

In the *Appendix B* we recall the connection between the supermanifolds and the linear bundles to the extent necessary for our purposes. In this Appendix we suggest a natural lifting in the general case of the reduction procedure from the manifolds to their corresponding supermanifolds with odd symplectic structures.

For rigorous definitions and conventions in supermathematics used here we refer too [1].

The preliminary results of this article were published in [17].

2 Examples of Kählerian Supermanifolds and Hamiltonian Reduction

We mostly consider symplectic structures (odd or even one) as the part of corresponding Kählerian structures. In the same way as in the bosonic case [18] complex supermanifold is provided by even (odd) Kählerian structure if symplectic structure is defined by real closed nondegenerated even (odd) two-form Ω^{κ} which in local complex coordinates z^A; is given by the following expression

$$\Omega^{\kappa} = i(-1)^{p(A)(p(B)+\kappa+1)} g^{\kappa}_{A\bar{B}} dz^{A} \wedge d\bar{z}^{B}, \tag{2.1}$$

where

$$g^{\kappa}_{A\bar{B}} = (-1)^{(p(A)+\kappa+1)(p(B)+\kappa+1)+\kappa+1}\overline{g^{\kappa}_{B\bar{A}}}, \quad p(g^{\kappa}_{A\bar{B}}) = p_A + p_B + \kappa .$$

Here and further index $\kappa = 0(1)$ denote even(odd) case.

Then there exists a local real even (odd) function $K^{\kappa}(z,\bar{z})$ (Kählerian potential), such that

$$g^{\kappa}_{A\bar{B}} = \frac{\partial^L}{\partial z^A}\frac{\partial^R}{\partial \bar{z}^B} K^{\kappa}(z,\bar{z}) . \tag{2.2}$$

(As well as in usual case [18] the potential K is defined with precision define up to arbitrary analytic and antianalytic functions.)

To even (odd) form Ω^{κ} there corresponds the even (odd) Poisson bracket

$$\{f,g\}_{\kappa} = i\left(\frac{\partial^R f}{\partial \bar{z}^A} g^{\bar{A}B}_{\kappa} \frac{\partial^L g}{\partial z^B} - (-1)^{(p(A)+\kappa)(p(B)+\kappa)} \frac{\partial^R f}{\partial z^A} g^{\bar{A}B}_{\kappa} \frac{\partial^L g}{\partial \bar{z}^B}\right), \tag{2.3}$$

where

$$g^{\bar{A}B}_{\kappa} g^{\kappa}_{B\bar{C}} = \delta^{\bar{A}}_{\bar{C}} , \quad \overline{g^{\bar{A}B}_{\kappa}} = (-1)^{(p(A)+\kappa)(p(B)+\kappa)} g^{\bar{B}A}_{\kappa}.$$

Its satisfied to conditions of reality and "antisimmetricity"

$$\overline{\{f,g\}_{\kappa}} = \{\bar{f},\bar{g}\}_{\kappa}, \quad \{f,g\}_{\kappa} = -(-1)^{(p(f)+\kappa)(p(g)+\kappa)}\{g,f\}_{\kappa}, \tag{2.4}$$

and Jacobi identities :

$$(-1)^{(p(f)+\kappa)(p(h)+\kappa)}\{f,\{g,h\}_{\kappa}\}_{\kappa} + (\text{cicl.perm.}) = 0 . \tag{2.5}$$

On the complex superspace $\mathbb{C}^{N+1,N+1}$ with complex coordinates $z = (z^n, \eta^n)$, $n = 0,1,...,N$ canonical symplectic structure

$$\Omega^0 = i(dz^n \wedge d\bar{z}^n - i d\eta^n \wedge d\bar{\eta}^n)$$

with corresponding even Poisson bracket

$$\{f,g\}_0 = i\left(\frac{\partial f}{\partial z^n}\frac{\partial g}{\partial \bar{z}^n} - \frac{\partial f}{\partial \bar{z}^n}\frac{\partial g}{\partial z^n}\right) + \frac{\partial^R f}{\partial \eta^n}\frac{\partial^L g}{\partial \bar{\eta}^n} + \frac{\partial^R f}{\partial \bar{\eta}^n}\frac{\partial^L g}{\partial \eta^n} \tag{2.6}$$

defines even Kählerian structure, and canonical odd symplectic structure

$$\Omega^1 = dz^n \wedge d\bar{\eta}^n + d\bar{z}^n \wedge d\eta^n$$

with corresronding odd bracket

$$\{f,g\}_1 = \frac{\partial f}{\partial z^n}\frac{\partial^L g}{\partial \bar{\eta}^n} + \frac{\partial f}{\partial \bar{z}^n}\frac{\partial^L g}{\partial \eta^n} - \frac{\partial^R f}{\partial \bar{\eta}^n}\frac{\partial g}{\partial z^n} - \frac{\partial^R f}{\partial \eta^n}\frac{\partial g}{\partial \bar{z}^n} \tag{2.7}$$

defines odd Kählerian structure. One can obtaines more nontrivial examples by Hamiltonian reduction.

It is well known that for the harmonic oscillator in $(N+1)_{\mathbb{C}}$–dimensional phase space using the energy integral for decreasing by one the complex degrees of freedom we go to N-dimensional complex projective space and Kählerian metric corresponding to reduced symplectic structure on it coincides with canonical one [19]. The straightforward generalization of this procedure on supercase gives us the following example.

Let

$$H = z^n\bar{z}^n - i\eta^n\bar{\eta}^n \tag{2.8}$$

be the Hamiltonian of the superoscillator in the complex phase superspace $\mathbb{C}^{N+1,N+1}$ with even Poisson bracket (2.6). H defines Hamiltonian action of group $U(1)$ on $\mathbb{C}^{N+1,N+1}$ via motion equations

$$\dot{f} = \{H,f\}_0 \quad , \quad z \to e^{it}z . \tag{2.9}$$

As well as in the ordinary case the $(N.N+1)$ - dimensional complex projective superspace $\mathbb{C}P(N.N+1)$ (the manifold of (1.0) - dimension complex subspaces in the $\mathbb{C}^{(N+1.N+1)}$) is obtained as the factorization of the $(2N+1.2N+2)_{\mathbb{R}}$ -dimensional level supersurface

$$H = h \tag{2.10}$$

by Hamiltonian action (2.9) of the group $U(1)$. One can choose as the local coordinates of the supermanifold $\mathbb{C}P(N, N+1)$ in the map $z^m \neq 0$ the functions $w^A_{(m)} = (w^a_{(m)}, \eta^k_{(m)}), a \neq m$, where

$$w^a_{(m)} = \frac{z^a}{z^m} \quad , \quad \theta^k_{(m)} = \frac{\eta^k}{z^m} \tag{2.11}$$

restricted on the supersurface (2.10). The transition functions for these coordinates from the map $z^n \neq 0$ to the map $z^m \neq 0$ are

$$w^a_{(m)} = \frac{w^a_{(n)}}{w^m_{(n)}}, \qquad \theta^k_{(m)} = \frac{\theta^k_{(n)}}{w^m_{(n)}}, \quad \text{where} \quad w^m_{(n)} = (w^a_{(n)}, w^n_{(n)} = 1). \quad (2.12)$$

These coordinates are invariant under $U(1)$ group action:

$$\{w^a_{(m)}, H\}_0 = \{\theta^k_{(m)}, H\}_0 = 0.$$

So the inherited Poisson bracket on $\mathbb{C}\mathrm{P}(\mathrm{N}, \mathrm{N}+1)$ is naturally defined by the relation

$$\{f, g\}^{\rm red}_0 = \{f, g\}_0 \mid_{H=h},$$

where f, g are functions depending on the coordinates $w^A_{(m)}, \bar{w}^A_{(m)}$ (see for details Appendix 1 or [19]).

The calculations give us

$$\begin{aligned}
&\{w^A_{(m)}, w^B_{(m)}\}^{\rm red}_0 = \{\bar{w}^A_{(m)}, \bar{w}^B_{(m)}\}^{\rm red}_0 = 0, \\
&\{w^A_{(m)}, \bar{w}^B_{(m)}\}^{\rm red}_0 = (-1)^{p_A p_B + 1}\{\bar{w}^B_{(m)}, w^A_{(m)}\}^{\rm red}_0 = \qquad (2.13)\\
&= (i)^{p_A p_B + 1}\frac{1 + (-i)^{p_C} w^C_{(m)}\bar{w}^C_{(m)}}{h}(\delta^{AB} + (-i)^{p_A p_B} w^A_{(m)}\bar{w}^B_{(m)}).
\end{aligned}$$

From (2.12) one obtain that the coordinates $w^A_{(m)}$ provide $\mathbb{C}\mathrm{P}(\mathrm{N}.\mathrm{N}+1)$ by complex structure and to Poisson bracket (2.13) correspond Kählerian structure with potential:

$$K_{(m)} = h \log(1 + (-i)^{p_C} w^C_{(m)}\bar{w}^C_{(m)}).$$

Let us consider now the reduction of the odd Poisson bracket (2.7) on the $\mathbb{C}^{\mathrm{N}+1,\mathrm{N}+1}$ by Hamiltonian action of $U(1)$ group. It is easy to check that it defined by an odd Hamiltonian

$$Q_2 = i(z^k\bar{\eta}^k - \bar{z}^k\eta^k), \qquad (2.14)$$

(which is supercharge of previous one), because it is easy to check that for arbitrary function f:

$$\dot{f} = \{H, f\}_0 = \{Q_2, f\}_1$$

where $\{\ ,\ \}_1$ is odd Poisson bracket (2.7). Performingng the reduction as above we obtain the supermanifold $M^{2N+1.2N+1}_{\mathbb{R}}$ of real dimension $(2N+1, 2N+1)$ which evidently can not has (even) complex structure. We define an odd symplectic structure on it similary to even case : the $U(1)$ -invariant functions $(w^a, \theta^a, \bar{w}^a, \bar{\theta}^a, H_0, Q_1)$ where w^a, θ^a, are defined by (2.11) ,

$$\begin{aligned}
H_0 &= z^k\bar{z}^k, & (2.15)\\
Q_1 &= z^k\bar{\eta}^k + \bar{z}^k\eta^k, & (2.16)
\end{aligned}$$

restricted on the level supersurface

$$Q_2 = q_2$$

can be seen as local coordinates of $M^{2N+1.2N+1}_{\mathbb{R}}$. In these coordinates the odd Poisson bracket is defined by following basic relations

$$\{w^a_{(m)}, \bar{\theta}^b_{(m)}\}^{\rm red}_1 = \frac{i(1 + w^c_{(m)}\bar{w}^c_{(m)})}{H_0}\delta^{ab},$$

$$\{\theta^a_{(m)}, H_0\}^{\rm red}_1 = -w^a_{(m)}, \quad \{Q_1, H_0\}^{\rm red}_1 = H_0.$$

We see that the same transformations group $U(1)$ of the complex superspace $\mathbb{C}^{(\mathrm{N}+1,\mathrm{N}+1)}$ which Hamiltonian action in both cases is defined by (2.9) reduces this superspace to rather different symplectic supermanifolds.

In the following section by Hamiltonian reduction we construct a complex supermanifold which can be considered as a reduction of both of them and which have naturally defined even and odd Kählerian structures.

3 Supergeneralization of CP(N) with Even and Odd Kählerian Structures

In this section we do Hamiltonian reduction of initial $\mathbb{C}^{\mathrm{N}+1,\mathrm{N}+1}$ with canonical even structure (2.6) by one generalization of $U(1)$ and the reduction of $\mathbb{C}^{\mathrm{N}+1,\mathrm{N}+1}$ with canonical odd structure (2.7) by another generalization of $U(1)$. The complex supermanifolds obtained in both cases appear to be the same (up to diffeomorfism) and can be considered as "intersection" of supermanifolds considered above. This supermanifold provided by even and odd Kählerian structures turns to be associated to the tangent bundle of complex projective space $\mathbb{C}P(N)$.

3.1 Reduction by Even Bracket

Now let us consider at first the reduction of even structure (2.6) on the superspace $\mathbb{C}^{N+1,N+1}$ by Hamiltonian H and its supercharges Q_1 and Q_2 (which defined by (2.8), (2.14), (2.16)). They form the superalgebra

$$\{Q_r, Q_s\}_0 = 2\delta_{rs}H, \quad \{Q_r, H\}_0 = \{H, H\}_0 = 0, \quad r, s = 1, 2. \qquad (3.1)$$

This superalgebra defines the Hamiltonian action of (1.2)– dimensional group of transformations of the $\mathbb{C}^{N+1,N+1}$. To every even element $\tilde{H} = \alpha H + \beta Q_1 + \gamma Q_2$ (where α is even and β and γ is odd constants) of this superalgebra corresponds one-parametric transformation $z \to \tilde{z}(t, z)$ via motions equations $\dot{z} = \{\tilde{H}, z\}_0$. The group of these transformations is the supergeneralization of the $U(1)$ group transformations (2.9). We denote it by $U^s(1)$. Lets define in $\mathbb{C}^{(N+1,N+1)}$ the level supersurface M_{h,q_1,q_2} by equations

$$H = h, \quad Q_1 = q_1, \quad Q_2 = q_2. \qquad (3.2)$$

Reduced phase superspace is the factorization of M_{h,q_1,q_2} by the action of $U(1)$ subgroup of $U^s(1)$, because transformations corresponding to Q_1 and to Q_2 do not preserve (3.2). For pulling down Poisson bracket (2.6) on it we have to choose convenient local coordinates which are $U^s(1)$ -invariant functions on $\mathbb{C}^{(N+1,N+1)}$ restricted on M_{h,q_1,q_2} (see for details Appendix 1) These coordinates are following

$$\sigma^a_{(m)} = -i\{w^a_{(m)}, Q_+\} = \theta^a_{(m)} - \theta^m_{(m)} w^a_{(m)}, \qquad (3.3)$$

$$x^a_{(m)} = w^a_{(m)} + i\frac{Q_-}{H}\sigma^a_{(m)}, \qquad (3.4)$$

where $w^a_{(m)}, \theta^a_{(m)}, \theta^m_{(m)}$ are defined by (2.11) and

$$Q_\pm = \frac{Q_1 \pm iQ_2}{2}.$$

These coordinates provide reduced superspace by complex structure (see Subsection 3.3).

If f and g are $U^s(1)$–invariant functions then $\{f, g\}$ is $U^s(1)$–invariant function too, so from (3.1), (3.3) using Jacoby identity (2.5) one can obtain that their Poisson brackets depend only on x^a, $\bar{x}^a$, σ^a $\bar{\sigma}^a$, and H.

The inherited Poisson bracket as well as in previous section is defined by the relation

$$\{f, g\}_0^{\rm red} = \{f, g\}_0 \,|_{H=h, Q_{1,2}=q_{1,2}},$$

where f, g are $U^s(1)$ -invariant functions, $\{\ \ ,\ \ \}_0$ is the canonical even bracket (2.6) on $\mathbb{C}^{N+1,N+1}$. Substituting (3.3), (3.4) in this relation and taking into account (2.13), (3.1), (3.2) and $U^s(1)$–invariance one obtain by straightforward calculations

$$\begin{aligned}
\{x^A, x^B\}_0^{\rm red} &= \{\bar{x}^A, \bar{x}^B\}_0^{\rm red} = 0, \quad \text{where} \quad x^A = (x^a, \sigma^a) \\
\{x^a, \bar{x}^b\}_0^{\rm red} &= i\frac{A}{h}(\delta^{ab} + x^a\bar{x}^b) - \frac{\sigma^a\bar{\sigma}^b}{h}, \\
\{x^a, \bar{\sigma}^b\}_0^{\rm red} &= i\frac{A}{h}\left(x^a\bar{\sigma}^b + \mu(\delta^{ab} + x^a\bar{x}^b)\right) \\
\{\sigma^a, \bar{\sigma}^b\}_0^{\rm red} &= \frac{A}{h}\left((1 + i\mu\bar{\mu})\delta^{ab} + x^a\bar{x}^b + i(\sigma^a + \mu x^a)(\bar{\sigma}^b + \bar{\mu}\bar{x}^b\right),
\end{aligned} \qquad (3.5)$$

(other relations are obtained from (3.5) taking into account (2.4)) where

$$A = 1 + x^a\bar{x}^a - i\sigma^a\bar{\sigma}^a + \frac{i\sigma^a\bar{x}^a\bar{\sigma}^b x^b}{1 + x^c\bar{x}^c}, \quad \mu = \frac{\bar{x}^a\sigma^a}{1 + x^b\bar{x}^b}.$$

One can show that to odd structure (3.5) corresponds Kählerian structure with potential

$$K = h\log\left(1 + x^a\bar{x}^a - i\sigma^a\bar{\sigma}^a + \frac{i\sigma^a\bar{x}^a\bar{\sigma}^b x^b}{1 + x^c\bar{x}^c}\right). \qquad (3.6)$$

3.2 Reduction by Odd Bracket

In the same way we consider the reduction of the $\mathbb{C}^{(N+1,N+1)}$ with odd structure (2.7) by another supergeneralization $U^{\bar{s}}(1)$ of the group $U(1)$ generated by Q_2 and H_0 (defined by (2.14), (2.15))(as it was mentioned above Q_2 defines $U(1)$ group action (2.9) in terms of odd bracket). This group is abelian :

$$\{H_0, Q_2\}_1 = \{H_0, H_0\}_1 = \{Q_2, Q_2\}_1 = 0,$$

so reduced phase superspace have real dimension $(2N.2N)$. The functions $w^a_{(m)}, \sigma^a_{(m)}$, defined by (2.11) and (3.3) commute with Q_2 and H_0 so their restriction on levels supermanifold

$$Q_2 = q_2, \quad H_0 = h_0$$

are the appropriate local coordinates for pulling down odd Poisson bracket on a reduced superspace. The inherited odd Poisson bracket is defined in the same way as (3.5):

$$\{f, g\}^{\rm red}_1 = \{f, g\}_1 \mid_{H_0=h_0, Q_2=q_2},$$

where f, g are $U^{\bar{s}}(1)$–invariant functions, $\{\ ,\ \}_1$ is the canonical odd bracket (2.7) on $\mathbb{C}^{N+1,N+1}$. The calculations give

$$\begin{aligned}
\{w^A, w^B\}^{\rm red}_1 &= \{\bar{w}^A, \bar{w}^B\}^{\rm red}_1 = 0, \quad \text{where} \quad w^A = (w^a, \sigma^a)\\
\{w^a, \bar{w}^b\}^{\rm red}_1 &= 0,\\
\{w^a, \bar{\sigma}^b\}^{\rm red}_1 &= \frac{1 + w^c\bar{w}^c}{h_0}(\delta^{ab} + w^a\bar{w}^b),\\
\{\sigma^a, \bar{\sigma}^b\}^{\rm red}_1 &= \frac{1 + w^c\bar{w}^c}{h_0}(\sigma^a\bar{w}^b - w^a\bar{\sigma}^b) +\\
&+ \left(\frac{\sigma^c\bar{w}^c - w^c\bar{\sigma}^c}{h_0} + iq_2(1 + w^c\bar{w}^c)\right)(\delta^{ab} + w^a\bar{w}^b)
\end{aligned} \tag{3.7}$$

(other relations are obtained from (3.7) taking into account (2.4)). Corresponding odd Kählerian structure (the local coordinates $(w^a_{(m)}, \sigma^a_{(m)})$ provide reduced superspace by complex structure (see Subsection 3.3)) is given by potential

$$K_1 = h_0 \frac{i(w^a\bar{\sigma}^a - \bar{w}^a\sigma^a)}{1 + w^b\bar{w}^b} + q_2 \log(1 + w^a\bar{w}^a). \tag{3.8}$$

3.3 Investigation of the Global Properties

We obtain two reduced superspaces one with coordinates x^a, σ^a and even Kählerian structure with potential (3.6), another with coordinates w^a, σ^a and odd Kählerian structure with potential (3.8) . Now we show that they coincide up to diffeomorphism and clarify their global structure. It

is not useless for these purposes to investigate the transitions functions from map to map for coordinates $w^a_{(m)}, \sigma^a_{(m)}$ and $x^a_{(m)}, \sigma^a_{(m)}$.

The coordinates $\sigma^a_{(m)}$ transform like differentiales of the $w^a_{(m)}$ according their definition (3.3).

$$\begin{aligned}
w^a_{(n)} \to w^a_{(m)} &= \frac{w^a_{(n)}}{w^m_{(n)}},\\
\sigma^a_{(n)} \to \sigma^a_{(m)} &= \frac{\sigma^a_{(n)}w^m_{(n)} - w^a_{(n)}\sigma^m_{(n)}}{(w^m_{(n)})^2}, \quad k = 0, ..., N,
\end{aligned} \tag{3.9}$$

where $(w^n_{(n)} = 1,\ \sigma^n_{(n)} = 0)$.

From (3.4) and (3.9) it is easy to see that the coordinates $(x^a_{(m)}; \sigma^a_{(m)})$ transform like $(w^a_{(m)}, \sigma^a_{(m)})$:

$$\begin{aligned}
x^a_{(n)} \to x^a_{(m)} &= \frac{x^a_{(n)}}{x^m_{(n)}},\\
\sigma^a_{(n)} \to \sigma^a_{(m)} &= \frac{\sigma^a_{(n)}x^m_{(n)} - x^a_{(n)}\sigma^m_{(n)}}{(x^m_{(n)})^2} \quad (x^n_{(n)} = 1, \sigma^n_{(n)} = 0).
\end{aligned} \tag{3.10}$$

As seen, this supermanifolds have global complex structures.

It allows us to consider these two reduced superspaces as the same because one can identify $(w^a_{(m)}, \sigma^a_{(m)})$ with $(x^a_{(m)}, \sigma^a_{(m)})$. The correspondence $(x^a_{(m)}, \sigma^a_{(m)}) \to (w^a_{(m)}, \sigma^a_{(m)})$ preserving under the transformations (3.9), (3.10) sets up isomorphism from the functions defining on the reduced superspace with even structure (3.5) on the functions defining on the reduced superspace with odd structure (3.7). The obtained phase superspace we denote by $S\mathbb{C}\mathrm{P}(\mathrm{N})$.

Now let us summarize our results . The phase superspace $\mathbb{C}\mathrm{P}(\mathrm{N.N}+1)$ which was constructed in the Section 2 as the reduction of $\mathbb{C}^{N+1.N+1}$ with even canonic structure by the Hamiltonian of superoscillator (without using its supercharges) and now constructed $S\mathbb{C}\mathrm{P}(\mathrm{N})$ have the same underlying manifold - N-dimensional complex projective space $\mathbb{C}\mathrm{P}(\mathrm{N})$. The Kählerian structure which corresponded to (2.13) on the $\mathbb{C}\mathrm{P}(\mathrm{N.N}+1)$ as well as the even Kählerian structure with (3.5) for $S\mathbb{C}\mathrm{P}(\mathrm{N})$ pull down to the standard Kählerian structure of underlying complex projective space. $S\mathbb{C}\mathrm{P}(\mathrm{N})$ can be considered as the further reduction of $\mathbb{C}\mathrm{P}(\mathrm{N.N}+1)$ by the supercharges. In contrary to $\mathbb{C}\mathrm{P}(\mathrm{N})$ it have naturally defined odd

Kählerian structure with potential (3.8) and can be considered as further reduction of $M_{\mathbb{R}}^{2N+1.2N+1}$ by H_0 too

$$\mathbb{C}^{(N+1.N+1)} \xrightarrow{H} \mathbb{C}P(N.N+1) \xrightarrow{Q_1,Q_2} S\mathbb{C}P(N) \quad (\text{even} \quad \text{reduction})$$

$$\mathbb{C}^{(N+1.N+1)} \xrightarrow{Q_2} M_{\mathbb{R}}^{2N+1.2N+1} \xrightarrow{H_0} S\mathbb{C}P(N) \quad (\text{odd} \quad \text{reduction}).$$

Moreover from the equations (3.9), (3.10) it is easy to see that $S\mathbb{C}P(N)$ with local coordinates $x^a_{(m)}, \sigma^a_{(m)}$ is associated to the $T\mathbb{C}P(N)$ - tangent bundle of the underlying manifold $\mathbb{C}P(N)$ because the even coordinates from map to map transform through themself only and odd coordinates transform as differentials of even ones [1] (see also Appendix 2).

From this point of view it becomes natural the following property of the odd symplectic structure (3.7). One can show that in the coordinates

$$\tilde{\sigma}_a = g_{a\bar{b}}\bar{\sigma}^b,$$

where g_{ab} is the Kählerian metric of the underlyind projective space , the odd symplectic structure turns out to be canonical one if $Q_2 = 0$ (for general case, if $Q_2 \neq 0$ see Appendix 2).

$$\tilde{\Omega}^1 = dw^a \wedge d\tilde{\sigma}_a + d\bar{w}^a \wedge d\bar{\tilde{\sigma}}_a \; .$$

Indeed in the coordinates (w^a, σ_a) $S\mathbb{C}P(N)$ is associated to $T^*\mathbb{C}P(N)$ - cotangent bundle of $\mathbb{C}P(N)$, which have naturally defined canonical symplectic structure [19].

It has been mentioned in Introduction that these constructions have general meaning. Indeed for every Kählerian manifold M with local complex coordinates w^a one can consider the complex supermanifold SM $(dim_{\mathbb{C}}SM = (dim_{\mathbb{C}}M.dim_{\mathbb{C}}M))$ with local coordinates w^a, σ^a which is associated to TM. Then the local functions

$$K_0(w,\bar{w},\sigma,\bar{\sigma}) = \quad K(w,\bar{w}) \quad + F(ig_{a\bar{b}}(w,\bar{w})\sigma^a\bar{\sigma}^b), \tag{3.11}$$

$$K_1(w,\bar{w},\sigma,\bar{\sigma}) = \quad \epsilon\frac{\partial K(w,\bar{w})}{\partial w^a}\sigma^a \quad + \bar{\epsilon}\frac{\partial K(w,\bar{w})}{\partial \bar{w}^a}\bar{\sigma}^a + \alpha K(w,\bar{w}), \tag{3.12}$$

(where $K(w,\bar{w})$ is the Kählerian potential of M , $g_{a\bar{b}}$- corresponding Riemannian metric, $F(r)$-arbitrary scalar function such that $F'(0) \neq 0$, ϵ is even complex constant an α is real odd one) can be considered as the potentials which correctly define global even and odd Kählerian structures on SM [16].

In the case $M = \mathbb{C}P(N)$ we obtain immediately the structures constructed above putting in (3.11), (3.12) $K(w,\bar{w}) = \log(1+w^a\bar{w}^a), F(r) = \log(1-r), \epsilon = i, \alpha = q_2$.

4 Operator Δ and bi-Hamiltonian Mechanics

Now we want to discuss the properties of some supergeometrical constructions which can be defined in natural way on the supermanifolds provided by even and odd symplectic structures studying them on the supermanifold constructed above.

The supermanifolds which are associated in some coordinates to tangent bundle (see Appendix 2) can be considered as "gauge fixing" objects for the studying the supergeometrical constructions which in this case have to reduce to the well-known geometrical objects. So this constructions can be considered as the generalization on supercase of the corresponding geometrical objects.

From this point of view it is interesting to look at the explicit expressions for the "operator Δ" and the bi-Hamiltonian mechanics on the $S\mathbb{C}P(N)$ provided by odd and even brackets (3.6), (3.9) (similar expressions for the supermanifolds provided by two Kählerian structures with potentials (3.11), (3.12) see in [16]).

4.1 Operator Δ on $S\mathbb{C}P(N)$

On the supermanifold $\mathcal{M}^{m.m}$ with coordinates $z^A = (x^i, \theta^i)$ which is provided by odd symplectic structure with Poisson bracket $\{\ ,\ \}_1$ and the volume form $dv = \rho(x,\theta)d^m x d^m\theta$ one can invariantly define the odd differential operator of the second order, so-called "operator Δ" which is invariant under the transformations preserving the symplectic structure and the volume form [2, 11]. Its action on the function $f(x,\theta)$ is the divergence of the Hamiltonian vector field $\mathbf{D}_f = \{f, z^A\}_1\frac{\partial^L}{\partial z^A}$ with the volume form dv:

$$\Delta f = div^{\rho}\mathbf{D}_f \equiv \frac{\mathcal{L}_{\mathbf{D}_f} dv}{dv}, \qquad (4.1)$$

where $\mathcal{L}_{\mathbf{D}f}$ — Lie derivative along $\mathbf{D}_f$ [1]. In coordinate form

$$\Delta f = \frac{1}{\rho}(-1)^{p(A)}\frac{\partial^L}{\partial z^A}\left(\rho\{z^A, f\}_1\right). \qquad (4.2)$$

The "operator Δ" have no analogs with even symplectic structures — the oddness of the Poisson bracket $\{\ ,\ \}_1$ which force that operator (4.2) to have dependence of second derivatives.

If the density $\rho = 1$ and $\{\ ,\ \}_1$ has the canonical form (1.2) then Δ is in the canonical form

$$\Delta^{\text{can}} f = 2\frac{\partial^2 f}{\partial x^i \partial\theta^i}, \qquad (4.3)$$

which is well-known from BV-formalism [2-4].

It is easy to obtain from (4.1) using Jacobi identities, Leibnitz rules and the transformation law of integral density $\rho(z)$ that generalized operator Δ (4.2) is connected with corresponding odd bracket by the same expressions as canonical operator Δ^{can} (4.3) connected with canonical odd bracket (1.2) [3, 4]:

$$\begin{aligned} \Delta\{f,g\}_1 &= \{f,\Delta g\}_1 + (-1)^{p(g)+1}\{\Delta f, g\}_1 \\ (-1)^{p(g)}\{f,g\}_1 &= \frac{1}{2}\left(\Delta(fg) - f\Delta g - (-1)^{p(g)}(\Delta f)g\right) \\ \Delta' f &= \Delta f + \{\log\mathcal{J}, f\}_1, \end{aligned}$$

where $\mathcal{J}$-Jacobian of canonical transformation of odd bracket, Δ'- operator Δ in new coordinates. However the nilpotency condition

$$\Delta^2 = 0 \qquad (4.4)$$

are violated for arbitrary $\rho(x,\theta)$.

For example , if symplectic structure is canonical, (4.4) hold if $\rho(x,\theta)$ satisfy to the equation

$$\Delta\rho = 0$$

which is master equation of BV-formalism for the action $S = \log\rho$. Then Δ corresponding to operator of BRST transformation [2-4]. It is

interesting to study the connection between the condition (4.4) and the possibility to reduce (4.2) to (4.3) by the suitable transformation of the coordinates.

If the supermanifold $\mathcal{M}$ provided by even symplectic structure Ω^0 also here one can put into (4.2) the density ρ , which is invariant under canonical transformations of Ω^0 [19, 20]:

$$\rho(z) = \sqrt{\mathrm{Ber}\Omega^0_{AB}}\,. \qquad (4.5)$$

Let $\mathcal{M} = S\mathbb{C}\mathrm{P(N)}$ provided by odd Poisson bracket (3.7) (with $q_2 = 0$) and even one (3.5).The invariant (under canonical transformations of (3.5)) density ρ on it has the form

$$\rho(w,\bar w,\theta,\bar\theta) = (1-r)^2 \equiv (1 - ig_{a\bar b}\theta^a\bar\theta^b)^2 \qquad (4.6)$$

where

$$g_{a\bar b} = \frac{1}{1+w^c\bar w^c}\left(\delta_{a\bar b} - \frac{\bar w^a w^b}{1+w^c\bar w^c}\right) \qquad (4.7)$$

–Kählerian metric of $\mathbb{C}\mathrm{P(N)}$ (r corresponds to cohomologies on $\mathbb{C}\mathrm{P(N)}$).

The operator Δ on $S\mathbb{C}\mathrm{P(N)}$ with this density takes the folowing form

$$\Delta f = \frac{1}{\rho}\left(\nabla^a\frac{\partial^L}{\partial\theta^a} + \overline{\nabla^a}\frac{\partial^L}{\partial\bar\theta^a}\right)(\rho f), \qquad (4.8)$$

where

$$\nabla_a = \frac{\partial}{\partial w^a} - \Gamma^c_{ab}\theta^b\frac{\partial^L}{\partial\theta^c}, \quad \overline{\nabla^a} = g^{\bar a b}\nabla_b,$$

$\Gamma^c_{ab} = g^{\bar d c}g_{a\bar d,b} \equiv -\frac{\bar w^a\delta^c_b + \bar w^b\delta^c_a}{1+w^d\bar w^d}$ - the Christoffel symbols of the Kählerian metric (4.7) on $\mathbb{C}\mathrm{P(N)}$. Nilpotency condition (4.4) is satisfied obviously.

The operator (4.8) corresponds to the operator of covariant divergency $\delta = *d*$ on $\mathbb{C}\mathrm{P(N)}$.

Since $\mathcal{M} = SM$ with Kählerian potentials (3.11), (3.12) ($\epsilon = i, \alpha = 0$) operator Δ is also defined by the expression (4.8) [16], where Γ^c_{ab}- the Christoffel symbols of the Kählerian metric on underlying manifold M,

$$\rho = \frac{\det(\delta^a_b + iF'(r)R^a_{bcd}\theta^c\theta^d)}{F'(r)^{N-1}(F'(r) + F''(r)r)} \qquad (4.9)$$

where $R^a_{bc\bar d} = (\Gamma^a_{bc})_{,\bar d}$ is the curvature tensor on M, $r = ig_{a\bar b}$. (We see that ρ depends on Chern classes of the underlying Kählerian manifold.It is interesting to compare (4.9) with the general formulas for characteristic classes on the supermanifolds [20].) To operator Δ on SM is corresponds the covariant divergence on the underlying Kählerian manifold M.

4.2 Bi-Hamiltonian Mechanics on $S\mathbb{C}$P(N)

Here we deliver explicit formulae for the even vector fields preserving even and odd Poisson brackets (bi-Hamiltonian mechanics) (3.5), (3.7) on $S\mathbb{C}$P(N). In other words we have to find the pairs of the functions (H,Q) $(p(H)=0,\ p(Q)=1)$ on $S\mathbb{C}$P(N) such that for arbitraty function f:

$$\{H,f\}_0=\{Q,f\}_1, \qquad (4.10)$$

where $\{\ ,\ \}_0$ ($\{\ ,\ \}_1$) defines by (3.5), (3.7). To every pair (H,Q) the solution of (4.10) corresponds vector field

$$\mathbf{D}_{H,Q}=\{H,z^A\}_0\frac{\partial^L}{\partial z^A}=\{Q,z^A\}_1\frac{\partial^L}{\partial z^A}\ .$$

These fields form a finite-dimensional Lie algebra [13] and they are defined by Killing vectors of the underlying manifold M [16]. The solutions of the (4.10) is following:

$$\begin{aligned} H &= H_0-\frac{i}{1-r}\frac{\partial^2 H_0}{\partial w^a\partial\bar w^b}\theta^a\bar\theta^b,\\ Q &= i\left(\frac{\partial H_0}{\partial w^a}\theta^a-\frac{\partial H_0}{\partial \bar w^a}\bar\theta^a\right), \end{aligned}$$

where

$$H_0=\frac{h_{a\bar b}w^a\bar w^b-\mathrm{tr}h+h_a w^a+\overline{h_a w^a}}{1+w^c\bar w^c},$$

$h_{a\bar b}$ are arbitrary Hermitian matrices and h_a – arbitrary complex numbers. Corresponding vector field

$$\mathbf{D}_{H,Q}=V^a(w)\frac{\partial}{\partial w^a}+V^a_c(w)\theta^c\frac{\partial}{\partial\theta^a}, \qquad (4.11)$$

where

$$V^a(w)=ig^{\bar b a}\frac{\partial H_0(w,\bar w)}{\partial\bar w^b}$$

is the Killing vector of $\mathbb{C}$P(N). Since $\mathbf{D}_{H,Q}$ defined by (4.11) is holomorphic and Hamiltonian for the both brackets, it is the Killing vector for both Kählerian structures.

Bi-Hamiltonian mechanics on supermanifold SM with symplectic structures, defining by (3.11), (3.12) have a similar form (4.11), where V^a is Killing vector of underlying Kählerian manifold M [16].

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Appendixes

A On Procedure of Hamiltonian Reduction

In this Appendix we retell the main algebraic notions of Hamitonian reduction mechanism using language which is maximally adapted for our purposes and can be evidently generalized on supercase.(The detailed considerations see in [19]). Let M be symplectic space with symplectic structure Ω and $\Gamma(M)$ be an algebra of functions on M . Poisson bracket $\{\ ,\ \}$ corresponding to Ω defined by the following relation $\{f,g\}=\Omega(\mathbf{D}_f,\mathbf{D}_g)$ where $\mathbf{D}_f$ is the vector field corresponding to f via the equation

$$\Omega(\mathbf{D}_f,\mathbf{V})=df(\mathbf{V})$$

for arbitrary vector field $\mathbf{V}$.

Let C be an subalgebra in $\Gamma(M)$ which is closed under $\{\ ,\ \}$. The algebra of functions $\Gamma(M)$ has two algebraic operations – usual multiplicative structure and Lie algebra multiplication provided by Poisson bracket $\{\ ,\ \}$. Further if it is not pointed we suppose the first operation only.)

Let the functions $F_1,\dots,F_k$ be generators of C. In this case $\{F_i,F_j\}=c^k_{ij}F_k$ where c^k_{ij} are constants, the functions $\{F_i\}$ generate Hamiltonian action of the group G (corresponding to Lie algebra with structure constants c^k_{ij}) on the M . To every function F_i corresponds G group infinitesimal transformation via the vector field $\mathbf{D}_{F_i}$. and $\mathbf{D}_{\{F_i,F_j\}}=[\mathbf{D}_{F_i},\mathbf{D}_{F_j}]$.

To subalgebra $\dot{C}$ corresponds the reduction procedure from M to symplectic manifold M^{red}.

Let M_p be the level manifold in M defined by

$$F_i = p_i$$

and G_p – its isotropy group: $G_p = \{g \in G : gp_ig^{-1} = p_i\}$. Then $M^{\rm red} = M_p/G_{p_i}$ and Ω is pulling down on the $\Omega^{\rm red}$ on $M^{\rm red}$ defining its symplectic structure.

For supercase it is more convenient to describe $M^{\rm red}$ and $\Omega^{\rm red}$ correspondingly in the terms of $\Gamma^{\rm red}$ – algebra of the functions on it and $\{\ ,\ \}^{\rm red}$ – Poisson bracket corresponding to $\Omega^{\rm red}$. (The generators of $\Gamma^{\rm red}$ are the coordinates of $M^{\rm red}$.) Let $B(M)$ be an subalgebra of the functions which is "orthogonal" to subalgebra C by Poisson bracket $\{\ ,\ \}$.

$$B = \{\Gamma \ni f : \{f, g\} = 0 \quad \forall g \in C\}$$

in other words $f \in B$ iff $\{f, F_i\} = 0$.

Because of Jacoby identity B is Lie algebra too:

$$f, g \in B \Rightarrow \{f, g\} \in B,$$

so B is the subalgera of G — invariant functions of Γ.

To level manifold M_p corresponds the ideal $\mathcal{J}$ in the algebra Γ generating by the functions $F_i - p_i$

$$\mathcal{J} = \{\Gamma \ni f : f = \sum \alpha_i(F_i - p_i) \quad \text{where} \quad \alpha_i \in \Gamma\}$$

$B \cap \mathcal{J}$ is the ideal in B too, so on can consider subalgebra

$$\Gamma^{\rm red} = B/B \cap \mathcal{J}.$$

It is the algebra of functions on reducted space $M^{\rm red}$. The Poisson bracket $\{\ ,\ \}^{\rm red}$ on $\Gamma^{\rm red}$ is defined in the following way. For any $[f], [g] \in B$, where $[f]$ is the equivalence class of the function $f \in B$ in $\Gamma^{\rm red}$

$$\{[f], [g]\}^{\rm red} = [\{f, g\}].$$

To check the correctness of this definition we note that if $f, g \in B$ then $\{f, g\} \in B$ too . If $f \to \tilde{f} = f + h$ where $h = \sum \alpha_i(F_i - p_i) \in B \cap \mathcal{J}$ then $\{h, g\} \in B$ and

$$\{h, g\} = \{\sum \alpha_i(F_i - p_i), g\} = \sum \{\alpha_i, g\}(F_i - p_i) \in \mathcal{J}$$

because $\{F_i - p_i, g\} = 0$. So $\{\tilde{f}, g\} - \{f, g\} \in B \cap \mathcal{J}$.

The reduction procedure leads to the fact that if dynamical system on M is described by Hamiltonian H which is G - invariant ($H \in B$) and at $t = 0$ the conditions $F_i = p_i$ hold then

i) these conditions preserve in a time,

ii) $[f^t] = [f]^t$,

where h^t we denote the evolution of the function h in the time t via motion equations $\dot{h} = \{H, h\}$ $[\dot{h}] = \{[H], [h]\}^{\rm red}$.

As example we retell in these terms the reduction procedure performed in the Subsection 3.1.

We consider as C the algebra of functions on the $\mathbb{C}^{(N+1.N+1)}$ which explicitly depend on the functions H, Q_1, Q_2 playing the role of generators F_i. The "orthogonal" subalgebra B of $U^s(1)$ - invariant functions is the algebra of functions explicitly depending on x^a, $\bar{x}^a$, σ^a $\bar{\sigma}^a$, and H functions. The functions $f(H - h) + g(Q_1 - q_1) + r(Q_2 - q_2)$ where f, g, r are arbitrary functions consist the ideal $\mathcal{J}$. $B \cap \mathcal{J}$ - the $U^s(1)$ - invariant part of this ideal consists on the functions depending only on H. So the generators of the algebra $\Gamma^{red} = B/B \cap \mathcal{J}$ are $[x^a]$, $[\bar{x}^a]$, $[\sigma^a]$ $[\bar{\sigma}^a]$ and the functions (coordinates) x^a, $\bar{x}^a$, σ^a $\bar{\sigma}^a$ are their corresponding representatives.

B Supermanifolds and Linear Bundles

In this Appendix we briefly mention the connection between supermanifolds and corresponding linear bundles to the extent necessary for our purposes. (See in details in [1].)

Let TM be the tangent bundle to the manifold M. $x^a_{(m)}$ are the local coordinates on the M in m-th map and the $(x^a_{(m)}, v^a_{(m)})$ are the corresponding local coordinates on TM ($v^a_{(m)}$ are coordinates of tangent space in the basic $\frac{\partial}{\partial x^a_{(m)}}$). From map to map

$$x^a_{(k)} \to x^a_{(m)} = x^a_{(m)}(x^a_{(k)}), \quad v^a_{(k)} \to v^a_{(m)} = \frac{\partial x^a_{(m)}}{\partial x^b_{(k)}} v^b_{(k)}. \tag{A2.1}$$

Considering for every map the superalgebra generating by $(x^a_{(m)}, \theta^a_{(m)})$ where $x^a_{(m)}$ are even and $\theta^a_{(m)}$ are odd, transforming from map to map like $(x^a_{(m)}, v^a_{(m)})$ in the (A2.1) $(v \leftrightarrow \theta)$ we go to supermanifold $\mathcal{M}$ which is associated to TM in the coordinates $(x^a_{(m)}, \theta^a_{(m)})$. For the coordinates $(x^a_{(m)}, \theta^a_{(m)})$ on the $\mathcal{M}$ the more general class of transformations is admittable :

$$x^a \to \tilde{x}^a(x^a, \theta^a) \quad \theta^a \to \tilde{\theta}^a(x^a, \theta^a)$$

which do not correspond to (A2.1). In particular if $\theta^a \to \tilde{\theta}_a = g_{ab}\theta^b$, where g_{ab} is some Riemanian metric on M then the supermanifold $\mathcal{M}$ in the coordinates $(x^a, \tilde{\theta}_a)$ is associated to the cotangent bundle T^*M of M.

On the supermanifolds which can be associated in some coordinates to tangent or cotangent bundle the superstructures evidently are reduced to the standard geometrical objects.

For example on the supermanifold $\mathcal{M}$ considered here the canonical odd (Buttin) bracket $\{\ ,\ \}_1$ (defined by basic relations $\{x^i, \tilde{\theta}_j\}_1 = \delta^i_j$) is corresponding to the Schouten bracket $[\ ,\ ]$ of the polyvector fields on M: To polyvector field $\mathbf{T} = T^{j_1,\dots,j_k}(x)$ on M corresponds the function $\rho\mathbf{T} = T(x,\theta) = T^{j_1,\dots,j_k}(x)\theta_{j_1}\dots\theta_{j_k}$ on the $\mathcal{M}$ such that

$$\{\rho\mathbf{T}, \rho\mathbf{U}\}_1 = \rho[\mathbf{T}, \mathbf{U}].$$

Similarly operator $D = \theta^a \frac{\partial}{\partial x^a}$ on the $\mathcal{M}$ corresponds to the exterior differentiation operator on T^*M and operator Δ to the divergence [1].

On one hand these type supermanifolds can be served as the good tests for studying superstructures on other hand we can use them as condensed language for constructed the geometrical structures in superterms. We deliver one example which is strightly connected with the considerations in the Subsection 3.2.

The reduction procedure performed in Section 3 was indeed the prolongation of the $M \stackrel{\text{reduction}}{\to} M^{\text{red}}$ to the $TM \stackrel{\text{reduction}}{\to} TM^{\text{red}}$ in the case $M = \mathbb{C}^{N+1}$, $M^{\text{red}} = \mathbb{C}P(N)$.

Now for the odd structure reduction we show that in the general case. Let M be the symplectic manifold with symplectic structure defined by Poisson bracket $\{\ ,\ \}$ and the functions I_r. generate Hamiltonian action of the Lie group G on it:

$$\{I_r, I_s\} = c^t_{rs} I_m$$

where c^t_{rs} are the structure constants of the Lie algebra $\mathcal{G}$ of G. Let M^{red} be the manifold obtained by reduction: $M^{\text{red}} = M_p/G_p$ where

$M_p = \{x \in M : I_r(x) = p_r\}$ is the level manifold and G_p - is its isotropy group.

Let x^i be the local coordinates on M and y^a - the local ones on M^{red} in which the reduction was performed : $\{y^a(x), I_r(x)\} = 0$. Then

$$w^{ab} = \{y^a, y^b\}^{\text{red}} = \{y^a(x), y^b(x)\} \mid_{I_r(x)=p_r} \qquad (A2.2)$$

defines the reduced Poisson bracket (and symplectic structure) on M^{red}.

If $\mathcal{M}$ is supermanifold associated to T^*M in the local coordinates (x^i, θ_i) and Poisson bracket $\{\ ,\ \}_1$ defines the odd canonical structure on it then it is easy to see that the functions

$$Q_r = \{I_r(x), x^i\}\theta_i = \{I_r, F\}, \quad \text{where} \quad F = \frac{1}{2}\{x^i, x^j\}\theta_i\theta_j$$

define the same Hamiltonian action of the group G on the M in the terms of odd bracket: for arbitrary function $f(x)$ on M

$$\{f(x), I_r(x)\} = \{f(x), Q_r(x,\theta)\}_1.$$

Moreover the functions (Q_r, I_r) define the Hamiltonian action of the supergeneralization of the group G on the $\mathcal{M}$ in the terms of odd bracket:

$$\{Q_r, Q_s\}_1 = c^t_{rs} Q_t, \quad \{Q_r, I_s\}_1 = c^t_{rs} I_t, \quad \{I_r, I_s\}_1 = 0.$$

One can show that the functions $y^A = (y^a, \eta^a = \{y^a(x), x^i\}\theta_i = \{y^a(x), F(x,\theta)\}_{-1})$ play the role of local coordinates on reduced supermanifold $\mathcal{M}^{\text{red}}$ $(dim\mathcal{M}^{\text{red}} = (dimM^{\text{red}}.dimM^{\text{red}})$) :

$$\{y^A(x,\theta), I_r\}_1 = \{y^A(x,\theta), Q_r\}_1 = 0,$$

and in this coordinates $\mathcal{M}^{\text{red}}$ associated to TM^{red}. The functions $y^A = (y^a, \eta^a)$ one can used for reduction of odd bracket $\{\ ,\ \}_1$ on $\mathcal{M}^{\text{red}}$:

$$\{y^a, y^b\}^{\text{red}}_1 = 0, \quad \{y^a, \eta^b\}^{\text{red}}_1 = w^{ab},$$

$$\{\eta^a, \eta^b\}^{\text{red}}_1 = \frac{\partial w^{ab}}{\partial y^c}\eta^c + \frac{\partial w^{ab}}{\partial p^r} q^r$$

where $w^{ab}(y,p)$ is given by (A2.2) and $I_r = p_r$, $Q_r = q_r$ define the level supermanifold in $\mathcal{M}$.

One can construct local coordinates $(y^a, \tilde{\eta}_a)$ such that in these coordinates $\mathcal{M}^{\mathrm{red}}$ is associated to $T^* M^{\mathrm{red}}$:

$$\tilde{\eta}_a = w_{ab}\eta^b - \frac{\partial A_a}{\partial p^k} q^k,$$

where

$$w_{bc}w^{ca} = \delta_b^a \quad \text{and} \quad \frac{\partial A_a}{\partial y^b} - \frac{\partial A_b}{\partial y^a} = w_{ab}.$$

In this coordinates reduced symplectic structure coincides with canonical one :

$$\{y^a, y^b\}_1^{\mathrm{red}} = 0, \quad \{\tilde{\eta}^a, \tilde{\eta}^b\}_1^{\mathrm{red}} = 0, \{y^a, \tilde{\eta}^b\}_1^{\mathrm{red}} = \delta_b^a \quad .$$

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