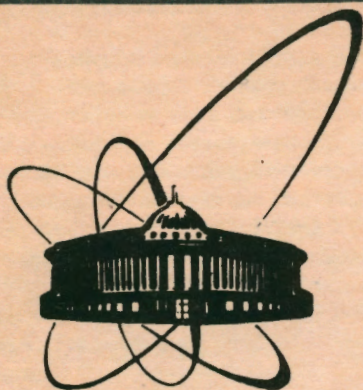


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BOUSSINESQ-TYPE EQUATIONS  
FROM NONLINEAR REALIZATIONS OF  $W_3$

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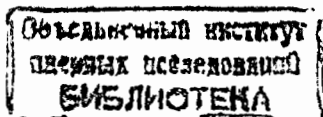
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# 1 Introduction

The existence of intimate relationships between  $W$ -algebras on one hand and conformal field theories and integrable systems in  $1+1$  dimensions on the other (see, e.g. [1-12]) is a fairly well-established fact which has profound and far-reaching implications in modern mathematical physics, especially in what concerns the string theory and  $2D$  gravity. One of the most exciting discoveries in this area is the understanding of the property that various  $W$  algebras and their superextensions provide the second Hamiltonian structures for the generalized KdV and KP hierarchies as well as superextensions of the latter. For instance, the  $W_2$  (Virasoro) algebra defines the second Hamiltonian structure for the KdV hierarchy [5],  $W_3$  for the Boussinesq one [6],  $W_{1+\infty}$  for the KP hierarchy [7], etc. A natural description of the correspondence between the  $W$  algebras and  $1+1$  integrable hierarchies is achieved in terms of pseudo-differential operators [8]. It is of interest to understand these remarkable relationships proceeding directly from the intrinsic geometries of  $W$  symmetries. Besides providing new insights into the geometric origin of the above hierarchies, this could shed more light on the geometry of the associated theories, such as  $W$  gravities,  $W$  strings, etc.

To understand the geometry behind a symmetry group  $G$ , the key concept is to consider it as a group of transformations acting on the coset space  $G/H$  with an appropriately chosen stability subgroup  $H$ . This is the content of the famous nonlinear (or coset) realization method [13]. In the papers of two of us (E.I. & S.K.) [12, 14, 15], it has been argued that the most direct and fruitful way of revealing geometric features of  $W$  symmetries is via this method. However, it had been originally invented to treat the Lie type symmetries, therefore, its application to  $W_N$  symmetries encounters difficulties because  $W_N$  algebras for  $N \geq 3$  are not Lie ones. A way out of this difficulty has been proposed in ref. [12]. It consists in replacing  $W_N$  algebras by some associate infinite-dimensional linear algebras  $W_N^\infty$  which appear if one treats as new independent generators all the higher-spin composite generators present in the enveloping algebra of the basic  $W_N$  generators (these are the spin 2 and 3 generators in the  $W_3$  case, the spin 2,3 and 4 ones in the  $W_4$  case, etc).  $W_N$  symmetries can then be viewed as particular realizations of linear  $W_N^\infty$  symmetries. To the latter, one may apply the entire arsenal of the coset realization techniques. The authors of ref.[12] have constructed a coset realization of the product of two light-cone copies of  $W_3^\infty$  and have shown that after imposing an infinite number of the inverse Higgs [16] type covariant constraints on the relevant Cartan forms (this is called the covariant reduction of a given coset [12, 14]), one is left with the realization of  $W_3^\infty$  on two scalar 2D fields which coincides with the well-known  $sl_3$  Toda realization of  $W_3$  [4, 10]. As a consequence of the covariant reduction constraints, the scalar fields turned out to satisfy the  $sl_3$  Toda lattice field equations (or their free version, depending on the choice of the vacuum stability subgroup) which thus had proven to be intimately related to the intrinsic geometry of  $W_3^\infty$ . An analogous treatment of the  $sl_2$  Toda system (Liouville theory) in the framework of much simpler nonlinear realization of  $W_2$  (Virasoro) symmetry has been earlier given in [14]<sup>1</sup>.

<sup>1</sup> $W_2$  symmetry is linear, so there is no direct necessity to pass to something like  $W_2^\infty$  while constructing its nonlinear realization. However, this necessity comes out if one tries to understand the KdV hierarchy in the nonlinear realization approach [17].



In the present paper we construct coset realizations of  $W_3^\infty$  (its one copy) which are *different* from those found in [12]. We demonstrate that there exists a set of the covariant reduction constraints which reduces the number of independent coset parameters-fields to the two fields of conformal spins 2 and 3 identified with the currents of  $W_3$  and simultaneously gives rise to the Boussinesq equation for these fields. Both the spatial ( $x$ ) and evolution ( $t$ ) coordinates of these fields naturally appear as the parameters of the coset considered. The generator to which  $t$  is attached coincides with the Hamiltonian used in the standard Hamiltonian approach to the Boussinesq equation, thus establishing a link between the second Hamiltonian structure for this equation [6] and our geometric approach. The Miura map for the Boussinesq equation gets also a simple geometric interpretation. One enlarges the coset by adding two spin 1 generators from the stability subgroup and then imposes additional covariant constraints which covariantly express the spin 2 and spin 3 coset fields in terms of the two new spin 1 fields. The resulting expressions are just the Miura transformations relating the Boussinesq equation to a "modified" Boussinesq equation. Thus the Miura map arises as a manifestly covariant relation between parameters of a coset of  $W_3^\infty$ . Quite analogously one may construct further Miura map onto the two spin 0 fields which leads to the Feigin-Fuchs type representation for the spin 2 and spin 3 fields. One should transfer two spin 0 generators from the stability group to the coset (the remaining generators still form a subalgebra) and impose appropriate inverse Higgs constraints. The zero-curvature representation for the ordinary and modified Boussinesq equations, as well as the relevant matrix Lax pairs, appear very naturally in this picture, basically as the Maurer-Cartan equations for the reduced cosets. The  $W_3$  symmetry of the Boussinesq equations established recently in [18] also immediately follows, it is recognized as left  $W_3^\infty$  shifts on the relevant coset manifolds. The higher-order equations from the Boussinesq hierarchy can be produced by considering more general cosets of  $W_3^\infty$  with additional evolution parameters associated with the higher-spin generators.

The paper is organized as follows.

In Sect.2 we consider a toy example of the nonlinear realizations of Virasoro ( $W_2$ ) symmetry and show that some essential features of the entire construction can be seen already in this simplified situation. In particular, the holomorphic component of the conformal stress-tensor comes out as a parameter of some coset manifold of  $W_2$ , the Miura maps amount to covariant constraints on the coset parameters, etc.

In Sect.3 we recapitulate the basic facts about the linear algebra  $W_3^\infty$  following ref. [12] and list its some subalgebras which are utilized while constructing coset realizations of  $W_3^\infty$  symmetry in the next Section.

In Sect.4 we construct three coset realizations of  $W_3^\infty$  and give necessary technical details (parametrizations of the coset elements, Cartan forms).

Sect.5 is the central, there we apply the covariant reduction procedure to the cosets constructed in Sect.4 and show that it results in expressing an infinite tower of the coset parameters-fields in terms of a few essential ones: either the spin 2 and spin 3 fields  $u$ ,  $v$ , or two spin 1 fields  $u_1$ ,  $v_1$ , or two spin 0 fields  $u_0$ ,  $v_0$ . Simultaneously one obtains dynamical equations for these fields, namely the Boussinesq equation and two types of the modified Boussinesq equations. We explain the geometric meaning of the appropriate Miura maps and zero-curvature representations and make contact with the Hamiltonian

formulation.

In Sect.6 we study transformation properties of the Boussinesq equation under left shifts of  $W_3^\infty$  on the original coset elements. We show that in the realization on the spin 2 and spin 3 fields, the  $W_3^\infty$  transformations constitute the  $W_3$  symmetry of the Boussinesq equation revealed in a recent paper [18].

## 2 A sketch of nonlinear realizations of Virasoro symmetry

For reader's convenience and to make more clear what kind of nonlinear realizations of  $W_3$  symmetry we are going to construct, we first dwell on a simpler case of Virasoro ( $W_2$ ) symmetry. We will demonstrate here that the holomorphic component of the conformal stress-tensor can be treated as the coset space parameter corresponding to a kind of the coset realization of one copy of this symmetry. The Miura map and the Feigin-Fuchs representation for this component naturally appear in the framework of some extended coset spaces of  $W_2$  as  $W_2$ -covariant constraints on the coset parameters.

We restrict our study, like in ref.[14], to the truncated  $W_2$  formed by the generators

$$W_2 = \{L_{-1}, L_0, L_1, \dots, L_n, \dots\} \quad n \geq -1; \quad [L_n, L_m] = (n-m)L_{n+m}. \quad (2.1)$$

In what follows we will denote by  $W_2$  just this algebra and, depending on the context, use the same term for the corresponding group of transformations.

As was observed in ref. [14], the standard realization of  $W_2$  as conformal transformations of  $R^1$  (or  $S^1$ ) coordinate  $x$  can be easily re-derived within the framework of coset realization method. It is induced by a left action of the group associated with the algebra (2.1) on the one-dimensional coset over the subgroup generated by

$$L_0, L_1, \dots, L_n, \dots; \quad n \geq 0. \quad (2.2)$$

Namely, parametrizing an element of this coset as

$$g(x) = e^{xL_{-1}}$$

and defining the group action on it following the standard rules of ref. [13]

$$g_0 g(x) = g(x') h(x, g_0), \quad g_0 = \exp \left( \sum_{n \geq -1} \lambda_n L_n \right), \quad (2.3)$$

where  $h$  is some induced transformation from the stability subgroup, one obtains for  $x$  the standard conformal transformations (regular at the origin)

$$\delta x \equiv \lambda(x) = \sum_{n \geq -1} \lambda_n (x)^{n+1}. \quad (2.4)$$

The above coset is not reductive in the sense that the coset generator  $L_{-1}$  is rotated by the stability group into the generators of the latter and this causes some difficulties in

applying the standard techniques of ref. [13] to the present case. The simplest reductive coset manifold of  $W_2$  is obtained by treating *all* generators (2.1) as the coset ones:

$$g(x) \Rightarrow \tilde{g}(x) = e^{xL_{-1}} \cdot \left( \prod_{n \geq 3} e^{u_n(x)L_n} \right) \cdot e^{u_1(x)L_1} e^{u(x)L_2} e^{u_0(x)L_0}, \quad (2.5)$$

where the coset parameters are regarded as fields given on the line manifold  $x$ , i.e. as a kind of goldstone fields, and the special arrangement of factors has been chosen for further convenience. Under the left  $W_2$  shifts the coordinate  $x$  as before transforms according to the rule (2.4), while the coset parameters-fields transform through themselves and the function  $\lambda(x)$ . For instance,

$$\begin{aligned} \delta u_0 &= -\lambda u'_0 + \lambda' \\ \delta u_1 &= -\lambda u'_1 - \lambda' u_1 + \frac{1}{2} \lambda'' \\ \delta u &= -\lambda u' - 2\lambda' u + \frac{1}{6} \lambda''', \text{ etc.} \end{aligned} \quad (2.6)$$

We observe that  $u_0(x)$  transforms as a  $2D$  dilaton, while  $u(x)$  as a holomorphic component of conformal stress-tensor. To see that the latter property is not accidental, let us look at the structure of the Cartan forms for the nonlinear realization in question.

The Cartan forms are introduced as usual by

$$\tilde{g}^{-1} d\tilde{g} = \sum_{n \geq -1} \omega_n L_n \quad (2.7)$$

and are invariant by construction under the left action of  $W_2$  symmetry. They can be easily evaluated using the commutation relations (2.1). A few first ones are as follows

$$\begin{aligned} \omega_{-1} &= e^{-u_0} dx \\ \omega_0 &= (u'_0 - 2u_1) dx \\ \omega_1 &= e^{u_0} (u'_1 + (u_1)^2 - 3u_2) dx. \end{aligned} \quad (2.8)$$

Note that the higher-order forms, like  $\omega_0$  and  $\omega_1$ , contain the pieces linear in the relevant parameters-fields (beginning with  $u_3$ ). Now, keeping in mind invariance of these forms, one may impose the manifestly covariant inverse Higgs type [16] constraints

$$\omega_n = 0, \quad \forall n \geq 0, \quad (2.9)$$

which can be looked upon as algebraic equations for expressing the parameters-fields  $u_1, u, u_n$  ( $n \geq 3$ ) in terms of  $u_0$  and its derivatives in a way compatible with the transformation properties (2.6). Thus  $u_0(x)$  is the only essential coset parameter-field in the present case. Using eqs.(2.8) one finds the coset fields  $u_1$  and  $u$  to be expressed by

$$u_1 = \frac{1}{2} u'_0, \quad u = \frac{1}{6} \left[ \frac{1}{2} (u'_0)^2 + u''_0 \right]. \quad (2.10)$$

We see that  $u$  indeed has the standard form of the conformal stress-tensor for the single scalar field in the Feigin-Fuchs representation (an arbitrary parameter that is usually

present in front of the Feigin-Fuchs term can be attributed to a rescaling of  $u_0$ ). Thus we have succeeded in deducing the Feigin-Fuchs representation for the stress-tensor as a covariant relation between the parameters of certain coset manifold of  $W_2$  symmetry.

The above coset realization of  $W_2$  is not unique, there exist other ones, with less trivial stability subgroups.

The first possibility is to factorize over a one-dimensional subgroup with the generator  $L_0$

$$\mathcal{H}_1 = \{L_0\}. \quad (2.11)$$

The relevant coset element and Cartan forms are obtained simply by putting  $u_0 = 0$  in eqs. (2.5), (2.7) and (2.8). The set of Cartan forms now consists of those living in the coset ( $\omega_{-1}, \omega_n$   $n \geq 1$ ) and in the stability subalgebra ( $\omega_0 = -2u_1 dx$ ). The coset forms now undergo homogeneous  $L_0$  rotations while  $\omega_0$  transforms inhomogeneously. The only essential coset field in this realization is  $u_1$ ; all others are expressed in terms of it via the covariant constraints

$$\omega_n = 0, \quad \forall n \geq 1. \quad (2.12)$$

For  $u$  one obtains now the representation

$$u = \frac{1}{3} [(u_1)^2 + u'_1], \quad (2.13)$$

which is known as the "Miura map" for the stress-tensor ( $u_1$  is interpreted as a  $U(1)$  Kac-Moody current). Thus the Miura map also naturally appears in the nonlinear realization approach to  $W_2$  as a covariant relation between the parameters of the coset of  $W_2$  over the subgroup with the algebra (2.11).

Finally, one may extend the stability group algebra by including the generator  $L_1$

$$\mathcal{H} = \{L_0, L_1\} \quad (2.14)$$

(further extension is impossible since adding, e.g., the generator  $L_2$  would immediately entail adding an infinite set of  $W_2$  generators and we would return to the non-reductive case discussed in the beginning of this Sect.). The set of the associate coset fields starts with  $u$  which is independent in this realization. All higher-order fields are expressed through  $u$  by the constraints

$$\omega_n = 0 \quad \forall n \geq 2, \quad (2.15)$$

which are still closed under the left action of  $W_2$ .

A few words are of need regarding the geometric meaning of the procedure of eliminating higher-order coset fields in the above examples. In all cases, after imposing the inverse Higgs constraints, we are left with the Cartan forms on the stability subalgebra and the form  $\omega_{-1}$ . The associate generators form subalgebras which are particular cases of what was called "the covariant reduction subalgebra" in ref. [12, 14, 15]. These are the one-generator subalgebra  $\{L_{-1}\}$  in the first example, the two-generator subalgebra  $\{L_{-1}, L_0\}$  in the second example and the algebra  $sl(3, R) = \{L_{-1}, L_0, L_1\}$  in the third example. The coordinate  $x$  parametrizes the one-dimensional cosets of the corresponding subgroups ("covariant reduction subgroups" in the terminology of ref. [12, 14, 15]) over the stability subgroups while the surviving coset fields ( $u_0(x)$ ,  $u_1(x)$  and  $u(x)$ , respectively) together with their derivatives of any order specify embedding of these curves

into the original infinite-dimensional cosets of  $W_2$ . As was argued in [19] on a simple finite-dimensional example, such curves (and two-dimensional hypersurfaces in the cases considered in ref. [12, 14] and Section 5 of the present paper) form fully geodesic submanifolds in the original coset manifolds. Thus the role of eqs. (2.9), (2.12), (2.15) is to single out one-dimensional geodesic submanifolds in the cosets of  $W_2$ .

The main points one learns from the above discussion are as follows:

i. The conformal stress-tensor  $u(x)$  as well as the 2D dilaton field  $u_0(x)$  and the  $U(1)$  Kac-Moody current  $u_1(x)$  can be given a nice geometric interpretation as the parameters of coset manifolds of the truncated Virasoro ( $W_2$ ) symmetry (2.1).

ii. The free-field Feigin-Fuchs type representation and Miura map for the stress-tensor appear in a geometric way as covariant constraints on the parameters of these cosets. These serve to single out geodesic curves in the original manifolds.

In the examples considered here the inverse Higgs constraints are purely kinematic, they do not imply any dynamics for the involved fields. To gain a dynamics, one should arrange the fields to depend, besides  $x$ , on an evolution parameter, a time coordinate. One way to do this is to add one more copy of  $W_2$  and to interpret the coset parameters associated with the two commuting generators  $L_{-1}$  as the light-cone 2D Minkowski coordinates. A nonlinear realization of such a symmetry generalizing first of the examples considered here has been studied in ref. [14]. It has been shown that the relevant inverse Higgs constraints not only serve to eliminate higher-order coset fields in terms of the 2D dilaton, but also give rise to the dynamical equations for the latter, in particular to the Liouville equation. Such a version of the inverse Higgs procedure has been called "the covariant reduction". An analogous construction for two light-cone copies of  $W_3$  symmetry has been given in ref.[12] and it has led to a new geometric interpretation of the  $sl_3$  Toda theory within the covariant reduction approach. In the next Sections we shall demonstrate that there exists another way to incorporate the evolution parameter into a nonlinear realization of  $W_3$  symmetry without doubling the algebra. The relevant cosets of  $W_3$  are a more or less direct generalization of those considered here. However, due to the presence of the time coordinate, it will turn out to be possible to obtain dynamical equations for the coset fields from the covariant reduction procedure: the Boussinesq and modified Boussinesq equations. But before discussing this we need to recall how to apply the notions of nonlinear realizations to the algebra  $W_3$ . As was shown in ref.[12], the only conceivable way to do this is to deal, instead of  $W_3$ , with some infinite-dimensional linear algebra  $W_3^\infty$  closely related to  $W_3$ .

### 3 $W_3^\infty$ and its subalgebras

In this Section we shall briefly recall salient features of the algebra  $W_3^\infty$  and its relation to  $W_3$ , closely following ref.[12]. To avoid a possible confusion, we point out that, like in [12], we start with the most general classical  $W_3$  algebra possessing an arbitrary central charge. Its commutation relations can be found, e.g., in ref.[6] (see eqs. (3.1) and (3.2) below).

The central idea invoked in [12] is to construct a linear algebra  $W_3^\infty$  from the nonlinear  $W_3$  by treating as independent all the higher-spin composite generators which appear while

considering successive commutators of the basic (spin 2 and 3)  $W_3$  generators.

Let us consider the defining relations of the classical  $W_3$  algebra [6]<sup>2</sup>

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \\ [L_n, J_m] &= (2n-m)J_{n+m} \\ [J_n, J_m] &= 16(n-m)J_{n+m}^{(4)} - \frac{8}{3}(n-m)\left(n^2+m^2 - \frac{1}{2}nm - 4\right)L_{n+m} - \\ &\quad - \frac{c}{9}(n^2-4)(n^2-1)n\delta_{n+m,0} \quad , \end{aligned} \quad (3.1)$$

where

$$J_n^{(4)} = -\frac{8}{c} \sum_m L_{n-m} L_m. \quad (3.2)$$

The structure relations of  $W_3^\infty$  are then defined as the full set of commutators between  $L_n$ ,  $J_n$ ,  $J_n^{(4)}$  and all higher-spin composites  $J_n^{(s)}$  ( $s \geq 5$ ) which come out in the commutators of lower-spin generators. All the composite generators, beginning with  $J_n^{(4)}$ , are treated as new independent ones. Thus  $W_3^\infty$  is formed by an infinite set of generators

$$L_n, J_n, J_n^{(4)}, J_n^{(s)}, \quad s \geq 5.$$

It should be pointed out that the full set of commutation relations of  $W_3^\infty$ , as is clear from the above definition, can be entirely deduced from the basic  $W_3$  relations (3.1), (3.2). For our purposes here it will be of no need to know the detailed structure of these commutation relations.

It will be of importance that the central charge  $c$  is non-zero in (3.1). The presence of this parameter strongly influences the structure of  $W_3^\infty$ . For example, the commutation relations of the basic generators  $L_m$ ,  $J_m$  with some (of the spins 4, 5 and 6) composite generators ( $\sim L_n L_m$ ,  $L_n J_m$ ,  $J_n J_m$ ) contain in the r.h.s., apart from the composite generators, also basic generators which appear just due to non-zero  $c$  in (3.1). In what follows the presence of such terms in the commutation relations will be very important (cf. [12]) and this is the main reason why we should keep  $c$  non-vanishing in (3.1)<sup>3</sup>. As an example we quote the commutator

$$[L_n, J_m^{(4)}] = (3n-m)J_{n+m}^{(4)} - \frac{4}{3}(n^3-n)L_{n+m}. \quad (3.3)$$

The second term in the r.h.s of (3.3) is owing to the presence of non-zero  $c$  in (3.1).

Just as in [12], while constructing a nonlinear realization of  $W_3^\infty$ , we will deal not with the whole algebra, but with its important subalgebra which contains all the spin  $s$  generators ( $s \geq 2$ ) with the indices varying from  $-(s-1)$  to  $\infty$  (this subalgebra is

<sup>2</sup>For correspondence with our forthcoming work on  $N=2$  super  $W_3$  algebra [20] we use a bit different normalizations of the spin 3 and spin 4 generators  $J_n$  and  $J_n^{(4)}$  compared to those used in [12]. These are the same as in ref. [21]

<sup>3</sup>One can regard  $c$  as a contraction parameter. After rescaling  $J_n \equiv c^{-\frac{1}{2}} \tilde{J}_n$ ,  $J_n^{(4)} \equiv c^{-1} \tilde{J}_n^{(4)}$ ,  $c$  can be put equal to zero. In this limit (3.1) contracts into the commutation relations of the "classical  $W_3$  algebra" of ref. [22].

a genuine generalization of the "truncated" Virasoro algebra, eq. (1.1)). If one thinks of  $W_3^\infty$  as an algebra of some  $2D$  field variations with holomorphic parameters (e.g., in the realization given in [12]), the above subalgebra corresponds to restricting to the parameters-functions regular at the origin. To simplify the terminology, in what follows just this truncated algebra will generally be referred to as  $W_3^\infty$ . We wish to point out that the higher-spin generators of this algebra, when treated as composite, still involve the basic generators with *all* conformal dimensions, both positive and negative. For instance, in eq. (3.2) one restricts the index  $n$  to vary in the range  $n \geq -3$ , however the summation still goes over the whole range  $-\infty < m < \infty$ .

In the rest of this Section we list some subalgebras of the truncated  $W_3^\infty$  algebra which will be employed in our further discussion. The proof of closeness of the relevant sets of generators in most of the cases goes by a direct inspection and essentially relies upon the property that all higher-spin generators in  $W_3^\infty$  (with spins  $\geq 4$ ) form an invariant subalgebra [12].

The reflection symmetry  $n \rightarrow -n$  of the original relations (3.1) implies the existence of a wedge subalgebra  $W_\wedge$  in  $W_3^\infty$  [12]. It is constituted by an infinite number of generators, with the indices varying from  $-(s-1)$  to  $(s-1)$  for each spin  $s$ <sup>4</sup>

$$W_\wedge = \left\{ \begin{array}{cccccccc} & & L_{-1} & L_0 & L_1 & & & \\ & & J_{-2} & J_{-1} & J_0 & J_1 & J_2 & \\ J_{-3}^{(4)} & J_{-2}^{(4)} & J_{-1}^{(4)} & J_0^{(4)} & J_1^{(4)} & J_2^{(4)} & J_3^{(4)} & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array} \right\} \quad (3.4)$$

(dots mean higher-spin generators with proper indices). An interesting factor-algebra of this wedge algebra is the  $sl(3, R)$  given by:

$$W_\wedge / \{J_{-3}^{(4)}, \dots, J_3^{(4)}, \dots\} \sim sl(3, R) \quad (3.5)$$

One more important subalgebra of  $W_3^\infty$  is constituted by the following generators

$$\mathcal{G} = \{J_{-2}, J_{-1}, L_{-1}, L_0, J_0, L_1, J_1, J_2, J_n^{(4)} (n \geq -3), J_n^{(s)} (s \geq 5, n \geq -s+1)\} \quad (3.6)$$

It is the maximal subalgebra of the truncated  $W_3^\infty$ . We see that it is obtained by adding to the wedge algebra (3.4) an infinite rest of the higher-spin generators with all positive conformal dimensions. All these generators still form an ideal and the factor-algebra over this ideal coincides with the  $sl(3, R)$  (3.5).

One may narrow the subalgebra (3.6) successively removing from it some generators. It is easy to check that the sets of generators

$$\mathcal{H} = \{J_{-1} + 2L_{-1}, L_0, J_0, L_1, J_1, J_2, J_n^{(4)} (n \geq -3), J_n^{(s)} (s \geq 5, n \geq -s+1)\} \quad (3.7)$$

$$\mathcal{H}_1 = \{J_{-1} + 2L_{-1}, L_0, J_0, J_n^{(4)} (n \geq -3), J_n^{(s)} (s \geq 5, n \geq -s+1)\} \quad (3.8)$$

$$\mathcal{H}_2 = \{J_{-1} + 2L_{-1}, J_n^{(4)} (n \geq -3), J_n^{(s)} (s \geq 5, n \geq -s+1)\} \quad (3.9)$$

<sup>4</sup>The precise relation of  $W_3^\infty$  and this wedge algebra to the  $W_\infty$ -type algebras and their wedge subalgebras (see, e.g. [23]) is not quite clear to us at present. For instance,  $W_\infty$  is known to contain each spin only once while this is not true for the algebras  $W_N^\infty$ . It is likely that  $W_\infty$  can be obtained as a  $N \rightarrow \infty$  limit and truncation of  $W_N^\infty$ .

still form subalgebras:

$$\mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{G} \quad (3.10)$$

Note that  $\mathcal{H}_1$  and  $\mathcal{H}_2$  can be extended by adding the generator  $J_{-2}$

$$\mathcal{H}'_1 = \mathcal{H}_1 \oplus J_{-2}, \quad \mathcal{H}'_2 = \mathcal{H}_2 \oplus J_{-2} \quad (3.11)$$

The algebra  $\mathcal{H}'_2$  has been already utilized as the stability subgroup algebra in the Toda type nonlinear realization of  $W_3^\infty$  [12] (under the restriction to one of two light-cone copies of  $W_3^\infty$  considered in [12], with the second  $2D$  light-cone coordinate regarded as an extra "time"). The algebras  $\mathcal{H}$ ,  $\mathcal{H}_1$  and  $\mathcal{H}_2$  will serve as the stability subgroup algebras in new nonlinear realizations of  $W_3^\infty$  we will construct in the next Section.

## 4 Nonlinear realizations of $W_3^\infty$

As has been argued in ref. [12], extending the nonlinear realization method to the  $W_N$  type symmetries implies replacing the latter by symmetries based on the linear algebras  $W_N^\infty$  and then constructing coset realizations of these  $W_N^\infty$  symmetries according to the general prescriptions of ref. [13]. Once this is done, the original  $W_N$  symmetry is expected to emerge as a particular field realization of  $W_N^\infty$ . So a nonlinear (coset) realization of  $W_3$  should always be understood as that of  $W_3^\infty$ . Just this point of view has been put forward in ref. [12] and we will pursue it here. In this way in [12] the  $sl_3$  Toda realization of  $W_3$  symmetry has been reproduced starting from a nonlinear realization of the product of two light-cone copies of  $W_3^\infty$  symmetries. Here we construct a set of nonlinear realizations of one  $W_3^\infty$ . In Sect. 6 we will prove that they also amount to some specific field realizations of the  $W_3$  algebra (3.1), (3.2). In the next Section we will show that these realizations bear a deep relation to the Boussinesq equation and Miura maps for the latter.

Any nonlinear realization is quite specified by the choice of the stability subgroup  $H$  or, equivalently, its subalgebra  $\mathcal{H}$ . So in the case at hand one should start by fixing the appropriate  $\mathcal{H} \subset W_3^\infty$ . Like in [12] we will always place the entire set of higher-spin generators (starting with the spin 4) in the stability subalgebra and consider first a nonlinear realization of  $W_3^\infty$  with (3.7) as such a subalgebra. This choice can be motivated by the following reasonings. In Sect. 2 we have learned that the spin 2 conformal stress-tensor  $u(x)$  can be interpreted as the essential coset field of the nonlinear realization of Virasoro symmetry  $W_2$  with the stability subgroup corresponding to the algebra (2.14). There arises a natural question whether it is possible to give an analogous coset interpretation to both spin 2 and spin 3 currents associated with the  $W_3$  algebra. Clearly, the appropriate stability subalgebra of  $W_3^\infty$  should become (2.14) upon reducing  $W_3 \rightarrow W_2$ , i.e. taking away all generators except for the  $W_2$  ones. The algebra (3.7) just satisfies this criterion.

Inspecting the set of the  $W_3^\infty$  generators which are out of (3.7), i.e. belong to the coset, we find that the lowest dimension ones are  $J_{-2}$  ( $cm^{-2}$ ),  $L_{-1}$  ( $cm^{-1}$ ),  $L_2$  ( $cm^2$ ) and  $J_3$  ( $cm^3$ ). With the generator  $L_{-1}$ , like in the  $W_2$  case, it is natural to associate the coordinate  $x$ . The last two generators have true dimensions for identifying the associate coset parameters, respectively  $u$  and  $v$ , with the spin 2 and spin 3 currents. All higher-order coset generators have growing negative dimensions so the corresponding parameters-fields are expected to be expressible in terms of  $u$  and  $v$  by the inverse Higgs effect. But there still remains

the generator  $J_{-2}$ . The dimension of the coset parameter related to it is inappropriate for treating this parameter as a field. On the other hand, one cannot put  $J_{-2}$  into the stability subgroup as its commutator with, e.g.,  $J_1$  yields the coset generator  $L_{-1}$  in the r.h.s. Thus, the only possibility one may conceive is to treat this parameter as an additional coordinate, we call it  $t$ , in parallel with  $x$  and to allow all coset fields to depend on it. One observes that  $t$  has the same dimension  $cm^2$  as the evolution parameter of the Boussinesq equation [6], so in what follows we will refer to it as to the "time" coordinate. Note that the interpretation of  $x$  and  $t$  as the coordinates parametrizing the "spatial" and "temporal" directions is quite natural from the physical point of view, for the translations along these directions are entirely independent as a consequence of commutativity of the generators  $L_{-1}$  and  $J_{-2}$ .

With all these remarks taken into account, an element of the coset space we are considering can be parametrized as follows:

$$g = e^{tJ_{-2}} e^{xL_{-1}} e^{\psi_3 L_3} \cdot \left( \prod_{n \geq 4} e^{\psi_n L_n} e^{\xi_n J_n} \right) \cdot e^{uL_2} e^{vJ_3} \quad (4.1)$$

As usual in nonlinear realizations, the group  $G$  (associated with  $W_3^\infty$  in the present case) acts as left multiplications of the coset element. This induces a group motion on the coset: the coordinates  $x, t$  together with the infinite tower of coset fields  $u(x, t), v(x, t), \psi_n(x, t), \xi_n(x, t)$  constitute a closed set under the group action. We are postponing the discussion of all symmetries induced on the coset parameters in this way to the Section 6 (their number is infinite in accordance with the infinite dimensionality of  $W_3^\infty$ ).

Besides the above minimal coset we will need also extended cosets with the stability subgroups generated by subalgebras  $\mathcal{H}_1$  and  $\mathcal{H}_2$  defined in eqs. (3.8), (3.9). The relevant coset elements are represented by

$$g_1 = ge^{u_1 L_1} e^{v_1 J_1} e^{v_2 J_2} \quad (4.2)$$

$$g_2 = g_1 e^{u_0 L_0} e^{v_0 J_0}, \quad (4.3)$$

where  $u_1, v_1, v_2, u_0, v_0$  are additional parameters-fields, all given on the space  $\{x, t\}$ . It is worth mentioning that the subalgebra (3.8) and the associated nonlinear realization of  $W_3^\infty$  generalize the subalgebra (2.11) and the realization of  $W_2$  related to this choice. In the realization of  $W_3^\infty$  on elements (4.3) all Virasoro generators belong to the coset, so it is an extension of the realization (2.5).

Let us now turn to constructing Cartan forms for the above cosets. Like in the  $W_2$  case these are defined by the generic relation

$$\Omega \equiv g^{-1} dg = \sum_{n \geq -1} \omega_n L_n + \sum_{n \geq -2} \theta_n J_n + \text{Higher-spin contributions}, \quad (4.4)$$

and by the analogous ones for two other cases, with  $g$  replaced by  $g_1$  or  $g_2$ .

The explicit expressions for the forms which we actually need can be obtained by using solely the commutation relations (3.1) and (3.3). For the realization (4.1) these are

$$\begin{aligned} \omega_0 &= 0, \quad \omega_1 = 160vdt - 3udx \\ \theta_{-1} &= 0, \quad \theta_0 = -6udt, \quad \theta_1 = -8\psi_3 dt, \quad \theta_2 = 12u^2 dt - (5vdx + 10\psi_4 dt) \end{aligned} \quad (4.5)$$

$$\begin{aligned} \omega_{-1} &= dx \\ \omega_2 &= du + 320\xi_4 dt - 4\psi_3 dx \\ \omega_3 &= d\psi_3 + \left(\frac{3}{2}u^2 - 5\psi_4\right) dx + (560\xi_5 - 240uv) dt \\ \omega_4 &= d\psi_4 - 6(\psi_5 - u\psi_3) dx + (896\xi_6 - 320v\psi_3 + 6384u\xi_4) dt \end{aligned} \quad (4.6)$$

$$\begin{aligned} \theta_{-2} &= dt \\ \theta_3 &= dv - 6\xi_4 dx + (36u\psi_3 - 12\psi_5) dt \\ \theta_4 &= d\xi_4 - (7\xi_5 - 2vu) dx + (20\psi_3^3 + 48u\psi_4 + 80v^2 + 8u^3 - 14\psi_6) dt. \end{aligned} \quad (4.7)$$

The Cartan forms (4.5) belong to the stability subalgebra and transform inhomogeneously under left  $W_3^\infty$  shifts. The forms (4.6), (4.7) are associated with the coset generators, so they are transformed into themselves and other coset forms.

In the realization corresponding to the coset (4.2) some Cartan forms on the stability subalgebra get contributions from the new coset fields  $u_1, v_1, v_2$  and, besides, the Cartan forms  $\omega_1, \theta_1, \theta_2$  become belonging to the coset. Due to the special arrangement of factors in (4.2) the forms associated with other coset generators are linear combinations of the previous coset forms with the coefficients depending on the new coset fields. We give here the explicit expressions only for the new coset forms

$$\omega_1 = du_1 + (u_1^2 + 12v_1^2 - 3u) dx + (160v + 48uv_1 - 48v_1u_1^2 - 64u_1v_2 + 64v_1^3) dt \quad (4.8)$$

$$\theta_1 = dv_1 + (2u_1v_1 - 4v_2) dx + (12uu_1 - 8\psi_3 - 4u_1^3 - 64v_1v_2 - 16u_1v_1^2) dt \quad (4.9)$$

$$\begin{aligned} \theta_2 &= dv_2 + du_1v_1 + (4v_1^3 + 4u_1v_2 + u_1^2v_1 - 3uv_1 - 5v) dx + (12u^2 - 10\psi_4 + u_1^4 \\ &\quad + 16v_1^4 - 64v_2^2 - 64u_1v_1v_2 - 24u_1^2v_1^2 - 6uu_1^2 + 24uv_1^2 + 160vv_1 + 8u_1\psi_3). \end{aligned} \quad (4.10)$$

Finally, the net effect of passing to the coset (4.3) is homogeneous rotations of the previously defined forms by the group factors with generators  $L_0$  and  $J_0$  and extension of the set of coset forms by  $\omega_0$  and  $\theta_0$ . The latter forms are given by

$$\begin{aligned} \omega_0 &= du_0 + (32u_1v_1 + 64v_2) dt - 2u_1 dx \\ \theta_0 &= dv_0 + (6u_1^2 - 24v_1^2 - 6u) dt - 3v_1 dx. \end{aligned} \quad (4.11)$$

As the last remark we mention that the form  $\Omega$  (4.4) (and its analogs for the cosets (4.2) and (4.3)  $\Omega_1$  and  $\Omega_2$ ) by definition satisfies the Maurer-Cartan equation

$$d^{ext} \Omega = \Omega \wedge \Omega. \quad (4.12)$$

## 5 Boussinesq equations and Miura maps from covariant reduction of the cosets of $W_3^\infty$

Here we generalize to the  $W_3^\infty$  cosets the inverse Higgs procedure (alias covariant reduction) which has been already applied in Sect.2 to simpler examples of nonlinear realizations of Virasoro symmetry. A new feature of this procedure in the case at hand is that it will

ead not only to the kinematic equations for expressing higher-order coset fields in terms of a few essential ones but also to the dynamical equations for the latter. This is directly related to the presence of the extra time coordinate  $t$ .

We begin with the coset (4.1). A natural generalization of the constraints (2.15) is as follows

$$\omega_n = 0, \quad \forall_{n \geq 2} \quad \theta_m = 0, \quad \forall_{m \geq 3}. \quad (5.1)$$

Upon imposing these constraints, the one-form  $\Omega$  (4.4) defined originally on the entire algebra  $W_3^\infty$  is reduced to the one-form valued in the algebra  $\mathcal{G}$  (3.6)

$$\Omega \Rightarrow \Omega^{red} \subset \mathcal{G}. \quad (5.2)$$

In accordance with the terminology explained in Sect.2,  $\mathcal{G}$  is the covariant reduction subalgebra in the present case. Taking into account that  $\mathcal{H}$  coincides with the  $sl(3, R)$  (3.5) modulo higher-spin generators, one may, without loss of generality, regard just this  $sl(3, R)$  as the reduction subalgebra and consider only the  $sl(3, R)$  part of  $\Omega^{red}$ . This part obeys the Maurer-Cartan equation (4.12) in its own right, without any contributions from the higher-spin generators. We will make use of this observation a bit later.

Let us now inspect eqs. (5.1). As opposed to the  $W_2$  constraints (2.15), each of eqs. (5.1) actually produces two equations, for the coefficients of the differentials  $dx$  and  $dt$ . Using the explicit structure of the lowest coset forms, eqs. (4.6) and (4.7), one finds

$$\begin{aligned} \psi_3 &= \frac{1}{4}u' \\ \psi_4 &= \frac{1}{5}\left(\frac{3}{2}u^2 + \frac{1}{4}u''\right) \\ \psi_5 &= \frac{1}{60}(21uu' + \frac{1}{2}u''') = \frac{1}{12}(\dot{v} + 9u\dot{u}) \end{aligned} \quad (5.3)$$

$$\xi_4 = -\frac{1}{320}\dot{u} = \frac{1}{6}v' \text{ etc.} \quad (5.4)$$

We see that the higher-order coset fields are expressed by the inverse Higgs constraints (5.1) in terms of two independent ones,  $u(x, t)$  and  $v(x, t)$ , thus generalizing an analogous phenomenon of the  $W_2$  case. However, for all coset fields, except  $\psi_3, \psi_4$  there simultaneously appear two expressions coming from equating to zero the coefficients of the differentials  $dx$  and  $dt$  in the appropriate forms. Requiring these expressions to be compatible amounts to the set of dynamical equations

$$\begin{aligned} \dot{u} &= -\frac{160}{3}v' \\ \dot{v} &= \frac{1}{10}u''' - \frac{24}{5}u'u, \end{aligned} \quad (5.5)$$

which is recognized as the Boussinesq equation [6] (after appropriate rescalings). Its another, second-order form is obtained by differentiating the first equation in (5.5) with respect to  $t$  and then using the second equation

$$\ddot{u} = -\frac{16}{3}u'''' + 128(u^2)'' \quad (5.6)$$

With making use of the Maurer-Cartan equation (4.12) one may show that the rest of constraints (5.1) does not imply any further dynamical restrictions on the fields  $u, v$  and serves only for the covariant elimination of higher-order coset fields.

Thus we have succeeded in deducing the Boussinesq equation from a nonlinear realization of  $W_3^\infty$  like it has been done in [12] for the  $sl_3$  Toda equations (starting with a nonlinear realization of two copies of  $W_3^\infty$ ). This shows a close relation of the Boussinesq equation to the intrinsic geometry of  $W_3^\infty$ : it reveals a nice geometric meaning as one of the constraints singling out a finite-dimensional geodesic hypersurface in the coset of  $W_3^\infty$  over the subgroup with the algebra  $\mathcal{H}$  (3.9). This hypersurface is homeomorphic to the two-dimensional coset of the group with the algebra  $\mathcal{G}$  (3.6) over the subgroup with the algebra  $\mathcal{H}$  (3.9). Taking account of the fact that the higher-spin generators drop out after such a factorization, this coset coincides with that of the group  $SL(3, R)$  with the generators (3.5) over its six-parameter Borel subgroup generated by  $\{J_{-1} + 2L_{-1}, J_0, L_0, L_1, J_1, J_2\}$ . The coordinates  $x$  and  $t$  parametrize this coset while the fields  $u$  and  $v$  describe the embedding of it as a hypersurface in the original coset space of  $W_3^\infty$ .

The Boussinesq equation is known to be completely integrable: it possesses a zero-curvature representation and the related Lax pair on the algebra  $sl(3, R)$  [9, 18]. It is instructive to see how these integrability properties are reproduced in the present geometric picture. After substitution of the expressions for higher coset fields, the most essential,  $sl(3, R)$  part of the one-form  $\Omega^{red}$  defined in eq. (5.2) reads

$$\Omega^{red} = (L_{-1} - 5vJ_2 - 3uL_1)dx + [160vL_1 + (9u^2 - \frac{1}{2}u'')J_2 + J_{-2} - 6uJ_0 - 2u'J_1]dt. \quad (5.7)$$

As has been mentioned before, the original Maurer-Cartan equations for this one-form are closed modulo higher-spin generators. Discarding the higher-spin pieces in the commutators of the  $sl(3, R)$  generators in  $\Omega^{red}$ , one easily establishes that the Maurer-Cartan equation

$$d^{ext}\Omega^{red} = \Omega^{red} \wedge \Omega^{red} \quad (5.8)$$

implies the Boussinesq equation (5.5) and so provides the zero-curvature representation for the latter. Recall that the original Maurer-Cartan equation (4.12) was purely kinematical. It becomes dynamical after invoking the covariant reduction constraints (5.1) (5.2). It should be emphasized that just these constraints are primary dynamical restrictions on the fields  $u$  and  $v$  in the present approach; the zero-curvature representation (5.8) is their consequence. This feature is typical for all other examples where the covariant reduction proved to be efficient [14, 15, 12].

To obtain a Lax representation from eqs. (5.7), (5.8), one introduces the "covariant derivatives"

$$\frac{\partial}{\partial t} + A_t, \quad \frac{\partial}{\partial x} + A_x,$$

where the  $sl(3, R)$  algebra valued connections  $A_t$  and  $A_x$  coincide with the coefficients of  $dt$  and  $dx$  in (5.7), and rewrites eq. (5.8) as the condition of commutativity of these covariant derivatives. Note that in this way one obtains just the Drinfel'd-Sokolov type Lax pair [9] for the Boussinesq equation (after choosing  $sl(3, R)$  generators in the fundamental  $3 \times 3$  matrix representation).



Let us turn to discussing the coset (4.2). As was already mentioned, it is an extension of the  $W_2$  coset associated with the stability subalgebra (2.11). So the relevant set of the covariant reduction constraints should be an appropriate generalization of the set (2.12):

$$\omega_n = 0, \quad \forall_{n \geq 1}, \quad \theta_m = 0, \quad \forall_{m \geq 1}. \quad (5.9)$$

It includes the previous set (5.1) and, in addition, implies vanishing of the Cartan forms  $\omega_1, \theta_1, \theta_2$ . The fields  $v$  and  $u$  still obey the Boussinesq equation (5.5) but now they are expressed (like  $v_2$ ) through the fields  $v_1$  and  $u_1$  which are the only independent coset fields for the realization at hand. Bearing in mind the explicit expressions for the additional coset forms (eqs. (4.8) - (4.10)), one finds:

$$u = \frac{1}{3}(u_1' + u_1^2 + 12v_1^2) \quad (5.10)$$

$$v = \frac{1}{5}\left(\frac{1}{4}v_1'' + \frac{1}{2}u_1'v_1 + \frac{3}{2}u_1v_1' + 2u_1^2v_1 - 8v_1^3\right). \quad (5.11)$$

By the same mechanism as in the previous case (compatibility between the equations coming from the coefficients of  $dx$  and  $dt$  in the appropriate forms) one also obtains the dynamical restrictions on the fields  $u_1$  and  $v_1$

$$\begin{aligned} \dot{u}_1 &= -8(v_1' + 4u_1v_1)' \\ \dot{v}_1 &= \frac{2}{3}(u_1' - 2u_1^2 + 24v_1^2)'. \end{aligned} \quad (5.12)$$

These equations can be easily checked to be consistent with the Boussinesq equation: differentiating (5.10), (5.11) with respect to  $t$  and making use of eqs. (5.12) one obtains just (5.5).

The expressions (5.10), (5.11) are a genuine generalization of eq. (2.13) and provide a Miura map of the  $W_3$  currents  $u$  and  $v$  onto the two independent  $U(1)$  Kac-Moody currents  $u_1$  and  $v_1$ . Thus in the present case this map also gets a geometric interpretation as the covariant relations between the fields parametrizing the coset of  $W_3^\infty$  symmetry.

By analogy with the modified KdV equation, it is natural to call eqs. (5.12) the modified Boussinesq equation. It can be rederived from the vanishing of the curvature of the reduced Cartan form  $\Omega_1^{red}$  (with the higher-spin generators factored out)

$$\begin{aligned} \Omega_1^{red} &= [J_{-2} - 4u_1J_{-1} + 16(v_1' + 4u_1v_1)L_0 + 2(2u_1^2 - 24v_1^2 - u_1')J_0 + 16v_1L_{-1}]dt \\ &+ (L_{-1} - 2u_1L_0 - 3v_1J_0)dx \end{aligned} \quad (5.13)$$

and so is integrable like the Boussinesq equation. One observes that  $\Omega_1$  is given on the five-dimensional subalgebra of the  $sl(3, R)$ . So in the present case the covariant reduction actually leaves us with the coset space of the group associated to this subalgebra over the subgroup generated by  $J_{-1}, J_0$  and  $L_0$ . Once again, the coordinates  $t$  and  $x$  are the parameters of this coset while the fields  $u_1$  and  $v_1$  specify how the latter is embedded into the original  $W_3^\infty$  coset.

Finally, let us see which new features are brought about by passing to the coset  $g_2$  defined in eq.(4.3). In this case the essential coset fields are  $u_0, v_0$  and, in addition to

the previous constraints, one should require vanishing of the two newly appearing coset Cartan forms  $\omega_0$  and  $\theta_0$  given by eq. (4.11)

$$\omega_0 = \theta_0 = 0, \Rightarrow \quad (5.14)$$

$$u_1 = \frac{1}{2}u_0', \quad v_1 = \frac{1}{3}v_0', \quad (5.15)$$

$$\dot{u}_0 = -\frac{16}{3}(v_0'' + 2u_0'v_0'), \quad \dot{v}_0 = u_0'' - (u_0')^2 + \frac{16}{3}(v_0')^2. \quad (5.16)$$

The covariant relations (5.15) are analogs of the first of eqs. (2.10), they give a further Miura transformation from the  $U(1)$  Kac-Moody currents  $u_1, v_1$  and the  $W_3$  currents  $u, v$  to the scalar fields  $u_0$  and  $v_0$ . For  $u$  and  $v$  one obtains the representation

$$\begin{aligned} u &= \frac{1}{6} \left[ u_0'' + \frac{1}{2}(u_0')^2 + \frac{8}{3}(v_0')^2 \right] \\ v &= \frac{1}{5} \left[ \frac{1}{12}v_0''' + \frac{1}{12}u_0''v_0' + \frac{1}{4}u_0'v_0'' + \frac{1}{6}(u_0')^2v_0' - \frac{8}{27}(v_0')^3 \right], \end{aligned} \quad (5.17)$$

which, after appropriate rescalings, is recognized as the free-field Feigin-Fuchs type representation for the  $W_3$  currents [4, 10]. Once again, the dynamical equation (5.16) induces for the sets  $u, v$  and  $u_1, v_1$  the Boussinesq equation (5.5) and the modified Boussinesq equation (5.12). It amounts to the zero-curvature representation for the reduced one-form

$$\Omega_2^{red} = \omega_{-1}L_{-1} + \theta_{-1}J_{-1} + \theta_{-2}J_{-2} \quad (5.18)$$

with

$$\begin{aligned} \omega_{-1} &= e^{-u_0} \left[ 4u_0' \sinh(4v_0)dt + (dx + \frac{16}{3}v_0'dt) \cosh(4v_0) \right] \\ \theta_{-1} &= -e^{-u_0} \left[ 2u_0' \cosh(4v_0)dt + \frac{1}{2}(dx + \frac{16}{3}v_0'dt) \sinh(4v_0) \right] \\ \theta_{-2} &= e^{-2u_0} dt. \end{aligned} \quad (5.19)$$

Thus in the present case the original coset of  $W_3^\infty$  has been covariantly reduced to the two-dimensional coset of the three-parameter subgroup with the generators  $L_{-1}, J_{-1}, J_{-2}$  over the one-parameter subgroup generated by  $J_{-1} + 2L_{-1}$ .

In conclusion of this Section we briefly discuss the relation to the Hamiltonian approach which provides one more link between the Boussinesq equation and the algebra  $W_3$ . It is known [6] that this equation can be interpreted as a Hamiltonian flow on  $W_3$ . Namely, it possesses the second Hamiltonian structure with the Poisson brackets between  $u$  and  $v$  forming  $W_3$

$$\dot{u} = \{u, H\}, \quad \dot{v} = \{v, H\}, \quad (5.20)$$

$$H = \frac{40c}{3} \int dx v(x, t) \quad (5.21)$$

$$\{u(x, t), u(y, t)\} = \frac{2}{c} \left[ \frac{1}{6} \frac{\partial^3}{\partial y^3} + 2u \frac{\partial}{\partial y} - u' \right] \delta(x - y)$$

$$\{u(x, t), v(y, t)\} = -\frac{2}{c} \left[ 3v \frac{\partial}{\partial y} + v' \right] \delta(x - y)$$

$$\{v(x,t), v(y,t)\} = \frac{3}{10c} \left[ -\frac{1}{48} \frac{\partial^5}{\partial y^5} + \frac{5}{4} u \frac{\partial^3}{\partial y^3} + \frac{15}{8} u' \frac{\partial^2}{\partial y^2} - \left( -\frac{9}{8} u'' + 12u^2 \right) \frac{\partial}{\partial y} - \left( -\frac{1}{4} u''' + 12uu' \right) \right] \delta(x-y) \quad (5.22)$$

where the fields in the r.h.s. are evaluated at the point  $y$ . Decomposing  $u$  and  $v$  in the Fourier modes with respect to  $x$ , one observes that the algebra (5.22) implies for these modes just the  $W_3$  algebra relations (3.1), (3.2).

We wish to point out that this Hamiltonian formalism matches very naturally with our nonlinear realization approach, though the precise relation between these two is as yet not clear to us. If one substitutes the Fourier decomposition of  $v$  in the Hamiltonian (5.21) and integrates over  $x$ ,  $H$  is recognized, up to a scale factor, as the generator  $J_{-2}$ , just the time translation generator in the nonlinear realization scheme.

As far as the modified Boussinesq equations (5.12), (5.16), are concerned they can be given the standard Hamiltonian form like (5.20) with the same Hamiltonian (5.21) expressed in terms of  $u_1, v_1$  or  $u_0, v_0$  by eqs. (5.11), (5.17) and with the following underlying Poisson structure

$$\{u_1(x,t), u_1(y,t)\} = \frac{3}{c} \frac{\partial}{\partial y} \delta(x-y), \quad \{v_1(x,t), v_1(y,t)\} = \frac{1}{4c} \frac{\partial}{\partial y} \delta(x-y), \quad (5.23)$$

$$\{u_1(x,t), v_1(y,t)\} = 0,$$

$$\{u_0(x,t), u_0(y,t)\} = -\frac{12}{c} \delta(x-y), \quad \{v_0(x,t), v_0(y,t)\} = -\frac{9}{4c} \delta(x-y), \quad (5.24)$$

$$\{u_0(x,t), v_0(y,t)\} = 0.$$

These Poisson brackets are characteristic of the  $U(1)$  Kac-Moody currents and free scalar fields.

## 6 $W_3$ symmetry of Boussinesq equations as left $W_3^\infty$ shifts

When studying integrable systems, one of the most important questions is which symmetries preserve the given equation. In the previous Sections we reformulated the Boussinesq equation in the framework of the nonlinear realizations approach as one of the covariant conditions which single out a two dimensional geodesic hypersurface in the coset (4.1) of  $W_3^\infty$ . Symmetries of Boussinesq equation are then the set of  $W_3^\infty$  transformations acting on the coset elements (4.1) from the left

$$g_0(\lambda) g(x, t, u, v, \dots) = g(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \dots) h(\lambda, x, t, u, v, \dots). \quad (6.1)$$

Here  $g_0(\lambda)$  is an arbitrary element of  $W_3^\infty$  with constant parameters and the induced element  $h$  belongs to the stability subgroup  $H$  generated by the set of generators (3.7).

In principle one could directly evaluate  $\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}$  by using eq.(6.1) and the commutation relations (3.1). However, in the case at hand even the infinitesimal transformations of the

fields  $\delta u, \delta v$  and coordinates  $\delta x, \delta t$  are very complicated functions of time  $t$  and all the higher-order coset fields  $\psi_n, \xi_m$ , so it is not too enlightening to try to give them explicitly. Below we will pursue another approach, in which transformation properties of the fields  $u, v$  and coordinates  $x, t$  are obtained together with an additional condition fixing the  $t$  dependence of the transformation parameters.

Let us begin by writing down the Cartan forms for the transformed coset (6.1)

$$\Omega = h^{-1} \tilde{\Omega} h + h^{-1} dh \quad (6.2)$$

$$\tilde{\Omega} \equiv g^{-1}(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \dots) dg(\tilde{x}, \tilde{t}, \tilde{u}, \tilde{v}, \dots), \quad (6.3)$$

where we have made use of the fact that the group parameters in (6.1) do not depend on  $x, t$ . Now, the induced element  $h$  of the stability subgroup can be parametrized as follows

$$h = e^{a_0 L_0} e^{a_1 L_1} e^{b_{-1} J_{-1}} e^{b_0 J_0} e^{b_1 J_1} e^{b_2 J_2} \tilde{h}, \quad (6.4)$$

where  $\tilde{h}$  stands for the factors spanned by higher-spin generators. Keeping in mind that the higher-spin generators form an ideal in the stability subalgebra and comparing the Cartan forms associated with the generators  $L_{-1}, L_0, L_1, J_{-2}, J_{-1}, J_0, J_1, J_2$  in both sides of (6.2) we immediately obtain the following set of equations (let us remind that the parameters  $a$  and  $b$  are infinitesimal)

$$\begin{aligned} \omega_{-1} &= \tilde{\omega}_{-1} - a_0 \tilde{\omega}_{-1} + 16b_1 \tilde{\theta}_{-2} + 8b_{-1} \tilde{\theta}_0 \\ 0 &= da_0 + 64b_2 \tilde{\theta}_{-2} + 8b_{-1} \tilde{\theta}_1 - 2a_1 \tilde{\omega}_{-1} \\ \omega_1 &= da - 1 + \tilde{\omega}_1 + a_0 \tilde{\omega}_1 - 8b_1 \tilde{\theta}_0 - 8b_0 \tilde{\theta}_1 - 16b_{-1} \tilde{\theta}_2 \\ \theta_{-2} &= \tilde{\theta}_{-2} - 2a_0 \tilde{\theta}_{-2} - b_{-1} \tilde{\omega}_{-1} \\ 0 &= db_{-1} - 4a_1 \tilde{\theta}_{-2} - 2b_0 \tilde{\omega}_{-1} \\ \theta_0 &= db_0 + \tilde{\theta}_0 + 3b_{-1} \tilde{\omega}_1 - 3b_1 \tilde{\omega}_{-1} \\ \theta_1 &= db_1 + \tilde{\theta}_1 + a_0 \tilde{\theta}_1 - 2a_1 \tilde{\theta}_0 - 4b_2 \tilde{\omega}_{-1} + 2b_0 \tilde{\omega}_1 \\ \theta_2 &= db_2 + \tilde{\theta}_2 + 2a_0 \tilde{\theta}_2 - a_1 \tilde{\theta}_1 + b_1 \tilde{\omega}_1 \end{aligned} \quad (6.5)$$

From the explicit expressions for the lowest Cartan forms (4.5), (4.6) and (4.7) we obtain the following equations for the variations of fields and coordinates:

$$\begin{aligned} \delta u &\equiv \tilde{u}(x + \delta x, t + \delta t) - u(x, t) = \frac{1}{6} \left[ \frac{1}{2} (\delta t)'' - 6u(\delta t) + 480v(\delta t)' \right] \\ \delta v &\equiv \tilde{v}(x + \delta x, t + \delta t) - v(x, t) \\ &= -\frac{1}{160} \left[ \frac{1}{4} (\delta t)' - 6u(\delta x) + 240v(\delta t) - 8u'(\delta t)'' + 8u''(\delta t)' \right] \\ (\delta t) &\equiv 2(\delta x)' \\ (\delta x) &\equiv 16 \left[ 8u(\delta t)' - \frac{1}{6} (\delta t)''' \right] \end{aligned} \quad (6.7)$$

Several comments are needed concerning the transformations properties (6.7), (6.8).

First of all, the time dependence of variations of the coordinates  $\delta x$ ,  $\delta t$  is controlled by the differential equations (6.8) which involve the field  $u(x, t)$ . It is hardly possible to find the general solution of these equations in a closed explicit form. Nonetheless, after expanding the coordinate variations and the field  $u$  in Taylor series with respect to  $t$

$$\delta t = \sum_{n=0}^{\infty} \delta_n t(x) t^n, \quad \delta x = \sum_{n=0}^{\infty} \delta_n x(x) t^n, \quad u(x, t) = \sum_{n=0}^{\infty} \frac{\partial^n u}{\partial t^n} t^n, \quad (6.9)$$

the role of eqs.(6.8) is reduced to expressing all functions  $\delta_n t(x)$ ,  $\delta_n x(x)$  through two independent functions  $\delta_0 t(x)$  and  $\delta_0 x(x)$ . So the transformations (6.7) are actually specified by the two functions of the coordinate  $x$ , much like the realization of one of the light-cone  $W_3$ 's in the  $sl_3$  Toda system [12]. Thus we have proven that the nonlinear realization of  $W_3^\infty$  in the coset considered is reduced to a kind of  $W_3$  transformations of the fields  $u$  and  $v$ . The same of course is true for other nonlinear realizations, with  $u_1$ ,  $v_1$  and  $u_0$ ,  $v_0$  as the essential coset parameters.

Secondly, after passing to the active form of the transformations of  $u, v$

$$\tilde{\delta}u = \delta u - \delta t \dot{u} - \delta x u', \quad \tilde{\delta}v = \delta v - \delta t \dot{v} - \delta x v'$$

and eliminating time derivatives by the Boussinesq equation (5.5) and constraints (6.8) we obtain the standard  $W_3$  transformations for the spin 2 and spin 3 currents:

$$u \equiv -T, \quad v \equiv -\frac{3}{80}J, \quad \delta x \equiv -f, \quad \delta t \equiv g$$

$$\begin{aligned} \tilde{\delta}_{conf} T &= \frac{1}{6}f''' + 2f'T + fT' \\ \tilde{\delta}_{conf} J &= 3f'J + fJ' \\ \tilde{\delta}_w T &= 3g'J + 2gJ' \\ \tilde{\delta}_w J &= -\frac{2}{9}g'''' - \frac{40}{3}g'''T - 20g''T' - 12g'T'' + 32g'\Lambda - \frac{8}{3}gT''' + 16g\Lambda'' \end{aligned} \quad (6.10)$$

where

$$\Lambda = -4T^2$$

and parameters are still subject to the constraints (6.8).

These transformations and constraints are just those deduced in a recent paper [18] starting with a Lax representation for the Boussinesq equation. In our scheme they come out in a nice geometric way as the  $W_3^\infty$  group motions on the set of essential coset parameters  $\{x, t, u(x, t), v(x, t)\}$ . Invariance of the Boussinesq equation under these transformations does not need to be checked, it directly stems from the fact that this equation is a dynamical part of the inverse Higgs constraints (5.1) which are  $W_3^\infty$ -covariant by construction.

Before passing to concluding remarks let us comment on the cosets of  $W_3^\infty$  corresponding to the choice of  $\mathcal{H}'_1, \mathcal{H}'_2$  (eq.(3.11)) as the stability subgroup algebras. In these cases the generator  $J_{-2}$  belongs to the stability subgroup, so one is left with one coordinate  $x$ , on which all other coset parameters are assumed to depend. The covariant reduction

constraints, like in nonlinear realizations of Virasoro symmetry (Sect. 2), do not produce any dynamical restrictions and serve entirely for the covariant elimination of the higher-dimension coset fields via the essential ones ( $u_1(x), v_1(x)$  or  $u_0(x), v_0(x)$ ). In particular, the Miura maps (5.10), (5.11), (5.15) and (5.16) arise as before. For the independent coset fields the left  $W_3^\infty$  shifts generate the standard  $sl_3$  Toda-type  $W_3$  transformations parametrized by two functions of  $x$  which collect the constant parameters associated with the spin 2 and spin 3 generators  $L_n$  and  $J_n$ . These transformations are just those deduced in [12] (as far as one light-cone copy of  $W_3^\infty$  is considered).

These realizations actually bear a tight relation to those associated with the cosets (4.2), (4.3). One may pass to a different parametrization of these coset elements where the time factor  $e^{iJ-2}$  stands from the right ( $\tilde{t}$  is related to  $t$  via a complicated field-dependent redefinition). Then one may check that the first-order time derivatives of the coset fields in this new parametrization are transformed into themselves under left  $W_3^\infty$  shifts and so can be self-consistently put equal to zero, thus eliminating any time dependence of the coset fields. This is equivalent to placing  $J_{-2}$  from the beginning in the stability group algebra.

## 7 Conclusion

In this paper we have revealed a new kind of the relationship between  $W_3$  symmetry and Boussinesq as well as modified Boussinesq equations: these have been found to emerge in a geometric way as covariant dynamical constraints on the parameters of some coset manifolds of  $W_3^\infty$  symmetry associated with  $W_3$ . The Miura maps relating these equations to each other arise as a sort of covariant kinematical constraints on the coset parameters. Put together, these constraints can be interpreted as the conditions singling out finite-dimensional geodesic hypersurfaces in the original infinite-dimensional coset manifolds. The spin 2 and spin 3  $W_3$  currents and the introduced via Miura maps two spin 1  $U(1)$  Kac-Moody currents and two spin 0 scalar fields come out as the essential parameters of three coset manifolds of  $W_3^\infty$  embedded into each other. Thus the considered Boussinesq-type equations, related Miura maps, involved currents and fields prove to be intimately linked to the intrinsic geometries of the coset manifolds of  $W_3^\infty$ , just like the  $sl_3$  Toda equations [12]. The common geometric origin of the latter equations and the Boussinesq ones suggests a deep connection between them which can hopefully be exposed most clearly within the present approach. The understanding of this relationship could have important implications, e.g. in  $W_3$  strings and  $W_3$  gravity.

An interpretation of Miura maps as the covariant relations between the fields parametrizing coset manifolds of the  $W_N^\infty$  type symmetries seems to be especially useful in searching for free-field representations of the currents generating more complicated  $W$  symmetries and their superextensions. Usually this is a subject of some guess-work. As we have argued in [12] and this paper, within the present approach finding such representations becomes more straightforward and algorithmic. One starts by defining the appropriate linear  $W_3^\infty$  type symmetry and its cosets, then construct the Cartan forms and finally impose suitable covariant reduction constraints. Doing so, we have recently found, e.g., Miura maps for the supercurrents of  $N = 2$  super  $W_3$  algebra [20, 24].

There remain some interesting problems with the Boussinesq equation itself. In particular, it is desirable to have a full understanding of the relationship with the Hamiltonian formulation and the formulation which uses the Gel'fand-Dikii brackets. Also it is as yet unknown how to incorporate in the present scheme in a simple way next equations from the Boussinesq hierarchy. To this end it seems natural to extend the coset spaces of  $W_3^\infty$  by placing in the coset some higher-spin generators  $J_n^{(s)}$  from the stability subgroup, e.g. the spin four one  $J_n^{(4)}$ , and to introduce additional time variables as the coset parameters associated with the generators  $J_{-s+1}^{(s)}$ , e.g.  $J_{-3}^{(4)}$ . New coset fields are expected to be removable by inverse Higgs effect, still leaving  $u$  and  $v$  (or  $u_1$  and  $v_1$  or  $u_0$  and  $v_0$ ) as the only essential fields of the theory. At the same time, due to the appearance of extra time variables, the essential fields could obey higher-order Boussinesq equations with respect to these variables as a consequence of appropriate extensions of the covariant reduction procedures employed above.

Finally, we would like to point out that the covariant reduction approach invented and applied first in the case of Liouville theory [14] mainly for the practical purpose of constructing higher superextensions of this theory [15] now turns out to possess a considerably wider range of applicability. It can be regarded as a universal tool for treating integrable systems in a manifestly geometric language of the coset space realizations of appropriate infinite-dimensional symmetries. The Toda systems [12], Boussinesq and KdV [17] hierarchies certainly admit an adequate geometric description in its framework. It would be of interest to consider along similar lines other classical integrable systems, such as the sine-Gordon and nonlinear Schrodinger equations, and to understand what are analogs of, say,  $W_3^\infty$  in all these cases. On the other hand, one of the problems ahead is to apply our nonlinear realization techniques to all known  $W$  type (super)algebras (e.g. Knizhnik-Bershadsky superalgebras) and to deduce the integrable equations associated with them. So our main goal is to provide a common geometrical basis for various integrable systems in 1+1 dimensions and the present work should be regarded as a step in this direction.

## References

- [1] A.B. Zamolodchikov, *Teor. Mat. Fiz.* 65 (1985) 347;  
V.A. Fateev and S. Lukyanov, *Int. Journ. Mod. Phys. A3* (1988) 507
- [2] M. Fukuma, H. Kawai and R. Nakayama, *Int. Journ. Mod. Phys. A6* (1991) 1385
- [3] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Nucl. Phys. B348* (1991) 435;  
E. Verlinde and H. Verlinde, *Nucl. Phys. B348* (1991) 457
- [4] A. Bilal and J.L. Gervais, *Phys. Lett. B* 206 (1988) 412; *Nucl. Phys. B* 314 (1989) 579, 646;  
I. Bakas, *Nucl. Phys. B* 302 (1988) 189
- [5] J.L. Gervais, *Phys. Lett. B* 160 (1985) 277
- [6] P. Mathieu, *Phys. Lett. B* 208 (1988) 101
- [7] K. Yamagishi, *Phys. Lett. B* 259 (1991) 436;  
F. Yu and Y.S. Wu, *Phys. Lett. B* 236 (1991) 220;  
A. Das, W.J. Huang and S. Panda, *Phys. Lett. B* 271 (1991) 109;  
A. Das, E. Sezgin and S.J. Sin, *Phys. Lett. B* 277 (1992) 435
- [8] I.M. Gel'fand and L.A. Dikii, *Math. Surv.* 30 (5) (1975) 77; *Funkt. Anal. Priloz.* 10 (1976) 13; 13 (1979) 13
- [9] V. Drinfel'd and V. Sokolov, *J. Sov. Math.* 30 (1985) 1975; *Sov. Math. Dokl.* 23 (1981) 457
- [10] S. Bellucci and E. Ivanov, *Mod. Phys. Lett. A6* (1991) 1269
- [11] E. Bergshoeff, A. Bilal and K.S. Stelle, *Int. J. Mod. Phys. 6* (1991) 5949
- [12] E. Ivanov, S. Krivonos and A. Pichugin, "Nonlinear realizations of  $W_3$  symmetry", preprint JINR E2-91-328, Dubna, 1991 (to appear in *Phys. Lett. B*)
- [13] S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* 177 B (1969) 2239;  
C. Callan, S. Coleman, J. Wess and B. Zumino, *Phys. Rev.* 177 B (1969) 2247;  
D.V. Volkov, *Sov. J. Part. Nucl.* 4 (1973) 3  
V.I. Ogievetsky, in *Proceedings of 10th Winter school of Theoretical Physics in Karpach*, v.1, p. 117 (Wroclaw 1974)
- [14] E.A. Ivanov and S.O. Krivonos, *Teor. Mat. Fiz.* 58 (1984) 200; *Lett. Math. Phys.* 8 (1984) 39
- [15] E.A. Ivanov and S.O. Krivonos, *Lett. Math. Phys.* 7 (1983) 523; *J. Phys. A: Math. Gen.* 17 (1984) L671;  
E.A. Ivanov, S.O. Krivonos and V.M. Leviant, *Nucl. Phys. B* 304 (1988) 601
- [16] E.A. Ivanov and V.I. Ogievetsky, *Teor. Mat. Fiz.* 25 (1975) 164
- [17] E. Ivanov, S. Krivonos and A. Pichugin, in preparation
- [18] B. Spence, *Phys. Lett. B* 276 (1992) 311
- [19] E.A. Ivanov, S.O. Krivonos and V.M. Leviant, *J. Phys. A: Math. Gen.* 22 (1989) 345
- [20] E. Ivanov, S. Krivonos and R.P. Malik, " $N = 2$  super  $W_3^\infty$  algebra and  $N = 2$  Boussinesq equations", preprint JINR E2-92-300, Dubna, July 1992
- [21] H. Lu, C.N. Pope, L.J. Romans, X. Shen and X.-J. Wang, *Phys. Lett. B* 264 (1991) 91
- [22] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, *Nucl. Phys. B* 349 (1991) 791;  
C.M. Hull, *Nucl. Phys. B* 353 (1991) 707

- [23] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. B 236 (1990) 173; B 242 (1990) 401; Nucl. Phys. B 339 (1990) 191;  
E. Bergshoeff, B. de Wit and M. Vasiliev, Nucl. Phys. B 366 (1991) 315;  
K.Stelle, " $w_\infty$ -Geometry and  $w_\infty$ -Gravity", Preprint JINR E2-91-90, Dubna, 1991;  
E. Bergshoeff, M.P. Blencowe and K.S. Stelle, Commun. Math. Phys. 128 (1990) 213;  
M. Bordemann, J. Hoppe and P. Schaller, Phys. Lett. B 232 (1989) 199;  
M.A. Vasiliev, Int. J. Mod. Phys. A6 (1991) 1115
- [24] E. Ivanov and S. Krivonos, "Superfield realizations of  $N = 2$  super- $W_3$ ", Preprint IC/92/64, Trieste, April 1992

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Построена новая реализация бесконечномерной линейной  $W_3$  симметрии, ассоциированной с  $W_3$  алгеброй, отличающаяся от преобразований инвариантности цепочки Toda. Уравнение Буссинеска и модифицированное уравнение Буссинеска возникают как условия, выделяющие двумерное подпространство в исходном фактор-пространстве группы  $W_3$ . Отметим следующие основные особенности построенной реализации: 1. Параметры фактор-пространства включают пространственно-временные координаты  $x$  и  $t$ , входящие в уравнение Буссинеска. 2. Токи спинов 2 и 3, генерирующие  $W_3$  алгебру, так же как  $U(1)$  токи Каца-Мууди и два скалярных поля, связанных с токами  $W_3$  преобразованиями Миуры, являются единственными существенными полями, параметризующими фактор-пространства; 3. Преобразования Миуры получают чисто геометрическую интерпретацию как  $W_3$  инвариантные связи, на поля, при переходе от одного фактор-пространства к другому; 4. Уравнение Буссинеска, так же как и два типа модифицированных уравнений Буссинеска, возникают как динамические условия, сопровождающие  $W_3$  ковариантную редукцию исходного фактор-пространства к двумерным геодезическим подпространствам; 5. Представления нулевой кривизны для этих уравнений возникают автоматически, как следствия ковариантной редукции; 6.  $W_3$  преобразования симметрии уравнения Буссинеска реализуются как левые  $W_3$  сдвиги на фактор-пространстве. Предложенный подход дает универсальное геометрическое описание связи между нелинейными алгебрами и интегрируемыми системами.

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We construct new coset realizations of infinite-dimensional linear  $W_3$  symmetry associated with Zamolodchikov's  $W_3$  algebra which are different from the previously explored  $sl_3$  Toda realization of  $W_3$ . We deduce the Boussinesq and modified Boussinesq equations as constraints on the geometry of the corresponding coset manifolds. The main characteristic features of these realizations are: i. Among the coset parameters there are the space and time coordinates  $x$  and  $t$  which enter the Boussinesq equations. ii. The spin 2 and 3 currents of  $W_3$  and two spin 1  $U(1)$  Kac-Moody currents as well as two spin 0 fields related to the  $W_3$  currents via Miura maps, come out as the only essential parameters-fields of these cosets. The remaining coset fields are covariantly expressed through them; iii. The Miura maps get a new geometric interpretation as  $W_3$  covariant constraints which relate the above fields while passing from one coset manifold to another; iv. The Boussinesq equation and two kinds of the modified Boussinesq equations appear geometrically as the dynamical constraints accomplishing  $W_3$  covariant reductions of original coset manifolds to their two-dimensional geodesic submanifolds; v. The zero-curvature representations for these equations arise automatically as a consequence of the covariant reduction; vi.  $W_3$  symmetry of the Boussinesq equations amounts to the left action of  $W_3$  symmetry on its cosets. The approach proposed could provide a universal geometric description of the relationship between  $W$ -type algebras and integrable hierarchies.

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