сообщения
объединенного
института
ядерных
исследований
дубна

E2-92-275

S.I.Kostadinov [1], G.Petrov [2]

IMPULSIVE MOVING MIRROR MODEL AND THE STABILITY OF SCHROEDINGER EQUATION WITH IMPULSE EFFECT IN A BANACH SPACE

[1]University of Plovdiv, Bulgaria
[2]Institute of Nuclear Research and Nuclear Energy, Sofia, Bulgaria

1992

1 Introduction

The equations with impulse effect are models of real processes which are subject to disturbances in its evolution acting in time very shot compared with the entire duration of the process.

The first paper in this direction is the paper by Milman and Myshkis[1](1960). The mathematical theory of ordinary differential equations with impulse effect has been further developed in the papers[2]-[4]. The paper [5] initiates the investigations of the equations with impulse effect in a Banach space. The first application of thes equations in quantum field theory is given in the papers [6]-[8].

The advantage of the local field theoretical approach for studing the properties of the vacuum is that it permits a different point of view and a deeper understanding of the nature of the vacuum energy. With the application of the equations with impulse effect in the impulsive moving mirror model in Minkowski space has been given another point of view in the local field theoretical approach for studing the properties of the vacuum.

In these problems the fundamental degrees of freedom of the Schroedinger picture necessarly acquire an implicit time dependence. Hence, the RHS of the Schroedinger equation, $H\Psi = id/dt\Psi$, seemingly becomes subject to interpretation by suddenly change of velocity $\dot{x}_n$ for $t = t_n$, (n = 0,1,2,...) of the moving mirror, e.g. $d/dt\Psi \to \Psi^+(t_n+0) - \Psi(t_n) = \int_{x_n t}^{\infty} dx[\phi^+(x) - \phi(x)]\delta/\delta\phi(x)\Psi(t_n)$.

We give a rigrous canonical analysis of thes impulse effect in paper [6] which confirms the latter.

2 Statement of the Problem

Consider the linear differential Schroedinger equation with impulse effect

$$d/dt\Psi(t) = H(t)\Psi(t) | t \neq t_n, (n = 0, 1, ...) \quad (1)$$

$$\Psi^+(t_n + 0) = F_n\Psi(t_n), t = t_n, (n = 1, 2. ...) \quad (2)$$

where $H(t) = -i/2\int_0^\infty d\omega \int_0^\infty d\omega' \{-T^{-1}_{\omega\omega'}\delta/\delta A_\omega.\delta/\delta A_{\omega'} + A_\omega(t)V_{\omega\omega'}(t)A_{\omega'}(t)\}$
and the impulsive operator $F_n = 1 - t_n/2\int_0^\infty d\omega \int_0^\infty d\omega' \int_0^\infty d\omega'' \{T^{-1}_{\omega\omega'}F_{\omega\omega''}A_{\omega''}\delta/\delta A_{\omega'} + T^{-1}_{\omega\omega'}F_{\omega'\omega''}A_{\omega''}\delta/\delta A_\omega\}$. [see [6]]

The Hamiltonian H(t), for $(t > t_0)$) is a continuous operator-function on t, the values of which are linear bounded operators mapping the complex Banach space into itself. The operators $F_n : B \to B$, for (n=0,1,...) are continuous and the moments t_n, for (n=0,1,...) of the impulse effect satisfy the condition $0 < t_0 < t_1 < \dots$ and $\lim_{n\to\infty}(t_n) = \infty$.

DEFINITION 1. We say that $\Psi(t)$ is a solution of the Schroedinger equation with impulse effect (1),(2) if $\Psi(t)$ is a piecewise right continuous function with first order disconinuities for $t = t_n$, (n=0,1,...) and such that

$$d/dt\Psi(t) = H(t)\Psi(t) \text{ for } t \neq t_n,\ (t \geq t_0),\ (n = 1,2,\dots)$$

and

$$\Psi^+(t_n + 0) = F_n\Psi(t_n). \text{ for } t = t_n.\ (n = 1.2...).$$

For $\Psi \in B$ and $t \geq t_0$ the Schroedinger equation with impulse effect (1),(2) has unique soulution $\Psi(t)$ which satisfies the condition

$$\Psi(t_0) = \Psi_0. \tag{3}$$

Then we can consider the family of operators W(t) defined by the formula

$$W(t)\Psi_0 = \Psi(t).\ (t_0 < t < \infty).$$

Denote by U(t,s) the evolution operator of the Schroedinger equation

$$d/dt\Psi(t) = H(t)\Psi(t),\ t \neq t_n. \tag{4}$$

We note that for $t_n < t \leq t_{n+1}$, (n=1,2,...) the following equality holds

$$W(t) = U(t,t_n)F_nU(t_n,t_{n-1})F_{n-1}\dots F_1U(t_1,t_0).$$

DEFINITION 2. The Schroedinger equation with impulse effect (1),(2) is called stable if

$$\sup_{t_0 \leq t < \infty} \| W(t)\| < \infty.$$

DEFINITION 3. The Schroedinger equation with impulse effect is called asymptotically stable if

$$\lim_{t\to\infty} \| W(t)\Psi\| = 0,\ (\Psi \in B).$$

DEFINITION 4. The Schroedinger equation with impulse effect (1),(2) is called uniformly stable if

$$\lim_{t\to\infty} \| W(t)\| = 0.$$

LEMMA 1. Let the following condition hold

$$H(t)\int_{t_0}^t dsH(s) = \int_{t_0}^t dsH(s).H(t).\ (t \geq t_0). \tag{5}$$

Then the solution $\Psi(t)$ of (4) for $t \geq t_0$ with initial condition has the form

$$\Psi(t) = \exp(\textstyle\int_{t_0}^t dsH(s))\Psi_0.$$

The proof of Lemma 1. is standard.

REMARK 1. For small $t \geq t_0$ the condition (5) is sufficient a well for the solution of (4) to have the form

$$\Psi(t) = \exp(\textstyle\int_{t_0}^t dsH(s))\Psi_0.$$

In fact, let $\Psi(t) = \exp(\int_{t_0}^t dsH(s))\Psi_0$. The equality

$$[exp \textstyle\int_{t_0}^t dsH(s)]' = H(t)\exp\int_{t_0}^t dsH(s)$$

can be presented as

$$\textstyle\sum_{n=0}^{\infty} 1/(n+1)!((\int_{t_0}^t dsH(s))^{n+1})' - 1/n!H(t)(\int_{t_0}^t dsH(s))^n) = 0.$$

We set $\Delta(t) = H(t)\int_{t_0}^t dsH(s) - \int_{t_0}^t dsH(s)H(t)$. Then the following equalities hold:

$$\sum_{n=0}^{\infty} 1/(n+1)!(\sum_{k=0}^{n}(\int_{t_0}^t dsH(s))^k H(t)(\int_{t_0}^t dsH(s))^{n-k} - (n+1)H(t)(\int_{t_0}^t dsH(s))^n) =$$

$$= \sum_{n=0}^{\infty} 1/(n+1)!\sum_{k=1}^{n}((\int_{t_0}^t dsH(s))^k H(t)(\int_{t_0}^t dsH(s))^{n-k} - H(t)(\int_{t_0}^t dsH(s))^n) =$$

$$= \sum_{n=0}^{\infty} 1/(n+1)! \sum_{k=1}^{n} k\left(\int_{t_0}^{t} dsH(s)\right)^{n-k} \Delta(t)\left(\int_{t_0}^{t} dsH(s)\right)^{k-1} = 0.$$

i.e.

$1/2\Delta(t) = -\sum_{n=2}^{\infty} 1/(n+1)! \sum_{k=1}^{n} k(\int_{t_0}^{t} dsH(s))^{n-k}\Delta(t)(\int_{t_0}^{t} dsH(s))^{k-1}$.

Finally we get

$1/2\| \Delta(t)\| \leq \sum_{n=2}^{\infty} \| \Delta(t)\|/(n+1)!.k(k+1)/2 \| \int_{t_0}^{t} dsH(s)\|^{n-1} =$

$= 1/2 \| \Delta(t)\|(exp \| \int_{t_0}^{t} dsH(s)\| - 1)$.

Then for sufficiently small t

$$|\exp \| \int_{t_0}^{t} dsH(s)\| - 1| < 1$$

and there exists an interval $[t_0, t*]$ such that $\Delta(t) = O(t \in [t_0, t*])$.

Applying the step method we come to the condition that the equality $\Delta(t) = 0$ holds as well if t belongs to an interval containing $[t_0, t*]$.

3 Main Result

We say that the condition (A) holds if the following condition is fulfilled:

$$(A)\ \int_{\tau}^{t} dsH(s).H(t) = H(t)\int_{\tau}^{t} dsH(s),\ (t_0 < \tau < t < \infty).$$

REMARK 2. Let the condition (A) hold. Then for $t_0 < \tau < t < \infty$ the following equality holds

$$H(t)H(\tau) = H(\tau)H(t).$$

We say that the conditions (B) are satisfied if:

B1: Condition (A) holds.

B2: There exists a constant $T > 0$ for which $t_{n+1} - t_n \leq T$ for n=1,2...

B3: $\exp \| \int_{\tau}^{t} dsH(s)\| \leq \exp\gamma(t-\tau)$ for $0 \leq t - \tau \leq T$ where γ is a constant.

B4. $\| F_n\Psi\| \leq q_n \| \Psi\| + h_n$ for n = 1,2,... and $\Psi \in B$ so that $q_1 q_2 \ldots q_{n(t)} \leq L\exp\delta t$ where $n(t) = n$ for $t_n \leq t < t_{n+1}$ and L and δ are constants.

If the conditions (B) hold, for the solution $\Psi(t)$ of the Schroedinger equation with impulse effect (1),(2) we obtain the following estimation

$$\begin{aligned}
\| \Psi(t)\| &\leq \exp\gamma(t-t_n)\{q_n \| \exp(\int_{t_{n-1}}^{t_n} dsH(s))F_{n-1}[\exp(\int_{t_{n-2}}^{t_{n-1}} dsH(s))]. \\
&\quad F_{n-2}\ldots F_1[\exp(\int_{t_0}^{t_1} dsH(s))]\Psi_0\| + h_n\} \leq \qquad (6) \\
&\leq [\exp\gamma(t-t_{n-1})]q_n\{q_{n-1} \| [\exp(\int_{t_{n-2}}^{t_{n-1}} dsH(s))]F_{n-2}\ldots F_1 \\
&\quad [\exp(\int_{t_0}^{t_1} dsH(s))]\Psi_0\| + h_{n-1}\} + [\exp\gamma(t-t_n)]h_n \leq \\
&\leq [\exp\gamma(t-t_{n-2})]q_n q_{n-1} \| F_{n-2}\ldots F_1[\exp(\int_{t_0}^{t_1} dsH(s))]\Psi_0\| + \\
&\quad + [\exp\gamma(t-t_{n-1})]q_n h_{n-1} + [\exp\gamma(t-t_n)]h_n \leq \\
&\leq \ldots \leq [\exp\gamma(t-t_0)]q_n q_{n-1}\ldots q_1 \| \Psi_0\| + \{[\exp\gamma(t-t_n)]h_n + \\
&\quad + [\exp\gamma(t-t_n)]q_n h_{n-1} + [\exp\gamma(t-t_n)]q_n q_{n-1}h_{n-2} + \ldots \\
&\quad + [\exp\gamma(t-t_1)]q_n q_{n-1}\ldots q_2 h_1\}.
\end{aligned}$$

Using this estimation we will find sufficient conditions under which the solutions of the Schroedinger equation with impulse effect are bounded.

THEOREM 1. Let the conditions (B) hold. Let $h_1 = h_2 = \ldots = 0$.

Then for $\delta + \gamma < 0, \lim_{t\to\infty} \Psi(t) = 0$ and for $\delta + \gamma = 0$ the solution $\Psi(t)$ is bounded for $t \geq t_0$.

The proof of the theorem follows immediately from the estimation (6).

THEOREM 2. Let the conditions (B) be satisfied. Suppose additionally that

$$q = q_1 = q_2 = \ldots;\ h = h_1 = h_2 = \ldots$$

and let

$$t_n = t_0 + nk\,(n = 1,2,\ldots).\ t = t_0 + \tau k.$$

[that is for nonrelativistic case] where $n \leq \tau \leq n+1$. $k \leq T$ is a constant and the inequality

$$qe^{\gamma k} < 1$$

holds. Then for $\delta + \gamma \leq 0$ the solution $\Psi(t)$ is bounded on the half axis $t \geq t_0$.

Proof. We denote by K the expression in the big brackets in (6) and after certain transformations we get

$$K = e^{\gamma(\tau - n)} \frac{q^n e^{\gamma k n} - 1}{q e^{\gamma k} - 1} h.$$

From the assumtions it follows that K is bounded. This conclusion combined with the condition B4 complets the proof.

THEOREM 3. Suppose that the following conditions hold:

1. The condition (A) is fulfilled.

2. There exist constants $p \geq 0, r \geq 0$ and $\sigma > 0, \rho > 0$ such that $0 \leq i(t_0, t) - p(t - t_0) \leq \sigma, 0 \leq i(t_0, t) - r(t - t_0) \leq \rho$ for $0 \leq t_0 \leq t < \infty$ where $i(a, b)$ is the number of the points t_n which lie in the interval $(a, b)(t_0 < a < b < \infty)$.

3. $F = F_1 = F_2 = \ldots$ where the operator F is linear and

$$[\int_\tau^t dsH(s)]F = C.F. \int_\tau^t dsH(s) \text{ for } t_0 \leq \tau \leq t < \infty \text{ and } C^{n(t)} \leq Re^{\rho t}$$
where $n(t) = n$ for the interval (a, b).

4. The operator F has a logaritm, i.e. there exists a linear and bounded operator $\ln F : B \to B$ such that $F = \exp \ln F$. Then the Schroedinger equation with impulse effect is bounded, asymptotically bounded, uniformly asimptotically bounded if and only if respectively

a) $\sup_{t \geq t_0} \| \exp \int_{t_0}^t ds(H(s) + p \ln F + r \ln C) \| < \infty$.

b) $\lim_{t \to \infty} \| [\exp \int_{t_0}^t ds(H(s) + p \ln F + r \ln C)] \Psi \| = 0 \, (\Psi \in B)$.

c) $\lim_{t \to \infty} \| \exp \int_{t_0}^t ds(H(s) + p \ln F + r \ln C) \| = 0$.

Proof. The operator-function W(t) has the form

$$\begin{aligned} W(t) &= [\exp \int_{t_0}^t dsH(s)][F.C]^{i(t_0,t)} = [\exp \int_{t_0}^t dsH(s)] F^{[p(t-t_0)]} F^{i(t_0,t)-[p(t-t_0)]}. \\ & C^{[r(t-t_0)]} C^{i(t_0,t)-[r(t-t_0)]} = [\exp \int_{t_0}^t dsH(s)][\exp \ln F[p(t - t_0)]] \\ & F^{i(t_0,t)-[p(t-t_0)]}[\exp \ln C[r(t - t_0)]] C^{i(t_0,t)-[r(t-t_0)]} = [\exp \int_{t_0}^t dsH(s)]. \end{aligned}$$

$$\begin{aligned} & [\exp \ln Fp(t - t_0)][\exp(\ln F[p(t - t_0)] - \ln Fp(t - t_0))] F^{i(t_0,t)-[p(t-t_0)]}. \\ & [\exp \ln Cr(t - t_0)][\exp(\ln C[r(t - t_0)] - \ln Cr(t - t_0)) C^{i(t_0,t)-[r(t-t_0)]} = \\ = \; & [\exp \int_{t_0}^t ds(H(s) + p \ln F + r \ln C)][\exp \ln F\{[p(t - t_0)] - p(t - t_0)\}]. \\ & F^{i(t_0,t)-[p(t-t_0)]}.[\exp \ln C\{[r(t - t_0)] - r(t - t_0)\}] C^{i(t_0,t)-[r(t-t_0)]}. \qquad (7) \end{aligned}$$

Taking ito account the representation (7) the assertions a), b) and c) follows immediately.

References

[1] Millman, V.D., Myshkis, A.D.,: On the stability of motion in presence of the impulses, Sibirski Math. J.,1 (1960), 233-237. (Russian)

[2] Myshkis, A.D., Samoilenko. A.M.:Systems with impulses in prescribed moments of time. Math. Sb., 74 (1967). (Russian).

[3] Samoilenko, A.M., Perestiuk, N.A.: Stability of the solutions of rifferential equations with impulse effect, Diff. Equations, 11,(1977). 1881- 1992, (Russian).

[4] Samoilenko, A.M., Perestiuk, N.A.: On the stability of the solutions of systems with impulse effect. Riff. Equations. 11. (1981) 1995 - 2001, (Russian).

[5] Zabreiko, P.P., Bainov, D.D., Kostadinov, S.I.: On the stability of class of linear homogeneous differential equations with impulse effect in a Banach space, Annales Universtatis Scient. Budapestiensis de Rolando Eoetvoes nominate,sect. Math. t. XXXI,(1988), 39-44.

[6] Petrov, G.:Impulsive Moving Mirrors and Impulsive differential Equations in Banach space, (submitted).

[7] Bainov, D.D., Kostadinov, S.I.,Petrov, G: Asymptotic behaviour of the solutions of Schroedinger equation with impulse effect in a Banach space, (submitted).

[8] Kostadinov,S.I.,Petrov, G.: Impulsive moving mirror model and the stability of linear homogeneous differential equations with impulse effect in a Banach space,(submitted).

Received by Publishing Department
on June 29, 1992.

Костадинов С.И., Петров Г. E2-92-275
Модель импульсно движущегося зеркала
и стабильность уравнения Шредингера
с импульсным эффектом в пространстве Банаха

Из специального класса систем получено уравнение Шредингера с импульсным эффектом в теории поля для пространства Минковского с зависящими от времени граничными условиями для движущегося зеркала. Полевой теоретический подход в исследовании свойств вакуума начался с анализа поведения локальных полевых величин в пространстве Минковского с равномерно движущимися зеркалами. Для модели импульсно движущегося зеркала реальный процесс взаимодействия между квантовым полем и внешним зеркалом протекает очень быстро по сравнению с внешним воздействием. Так, стабильность решений уравнения Шредингера для процесса есть стабильность вакуума Казимира.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1992

Kostadinov S.I., Petrov G. E2-92-275
Impulsive Moving Mirror Model
and the Stability of Schroedinger Equation with Impulse
Effect in a Banach Space

From a special class of systems has been used a Schroedinger equation with impulse effect in Minkowski space field theory with time dependent boundary conditions, i.e. those of moving mirrors. The field theoretical approach for studing the properties of the vacuum starts from an analysis of the behaviour of local field quantities in Minkowski space with uniformly moving mirrors. For the impulsive moving mirror model is the real process of interaction between the quantum field and the external mirror a subject to disturbances in its evolution acting in time very short compared with the entire duration of the process. So the stability of the solution of the Schroedinger evolution equation for the process is the stability of the vacuum of Casimir.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1992